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# Variable Exponent Spaces of Analytic Functions

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## Abstract

Variable exponent spaces are a generalization of Lebesgue spaces in which the exponent is a measurable function. Most of the research done in this topic has been situated under the context of real functions. In this work, we present two examples of variable exponent spaces of analytic functions: variable exponent Hardy spaces and variable exponent Bergman spaces. We will introduce the spaces together with some basic properties and the main techniques used in the context. We will show that in both cases, the boundedness of the evaluation functionals plays a key role in the theory. We also present a section of possible directions of research in this topic.

**Keywords:** variable exponents, Hardy spaces, Bergman spaces, Carleson measures, analytic functions

## 1. Introduction

In recent years the interest for nonstandard function spaces has risen due to several considerations including their need in some fields of applied mathematics, differential equations, and simply the possibility to study extensions and generalizations of classical spaces.

One of such type of spaces is the variable exponent Lebesgue spaces. A first introduction to them was due to Orlicz in 1931. He considered measurable functions  $u$  such that

$$\int_0^1 |u(x)|^{p(x)} dx < \infty,$$

the usual exponent  $p$  from the classical theory of Lebesgue spaces is replaced by a suitable function  $p(\cdot)$ . Orlicz was interested in the summability of Fourier series which led him to consider spaces of sequences  $(a_n)$  so that

$$\sum_{n=1}^{\infty} |\lambda a_n|^{p_n} < \infty,$$

for  $\lambda > 0$  and  $p_n \in [1, \infty)$ .

Later generalizations consisted on considering real functions  $u$  on a domain  $\Omega$  such that

$$\int_{\Omega} \varphi(x, |u(x)|) dx < \infty,$$

where

$$\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$$

is a suitable function. Such spaces are known nowadays as Musielak-Orlicz spaces. We are interested in the case

$$\varphi(x, y) = y^{p(x)},$$

which corresponds to variable exponent Lebesgue spaces, with some adequate restrictions for the *exponent*  $p$ .

The study of problems in spaces with nonstandard growth has been mainly related to Lebesgue spaces with variable order  $p(x)$ , to the corresponding Sobolev spaces, and to generalized Orlicz spaces (see surveys [1, 2]). The investigation of structural properties of variable exponent spaces and of operator theory in such spaces is of interest not only due to their intriguing mathematical structure—also worthy of investigation—but because such spaces appear in applications. One of these applications is the mathematical modeling of inhomogeneous materials, like electrorheological fluids. These are special viscous liquids that have the ability to change their mechanical properties significantly when they come in contact with an electric field. This can be explored in technological applications, like photonic crystals, smart inks, heterogeneous polymer composites, etc. [3–6].

Another known application is related to image restoration. Based on variable exponent spaces, the authors of [7] proposed a new model of image restoration combining the strength of the total variation approach and the isotropic diffusion approach. This model uses one value of  $p$  near the edge of images and another one near smooth regions, which leads to the variable exponent setting of the problem. Other references discussing image processing are [8–13]. This theory has also been applied to differential equations with nonstandard growth (see [14, 15]).

The theory of spaces of analytic functions on domains of the complex plane, or in general of  $\mathbb{C}^n$ , happens in parallel. Starting from the classical Hardy  $H^p(\mathbb{D})$  and Bergman  $A^p(\mathbb{D})$  spaces, the functional Banach spaces have had a preferential study.

A Banach space  $\mathcal{X}$  of complex-valued functions on a domain  $\Omega$  is said to be a *functional Banach space* (cf. [16]) if the following hold:

- Vectorial operations are precisely the pointwise operations on  $\Omega$ .
- If  $f(x) = g(x)$  for every  $x \in \Omega$ , then  $f = g$ .
- If  $f(x) = f(y)$  for every  $f \in \mathcal{X}$ , then  $x = y$ .
- Evaluation functionals  $x \mapsto f(x)$  are continuous on  $\mathcal{X}$ .

Classical Hardy and Bergman spaces are examples of functional Banach spaces. In the case of spaces of analytic functions, the evaluation functionals  $K_x(f) = f(x)$  ( $x \in \Omega$ ) are an extension of the well-known *reproducing kernels* in Hilbert spaces of analytic functions. The analyticity property of the functions in the space interacts with the Banach space structure by means of the continuity of evaluation functionals raising a fruitful relation between function theory and operator theory.

