Chapter

Existence of Open Loop Equilibria for Disturbed Stackelberg Games

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Abstract

In this work, we derive necessary and sufficient conditions for the existence of an hierarchic equilibrium of a disturbed two player linear quadratic game with open loop information structure. A convexity condition guarantees the existence of a unique Stackelberg equilibria; this solution is first obtained in terms of a pair of symmetric Riccati equations and also in terms of a coupled of system of Riccati equations. In this latter case, the obtained equilibrium controls are of feedback type.

Keywords: differential games, linear quadratic, Riccati differential equations, Stackelberg equilibrium, worst-case disturbance

1. Introduction

The study of linear quadratic (LQ) games has been addressed by many authors [1–4]. This type of games is often used as a benchmark to assess the game equilibrium strategies and its respective outcomes. In a disturbed differential game, each player calculates its strategy taking into account a worst-case unknown disturbance. In non-cooperative game theory, the concept of hierarchical or Stackelberg games is very important, since different applications in economics and engineering exist [1, 5]. This is also the case of gas networks where a hierarchy may be assigned to its controllable elements—compressors, sources, reductors, etc... Also, for this application, the modelling as a disturbed game makes a lot of sense, since the unknown offtakes of the network can be modelled as unknown disturbances. Further research on Stackelberg games can be found for instance in AbouKandil and Bertrand [6]; Medanic [7]; Yong [8]; Tolwinski [9].

No assumptions/constraints are made of the disturbance. To be easier to understand the hierarchical concept, we consider only two players. Therefore, we study a LQ game of two players with Open Loop (OL) information structure where the players choose its strategy according to a modified Stackelberg equilibrium. Player-1 is the follower and chooses its strategy after the nomination of the strategy of the leader. Player-2, the leader, chooses its strategy assuming rationality of the follower. Both players find their strategies assuming a worst-case disturbance.

In this work, we consider a finite time horizon, where for applications this is chosen according to the periodicity of the operation of the problem being studied.

The disturbed case of the representation of optimal equilibria for noncooperative games has been studied [10, 11] considering a Nash equilibrium. It is the aim of this paper to generalise the work of Jank and Kun to Stackelberg games and extend the results presented in Freiling and Jank [12]; Freiling et al. [13] to the disturbed case. To calculate the controls, we use a value function approach, appropriately guessed. Thence, we obtain sufficient conditions of existence of these controls and its representation in terms of the solution of certain Riccati equations. Furthermore, a feedback form of the worst-case Stackelberg equilibrium is obtained.

In a future paper, we expect to present analogous conditions using an operator approach.

In Section 2, we define the disturbed LQ game and define Stackelberg worst-case equilibrium. In Section 3, we derive sufficient conditions for the existence of a worst-case Stackelberg equilibrium under OL information structure and investigate how are these solutions related to certain Riccati differential equations. Section 4 concludes the paper and outlines some directions for future work.

2. Fundamental notions

We start with the concept of best reply:

Definition 2.1. (Best reply) Let Γ_N be a N-player differential game. For $i \in \{1, \dots, N\}$,

$$\gamma_{(-i)} := (\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_N) \in \otimes_{\; j \neq i} \mathcal{U}_j.$$

We say that $\tilde{\gamma}_i$ is the best reply against $\gamma_{(-i)}$ if

$$J_i(\gamma_1, \dots, \gamma_{i-1}, \tilde{\gamma}_i, \gamma_{i+1}, \dots, \gamma_N) \leq J_i(\gamma_1, \dots, \gamma_N)$$

holds for any strategy $\gamma_i \in \mathcal{U}_i$. We denote the set of all best replies by $\mathcal{R}_i\left(\gamma_{(-i)}\right)$.

We study games of quadratic criteria, defined in a finite time horizon $\left[t_0, t_f\right] \subset \mathbb{R}$ and subject to a linear dynamics, controlled players and also an unknown disturbance. Hereby also consider $u_j = \gamma_j\left(t, \eta_j\right)$, where η_j is the information structure of Player-j. In this case, $\eta_j, j = 1, \dots, N,$ is of OL type.

Definition 2.2. (Linear Quadratic (LQ) differential game) Let Γ_N be an $N-$ player differential game finite time horizon $T = \left[t_0, t_f\right]$. Suppose further that:

i. the dynamics of the game are assumed to obey a linear differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + \sum_{j=1}^{N} B_j(t)u_j(t) + C(t)w(t), \\ x(t_0) &= x_0. \end{aligned} \tag{1}$$

In this equation, $t \in \mathcal{T}$, where the initial t_0 and the final t_f are finite and fixed, the state $x(t)$ is an $n-$ dimension vector of continuous functions defined in $\mathcal{T}$ and with $x\left(t_f\right) = x_f$. The controls $u_i, i = 1, \dots, N,$ are square (Lebesgue) integrable and the m_i- dimension vector of continuous functions is also defined in $\mathcal{T}$. Also, the disturbance $w(t) \in \mathcal{L}^m(\mathcal{T})$. The different matrices are of adequate dimension and with elements continuous in $\mathcal{T}$.