The setting of nonstandard spaces of analytic functions is a young area of research; a lot is yet to be explored. In this chapter, an introduction to the subject of variable exponent spaces of analytic functions will be given, making emphasis in the special place that evaluation functionals have in the theory.

## 2. Variable exponents

Perhaps the most known function spaces in mathematical analysis are the *Lebesgue spaces*  $L^p$ . These are a fundamental basis of measure theory, functional analysis, harmonic analysis, and differential equations, among other areas. A recent generalization is the so-called *variable exponent Lebesgue spaces*. In this case, the defining parameter  $p$  is allowed to vary.

The basics on variable Lebesgue spaces may be found in the monographs [1, 17, 18]. For  $\Omega \subset \mathbb{R}^d$  we put  $p_\Omega^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x)$  and  $p_\Omega^- := \operatorname{ess\,inf}_{x \in \Omega} p(x)$ ; we use the abbreviations  $p^+ = p_\Omega^+$  and  $p^- = p_\Omega^-$ . For a measurable function  $p : \Omega \rightarrow [1, \infty)$ , we call it a *variable exponent* and define the set of all variable exponents with  $p^+ < \infty$  as  $\mathcal{P}(\Omega)$ .

For a complex-valued measurable function  $\varphi : \Omega \rightarrow \mathbb{C}$ , we define the *modular*  $\rho_{p(\cdot), \mu}$  by

$$\rho_{p(\cdot), \mu}(\varphi) := \int_{\Omega} |\varphi(x)|^{p(x)} d\mu(x)$$

and the *Luxemburg-Nakano norm* by

$$\|\varphi\|_{L^{p(\cdot)}(\Omega, \mu)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot), \mu} \left( \frac{\varphi}{\lambda} \right) \leq 1 \right\}. \quad (1)$$

Given  $p(\cdot) \in \mathcal{P}(\Omega)$ , the *variable Lebesgue space*  $L^{p(\cdot)}(\Omega, \mu)$  is defined as the set of all complex-valued measurable functions  $\varphi : \Omega \rightarrow \mathbb{C}$  for which the modular is finite, i.e.,  $\rho_{p(\cdot), \mu}(\varphi) < \infty$ . Equipped with the Luxemburg-Nakano norm (1) is a Banach space.

Most of the research being done on variable exponent spaces makes use of a regularity condition in the variable exponent in order to have a “fruitful” theory.

**Definition 1.** A function  $p : \Omega \rightarrow \mathbb{R}$  is said to be *log-Hölder continuous*—or satisfy the *Dini-Lipschitz condition* on  $\Omega$  if there exists a positive constant  $C_{\log}$  such that

$$|p(x) - p(y)| \leq \frac{C_{\log}}{\log(1/|x - y|)},$$

for all  $x, y \in \Omega$  such that  $|x - y| < 1/2$ , from which we obtain

$$|p(x) - p(y)| \leq \frac{2\ell C_{\log}}{\log\left(\frac{2\ell}{|x - y|}\right)}$$

for all  $x, y \in \Omega$  such that  $|x - y| < \ell$ . We will write  $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$  when  $p(\cdot)$  is log-Hölder continuous.

This condition has proven to be very useful in the theory since, among other things, it implies the boundedness of the Hardy-Littlewood maximal operator in  $L^{p(\cdot)}(\Omega)$ . Such operator is defined as

$$Mf(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy$$

and represents an important tool in harmonic analysis. We used it in [19] to show that the Bergman projection is bounded on variable exponent Bergman spaces.

Another consequence of the log-Hölder condition is the following inequality

$$|B|^{p_-(B)-p_+(B)} \leq C,$$

that hold for all balls  $B \subset \mathbb{C}$ . Here  $|B|$  stands for the Lebesgue measure of  $B$ ,  $p^+(B) = \operatorname{ess\,sup}_{z \in B} p(z)$ , and  $p^-(B) = \operatorname{ess\,inf}_{z \in B} p(z)$ . This will be a useful tool when studying complex function spaces since it allows to pass from an exponent that varies over a ball to a constant exponent at the center of such ball.

One interesting point of the theory of variable exponents is that, in general, the classical approach to Bergman spaces seems to fail in the variable framework. For example, in [19] it is shown that the Bergman projection is bounded from  $A^{p(\cdot)}(\mathbb{D})$  to  $A^{p(\cdot)}(\mathbb{D})$ . This is done by using the theory of Békollé-Bonami weights and a theorem that extends Rubio de Francia's extrapolation theory to variable Lebesgue spaces. This method is quite different from the usual ways of showing that the Bergman projection is bounded, which use either Schur's test or Calderón-Zygmund theory.

### 3. Variable exponent Hardy spaces

In this section we will give an introduction to the variable exponent Hardy spaces and the results and techniques that are usually found. We will follow the presentation as in [20].

**Definition 2.** For each  $z$  in the unit disk  $\mathbb{D}$ , the Poisson kernel  $P(z, \zeta)$  is defined as

$$P(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2},$$

and the Poisson transform of a function  $f \in L^{p(\cdot)}(\mathbb{T})$  is defined as

$$Pf(z) = \int_{\mathbb{T}} P(z, \zeta) f(\zeta) m(\zeta),$$

where  $m$  denotes the normalized Lebesgue measure on  $\mathbb{T}$ .

We will use the following result from [21]:

**Theorem 3.** Suppose that  $p$  is log-Hölder continuous. For each  $0 \leq r < 1$ , and  $f \in L^{p(\cdot)}(\mathbb{T})$ , the linear operator  $P_r : L^{p(\cdot)}(\mathbb{T}) \rightarrow L^{p(\cdot)}(\mathbb{T})$  defined as

$$P_r f(\zeta) = Pf(r\zeta)$$

is uniformly bounded on  $L^{p(\cdot)}(\mathbb{T})$ ,  $\|P_r(f)\|_{L^{p(\cdot)}(\mathbb{T})} \lesssim \|f\|_{L^{p(\cdot)}(\mathbb{T})}$ , and for every  $f \in L^{p(\cdot)}(\mathbb{T})$ ,

$$\|f - P_r f\|_{L^{p(\cdot)}(\mathbb{T})} \rightarrow 0, \quad \text{as } r \rightarrow 1^-.$$

We are now ready to define the harmonic Hardy spaces with variable exponents. Given  $f : \mathbb{D} \rightarrow \mathbb{C}$  and  $0 \leq r < 1$ , the dilations  $f_r : \mathbb{T} \rightarrow \mathbb{C}$  of  $f$  are defined as  $f_r(\zeta) = f(r\zeta)$ .

**Definition 4.** The variable exponent harmonic Hardy space  $h^{p(\cdot)}(\mathbb{D})$  is defined as the space of harmonic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{h^{p(\cdot)}(\mathbb{D})} = \sup_{0 \leq r < 1} \|f_r\|_{L^{p(\cdot)}(\mathbb{T})} < \infty.$$

Notice that, since  $Pf$  is harmonic for  $f \in L^{p(\cdot)}(\mathbb{T})$ , then Theorem 3 shows that the Poisson transform  $P : L^{p(\cdot)}(\mathbb{T}) \rightarrow h^{p(\cdot)}(\mathbb{D})$  is bounded. Moreover, we have the following theorem analogous to the constant exponent context.

**Theorem 5.** Suppose that  $p : [0, 2\pi] \rightarrow [1, \infty)$  is log-Hölder continuous. Then for every  $U \in h^{p(\cdot)}(\mathbb{D})$ , there exists  $u \in L^{p(\cdot)}(\mathbb{T})$  such that  $Pu = U$  and moreover,  $\|U\|_{h^{p(\cdot)}(\mathbb{D})} \sim \|u\|_{L^{p(\cdot)}(\mathbb{T})}$ .

**Theorem 6.** The following chain of inclusions holds

$$h^{p^+}(\mathbb{D}) \subset h^{p(\cdot)}(\mathbb{D}) \subset h^{p^-}(\mathbb{D}),$$

and moreover, the inclusions are continuous.

**Definition 7.** The variable exponent Hardy space  $H^{p(\cdot)}(\mathbb{D})$  is defined as the space of analytic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $f \in h^{p(\cdot)}(\mathbb{D})$ .

In an analogous way to the classical setting  $H^{p(\cdot)}(\mathbb{D})$  can be identified with the subspace of functions in  $L^{p(\cdot)}(\mathbb{T})$  whose negative Fourier coefficients are zero, and as such,  $H^{p(\cdot)}(\mathbb{D})$  is a Banach space.

Recall that for all functions  $f \in H^1(\mathbb{D})$ , we have the reproducing formula:

$$f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta}z} dm(\zeta).$$

For each  $z \in \mathbb{D}$ , the functions  $K_z : \mathbb{D} \rightarrow \mathbb{C}$  defined as

$$K_z(w) = \frac{1}{1 - \bar{z}w},$$

are called *reproducing kernels*. They are bounded on  $\mathbb{D}$  and consequently belong to every space  $H^{p(\cdot)}(\mathbb{D})$ . Moreover, as a consequence of the reproducing formula and the Hahn-Banach theorem, the linear span of  $\{K_z : z \in \mathbb{D}\}$  is dense in  $H^{p(\cdot)}(\mathbb{D})$ .

Consequently, the set of polynomials is also dense in  $H^{p(\cdot)}(\mathbb{D})$ .

**Theorem 8.** For each  $z \in \mathbb{D}$  consider the linear functional  $\gamma_z : H^{p(\cdot)}(\mathbb{D}) \rightarrow \mathbb{C}$  defined as  $\gamma_z(f) = f(z)$ . Then  $\gamma_z$  is a bounded operator for every  $z = |z|e^{i\theta} \in \mathbb{D}$  and

$$\|\gamma_z\| \lesssim \frac{1}{(1 - |z|)^{1/p(\theta)}}.$$

Consequently, the convergence in the  $H^{p(\cdot)}(\mathbb{D})$ -norm implies the uniform convergence on compact subsets of  $\mathbb{D}$ .

The main tool for proving the previous theorem is following the version of a Forelli-Rudin inequality, adapted to the case of variable exponents. Here, the Log-Hölder continuity plays a key role.

**Lemma 9.** Suppose  $p : [0, 2\pi] \rightarrow [1, \infty)$  and  $q : [0, 2\pi] \rightarrow [1, \infty)$  are such that  $\frac{1}{p(\theta)} + \frac{1}{q(\theta)} = 1$  for every  $\theta \in [0, 2\pi]$ . Let  $1/2 < r < 1$  and  $z = |z|e^{i\theta}$ . Define the function  $\varphi : [0, 2\pi] \rightarrow \mathbb{R}^+$  as

$$\varphi(t) = \frac{(1 - |z|)^{1/q(\theta)}}{|1 - |z|re^{i(t-\theta)}|}.$$



Then if  $\varphi(t) > 1$ , it holds that

$$\varphi(t)^{p(t)} \lesssim \varphi(t)^{p(\theta)}.$$

#### 4. Variable exponent Bergman spaces

The theory of Bergman spaces was introduced by S. Bergman in [22] and since the 1990s has gained a great deal of attention mainly due to some major breakthroughs at the time. For details on the theory of Bergman spaces, we refer to the books [5, 23].

**Definition 10.** Given a measurable function  $p \in \mathcal{P}(\mathbb{D})$ , we define the *variable exponent Bergman space*  $A^{p(\cdot)}(\mathbb{D})$  as the space of all analytic functions on  $\mathbb{D}$  that belong to the variable exponent Lebesgue space  $L^{p(\cdot)}(\mathbb{D})$  with respect to the area measure  $dA$  on the unit disk  $\mathbb{D}$ , i.e.,

$$A^{p(\cdot)}(\mathbb{D}) = \left\{ f \text{ is an analytic function and } \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z) < \infty \right\}.$$

One of the first results proven in [19] about this theory was the boundedness of the evaluation functionals that, as we saw before, will allow us to conclude that convergence in  $A^{p(\cdot)}(\mathbb{D})$  implies uniform convergence on compact subsets of  $\mathbb{D}$  and consequently  $L^{p(\cdot)}$ -limits of sequences in  $A^{p(\cdot)}(\mathbb{D})$  are analytic. Hence  $A^{p(\cdot)}(\mathbb{D})$  is a closed subspace of  $L^{p(\cdot)}(\mathbb{D})$  (and hence a Banach space). The first approach to such result used the following interpolation result based on Forelli-Rudin estimates:

**Lemma 11.** Let  $p \in \mathcal{P}(\mathbb{D})$ , then for every function  $f \in L^{p(\cdot)}(\mathbb{D})$  we have

$$\|f\|_{L^{p(\cdot)}(\mathbb{D})} \leq \|f\|_{L^1(\mathbb{D})}^{1/p^+} \|f\|_{L^\infty(\mathbb{D})}^{1-1/p^+}.$$

Using different techniques, the following sharp estimate was obtained in [24]:

**Theorem 12.** Let  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D})$ . Then for every  $z \in \mathbb{D}$  we have that

$$\|\gamma_z\|_{p(\cdot)} \asymp \frac{1}{(1 - |z|^2)^{2/p(z)}}. \quad (2)$$

The proof of this fact relies on the Log-Hölder condition and on the following version of a Jensen-type inequality for variable exponent spaces (see [17]).

**Lemma 13.** Suppose that  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D})$  and let

$$\frac{1}{q(z, y)} := \max \left\{ \frac{1}{p(z)} - \frac{1}{p(y)}, 0 \right\}.$$

Then for  $0 < \gamma < 1$ ,

$$\left( \int_{B(x, r)} |f(y)| dA(y) \right)^{p(z)} \lesssim \int_{B(x, r)} |f(y)|^{p(y)} dA(y) + \int_{B(x, r)} \gamma^{q(z, y)} dA(y)$$

for all  $z \in B(x, r)$  provided that  $\|f\|_{L^{p(\cdot)}(\mathbb{D})} \leq 1$ . The constant depends on  $\gamma$  and  $p(\cdot)$ .

The additional term that appears in the previous inequality makes the boundedness of the domain a necessary condition to use such technique. One question to address in future investigations is how the situation is for Bergman spaces defined on unbounded domains. In such case, the error term accumulates despite the regularity condition on  $p(\cdot)$ , and consequently a different technique is needed.

#### 4.1 Mollified dilations

One technique developed to study variable exponent Bergman spaces is a combination of mollifiers and dilations. Such concept was developed for studying the density of the set of polynomials in Bergman spaces. In the classical case, this is a consequence of the fact that convergence in  $A^p(\mathbb{D})$  implies convergence on compact subsets of  $\mathbb{D}$  and that for any  $f \in A^p(\mathbb{D})$ , its *radial dilations*

$$f_r(z) = f(rz)$$

converge in norm to  $f$  for  $r \rightarrow 1^-$ . This is a simple fact that depends on the dilation-invariant nature of Bergman Spaces, something that does not hold in the case of variable exponent Bergman spaces. This is one of the main difficulties for generalizing the classical theory.

One way of overcoming this problem in the case of variable exponent Lebesgue spaces is the use of a *mollification operator*. But in the case of the unit disk, changes need to be made. That is, when we had the idea to introduce a *mollified dilation*, given a function  $f \in A^{p(\cdot)}(\mathbb{D})$  and  $r \in [\frac{1}{2}, 1)$ , define the *mollified dilation*  $f_r : \frac{1+r}{2}\mathbb{D} \rightarrow \mathbb{C}$  of  $f$  as

$$f_r(z) := \int_{\mathbb{D}} f(rw) \eta_r(z - w) dA(w).$$

where  $\rho\mathbb{D}$  stands for the complex disk with radius  $\rho$ ,

$$\eta_r(z) := \frac{4r^2}{(1-r)^2} \eta\left(\frac{2rz}{1-r}\right)$$

and

$$\eta(z) := \begin{cases} \exp\left(\frac{1}{|z|^2 - 1}\right), & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1, \end{cases}$$

Mollified dilations allow us to approximate analytic functions in the variable exponent Bergman spaces by functions that are analytic on a neighborhood of the unit disk.

**Theorem 14.** Let  $p \in \mathcal{P}^{\log}(\mathbb{D})$  and let  $f \in A^{p(\cdot)}(\mathbb{D})$ . Then for  $\frac{1}{2} \leq r < 1$ ,  $f_r \in A^{p(\cdot)}(\mathbb{D})$ , and

$$\sup_{\frac{1}{2} \leq r < 1} \|f_r\|_{A^{p(\cdot)}(\mathbb{D})} \lesssim \|f\|_{A^{p(\cdot)}(\mathbb{D})}.$$

Moreover,  $\|f_r - f\|_{A^{p(\cdot)}(\mathbb{D})} \rightarrow 0$  as  $r \rightarrow 1^-$ .

As a consequence of the previous theorem, we can approximate a function  $f \in A^{p(\cdot)}(\mathbb{D})$  by a function  $f_r$  which is analytic in  $\frac{1+r}{2}\mathbb{D}$ . Hence, the sequence of Taylor



polynomials associated with  $f_r$  converges uniformly on  $\mathbb{D}$  to  $f_r$ . Therefore, given  $\varepsilon > 0$ , there exists  $1/2 \leq r < 1$  such that  $\|f - f_r\|_{A^{p(\cdot)}(\mathbb{D})} < \varepsilon/2$ . And there exists a polynomial  $p$  such that  $\|p - f_r\|_{A^{p(\cdot)}(\mathbb{D})} < \varepsilon/2$ .

## 4.2 Carleson measures

Another classical problem that has a variable exponent equivalent consists on finding a geometrical characterization of Carleson measures in variable exponent Bergman spaces. Given a positive Borel measure  $\mu$  defined on the unit disk  $\mathbb{D}$ , we say that  $\mu$  is a *Carleson measure* for the variable exponent Bergman space  $A^{p(\cdot)}(\mathbb{D})$  if there exists a constant  $C > 0$  such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{D}, \mu)} \leq C \|f\|_{A^{p(\cdot)}(\mathbb{D})}.$$

In other words,  $\mu$  is a Carleson measure on  $A^{p(\cdot)}(\mathbb{D})$  if  $A^{p(\cdot)}(\mathbb{D})$  is continuously embedded in  $L^{p(\cdot)}(\mathbb{D}, \mu)$ .

Lennart Carleson gave a geometric characterization of Carleson measures on Hardy spaces  $H^p$  and used this result in his proof of the Corona theorem and some interpolation problems (cf. [25]). After that, Carleson measures have gained interest, and a geometric characterization is usually important in spaces of analytic functions.

In the case of the classical Bergman spaces  $A^p(\mathbb{D})$ , Carleson measures are characterized in terms of the measure on pseudo-hyperbolic disks (see [26]). An interesting fact is the independence of the characterization on the exponent  $p$ . In the case of variable exponent Bergman spaces, the independence of  $p(\cdot)$  does not hold unless the  $p(\cdot)$  satisfies the Log-Hölder condition. The counterexample for the general case and the theorem for the restricted case were found in [24].

**Theorem 15.** *Let  $\mu$  be a finite, positive Borel measure on  $\mathbb{D}$  and  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D})$ . Then  $\mu$  is a Carleson measure for  $A^{p(\cdot)}(\mathbb{D})$  if and only if there exists a constant  $C > 0$  such that for every  $0 < r < 1$  and every  $a \in \mathbb{D}$ ,  $\mu(D_r(a)) \leq C|D_r(a)|$ , where  $D_r(a)$  denotes pseudo-hyperbolic disk with center at  $a \in \mathbb{D}$  and radius  $r$ :*

$$D_r(a) := \left\{ z \in \mathbb{D} : \left| \frac{a - z}{1 - \bar{a}z} \right| < r \right\}.$$

The main tool for the proof of the previous theorem is Eq. (2) that implies the following estimate for the reproducing kernels.

**Theorem 16.** *Let  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D})$ . Then for every  $z \in \mathbb{D}$  we have that*

$$\|k_z\|_{A^{p(\cdot)}(\mathbb{D})} \asymp \frac{1}{(1 - |z|^2)^{2(1-1/p(z))}}.$$

## 5. Open problems

### 5.1 Zero sets

An interesting area will be to study some analytic properties of functions belonging to variable exponent Bergman spaces. One starting point will be to study the structure of zero sets in the space.

A sequence of points  $\{z_n\} \subset \mathbb{D}$  is a zero set for the Bergman space  $A^p(\mathbb{D})$  if there exists  $f \in A^p(\mathbb{D})$  such that  $f(z_n) = 0$  for all  $n$  and  $f(z) \neq 0$  if  $z \notin \{z_n\}$ .

Although the problem of completely describing the zero sets in  $A^p(\mathbb{D})$  is still open, some information is available. For example, if  $n(r)$  denotes the number of zeroes of a function  $f \in A^p(\mathbb{D})$  having modulus smaller than  $r > 0$ , then it is known that

$$n(r) = O\left(\frac{1}{1-r} \log\left(\frac{1}{1-r}\right)\right), \quad r \rightarrow 1^-.$$

This result depends on studying the radial growth of functions in the Bergman space and on Jensen's formula for the sequence  $\{z_k\}$  of zeroes of  $f$  such that  $|z_k| < r$ .

$$\log |f(0)| = \sum_{k=1}^{n(r)} \log\left(\frac{|z_k|}{r}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

This equation holds for every analytic function on  $\mathbb{D}$ . Consequently, if we want to find out whether a similar result about the growth of the zeroes holds in the variable exponent case, then it is necessary to study the radial growth of functions in  $A^{p(\cdot)}(\mathbb{D})$ . Such results, in the case of a constant exponent again, depend on the subharmonicity of  $|f|^p$ .

Another result involving zero sets is that if  $f(0) \neq 0$ , then

$$\prod_{k=1}^{n(r)} \frac{1}{|z_k|} = O\left(n(r)^{1/p}\right), \quad r \rightarrow 1^-.$$

Finding an extension of such result to variable exponents would include making sense of the right-hand side term that is not even clear in this context.

## 5.2 Sampling sequences

A related concept that we would like to address in variable exponent Bergman spaces is that of sampling sequences. A sequence of points  $\{z_k\} \subset \mathbb{D}$  is sampling in  $A^p(\mathbb{D})$  if there exists constants  $c > 0$  and  $C > 0$  such that for all  $f \in A^p(\mathbb{D})$

$$c \|f\|_p^p \leq \sum_{k=1}^{\infty} \left(1 - |z_k|^2\right)^2 |f(z_k)|^p \leq C \|f\|_p^p.$$

One difference that is expected to occur in the case of variable exponents is that sampling sequences will probably not be invariant under automorphism of the unit disk. This is true for a constant exponent, and it will be interesting to find counter-examples if the exponent varies. Sampling sequences are related to the concept of “frames” in Hilbert spaces and can be thought as sequences that contain great information of the space. It is expected that a sampling sequence is a somehow “big” and “spread out” subset of the unit disk. The notion is clearly related in  $A^p$  to the concept of Carleson measures.

If we define a discrete measure  $\mu$  as

$$\mu = \sum_{k=1}^{\infty} \left(1 - |z_k|^2\right)^2 \delta_{z_k}$$

where  $\delta_{z_k}$  is the Dirac measure with a point mass at  $z_k$ , then the sequence  $\{z_k\}$  is sampling if  $\mu$  is a Carleson measure and satisfies a “reverse” Carleson condition. Then it becomes relevant to find a geometric characterization of the measures that satisfy a reverse Carleson condition (see [27–29]). The characterization depends on the concept of dominant sets and it is independent of  $p$ . The methods used by Luecking for Bergman spaces rely on a calculation of the norm of evaluation functionals and a submean value property. Different techniques must be developed for the variable exponent counterpart.

### 5.3 Operators in variable exponent Bergman spaces

The characterization of Carleson measures obtained in [24] opens the possibility of studying the boundedness and compactness of certain operators acting on variable exponent Bergman spaces.

Among those operators worth studying are multiplication operators. Those are formally defined on a functions space  $\mathcal{F}$  as

$$M_f(g) = fg$$

where  $g \in \mathcal{F}$ . One natural question is to find the set of symbols  $f$  that make the operator  $M_f$  map the space  $A^{p(\cdot)}(\mathbb{D})$  to itself.

Such question has been addressed in the case of weighted Bergman spaces in [23]. The proof relies on a geometric characterization of Carleson measures, a composition with a disk isomorphism, and a theorem of change of variables. Such tool is difficult to use in the setting of variable exponents since any change of variables will also affect the exponent and consequently the spaces are not necessarily invariant under composition with isomorphisms. This makes the situation different from the case of a constant exponent.

Other operators related to multiplication are Toeplitz operators. Those have already being studied in [30] for the case of weighted Bergman spaces with non-radial weights. Given a function  $\varphi \in L^1(\mathbb{D})$ , the Toeplitz operator  $T_\varphi$  is defined on the set of polynomials as

$$T_\varphi p(z) = \int_{\mathbb{D}} p(w) k_w(z) \varphi(w) dA(w).$$

One natural question is to ask whether this operator can be extended boundedly to the space  $A^{p(\cdot)}(\mathbb{D})$ . This question is addressed in [31] where a characterization of the boundedness and compactness of  $T_\varphi$  is found in terms of Carleson measures and the Berezin transform associated with  $T_\varphi$ . This type of results are sometimes referred to as Axler-Zheng theorems, and similar questions looking to establish the relation between the Berezin transform of a finite product of Toeplitz operators with its compactness have been addressed by several authors; a survey of this type of results is given in [32]. It is then reasonable to ask whether an Axler-Zheng type result for products of Toeplitz operators holds in the context of variable exponents.

The composition operators are another type of operators to be studied. Given an analytic self-map  $\varphi$  of the unit disk, the composition operator  $C_\varphi$  can be formally defined as

$$C_\varphi(f) = f \circ \varphi.$$

A natural question is to find function-theoretic conditions on  $\varphi$  that guarantee the boundedness and/or compactness of its associated composition operator acting

on  $A^{p(\cdot)}(\mathbb{D})$ . One approach to follow is to address the boundedness problems of linear operators acting on  $A^{p(\cdot)}(\mathbb{D})$  through the use Rubio de Francia extrapolation. Given an open set  $\Omega \subset \mathbb{C}$ , we denote by  $\mathcal{F}$  a family of pairs of non-negative, measurable functions defined on  $\Omega$ , and by  $A_1$  we denote the class of Muckenhoupt weights  $w$ . That is, the non-negative functions  $w$  such that

$$\operatorname{ess\,sup}_{z \in \Omega} \frac{Mw(z)}{w(z)} < \infty$$

where  $M$  denotes the Hardy-Littlewood maximal operator. Rubio de Francia extrapolation in the setting of variable exponent spaces can be stated as follows.

**Proposition 17** (Thm. 5.24 in [1]). *Suppose that for some  $p_0 \geq 1$  the family of pairs of functions  $\mathcal{F}$  is such that for all  $w \in A_1$ ,*

$$\int_{\Omega} F(x)^{p_0} w(x) dx \leq C_0 \int_{\Omega} G(x)^{p_0} w(x) dx, \quad (F, G) \in \mathcal{F}.$$

*Given  $p \in \mathcal{P}(\Omega)$ , if  $p_0 \leq p_- \leq p_+ < \infty$  and the maximal operator is bounded on  $L^{(p(\cdot)/p_0)'}(\Omega)$ , then*

$$\|F\|_{L^{p(\cdot)}(\Omega)} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}(\Omega)}.$$

This result allows, under certain conditions, to pass from the of studying operators defined on variable exponent spaces, to study operators defined on weighted constant exponent spaces, and therefore it is possible to use known results from the Muckenhoupt weights theory.

## 5.4 Analytic Besov spaces

Another type of spaces we are interested is the class of analytic Besov spaces  $B_p$ . These are the spaces of all analytic functions in the unit disk such that

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

For each  $p$  this is a normed space with norm

$$\|f\|_{B_p} = |f(0)| + \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right)^{1/p}.$$

One particular property of this space is that they are Möbius invariant in the sense that  $\|f \circ \varphi\|_{B_p} = \|f\|_{B_p}$  for every automorphism of the unit disk  $\varphi$ . This is probably not true in general for a variable exponent version due to the problems with the change of variables formula. A first question to address is to characterize the exponents that leave the space invariant under disk automorphisms.

One useful technique to study such spaces is a derivative-free characterization of functions in  $B_p$ . It can be seen (see, e.g., [33]) that  $f \in B_p$  if and only if

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - z\bar{w}|^4} dA(z) dA(w) < \infty.$$

A similar result in the case of variable exponents could serve as a starting point for investigating other properties of the space.

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