i. the performance criteria are of the form

$$J_i\left(u_i, u_{(-i)}, w\right) = \mathcal{K}\left(x\left(t_f\right)\right) + \int_{t_0}^{t_f} \Psi\left(u_i, u_{(-i)}, w\right) dt. \tag{2}$$

where

$$\mathcal{K}(x(t_f)) = x^T(t_f)K_{if}x(t_f) \tag{3}$$

$$\begin{aligned}\Psi(u_i, u_{(-i)}, w) &= x^T(t)Q_i(t)x(t) + w^T(t)P_i(t)w(t)\\ &+ \sum_{j=1}^{N} u_j^T(t)R_{ij}(t)u_j(t),\end{aligned} \tag{4}$$

with symmetric matrices $K_{if} \in \mathbb{R}^{n\times n}$ and symmetric, piecewise continuous and bounded matrix valued functions $Q_i(t) \in \mathbb{R}^{n\times n,}$ $R_{ij}(t) \in \mathbb{R}^{m_i\times m_j}$ and $P_i(t) \in \mathbb{R}^{m\times m}$, $i = 1, 2, \dots, N$.

We observe that no cost functional is assigned to the disturbance term because no constraints can be applied on an "unpredictable" parameter. In what follows, we consider $N = 2$. To extend the theory to $N > 2$, since this is an hierarchical solution, we need to define the structure of the leaders and followers in the game. We can even have more than two hierachy levels. We assume that Player-2 is the leader and Player-1 is the follower.

The leader seeks a strategy $u_2^*(t)$ in OL information structure and announces it before the game starts. This strategy is found knowing how the follower reacts to his choices. The follower calculates its strategy as a best reply to the strategy announced by the leader.

Problem 2.1. Find the control $u_i^* \in \mathcal{U}_i, i = 1, 2,$ in $\mathcal{T}$ for which $J_i^*\left(u_i, u_{(-i)}^*, w\right), i = 1, 2,$ is minimal when subject to constraints $u_i^*(t) = \gamma_i^*(t, \eta_i(t)), i = 1, 2,$ and (1) and considering a worst-case disturbance.

Consider $\mathcal{U}_i, i = 1, 2,$ the sets of functions such that (1) is solvable and J_i exists, with $u_i, i = 1, 2,$ in these conditions $\mathcal{U}_i, i = 1, 2, \mathcal{W}$ are said the sets of admissible controls and disturbance, respectively.

Definition 2.3. (Stackelberg equilibrium) Let Γ_2 be a 2-person differential game, we define the Stackelberg/worst-case equilibrium in two stages.

1. A function $\hat{w}_i(u) \in \mathcal{W}$ is called the worst-case disturbance, from the point of view of the ith player belonging to the set of admissible controls, if

$$J_i^*\left(u_i, u_{(-i)}^*, \hat{w}_i\right) \geq J_i^*\left(u_i, u_{(-i)}^*, w\right), \quad i = 1, 2, \tag{5}$$

holds for each $w \in \mathcal{W}$. There exists exactly one worst-case disturbance from the point of view of the ith player according to every set of controls.

2. We say that the controls (u_1^*, u_2^*) form a worst-case Stackelberg equilibrium if

i. The leader chooses u_2^* such that

$$\max_{\gamma_1 \in \mathcal{R}_1(u_2^*)} J_2(\gamma_1, u_2^*, \hat{w}_2) \leq \max_{\gamma_1 \in \mathcal{R}_1(u_2)} J_2(\gamma_1, u_2, \hat{w}_2)$$

for all $u_2 \in \mathcal{U}_2$.

ii. The follower then chooses u_1^* such that

$$\mathcal{R}_1(u_2) = \{u_1 | J_1(u_1, u_2, \hat{w}_1) \leq J_1(\gamma_1, u_2, \hat{w}_1)\}.$$

To guarantee the uniqueness of OL Stackelberg solutions, matrices are assumed to satisfy $K_{if} \geq 0, Q_i \geq 0, R_{ij} > 0, i \neq j$ and $R_{ii} \geq 0, i, j = 1, \ldots, N$ in $\mathcal{T}$ Simaan and Cruz [14].

In what follows, we drop the dependence of the parameters in t to reduce the length of the formulas.

3. Sufficient conditions for the existence of OL Stackelberg equilibria

In this section, we withdraw sufficient conditions for the existence of the worst-case Stackelberg equilibrium, using a value function approach.

A disturbed differential LQ game as defined in Definition 2.2 is said *playable* if there exists a unique Stackelberg worst-case equilibrium.

Theorem 3.1. Let the solution of the Riccati differential equation

$$\begin{aligned} \dot{E}_1 &= -E_1 A - A^T E_1 - Q_1 + E_1(S_1 + T_1)E_1, \\ E_1(t_f) &= K_{1f}, \end{aligned} \tag{6}$$

with $S_1 = B_1 R_{11}^{-1} B_1^T$ and $T_1 = C P_1^{-1} C^T$ exist on $\mathcal{T}$.

For any given admissible OL control of the leader, u_2, define $e_1 \in \mathbb{R}^n, d_1 \in \mathbb{R}$ by

$$\begin{aligned} \dot{e}_1 &= E_1(S_1 + T_1)e_1 - 2E_1 B_2 u_2 - A^T e_1^T, \\ e_1(t_f) &= 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{d}_1 &= -(u_2^T R_{12} + e_1^T B_2)u_2 + \frac{1}{4} e_1^T (S_1 + T_1)e_1, \\ d_1(t_f) &= 0. \end{aligned} \tag{8}$$

Then, the following identity holds:

$$\begin{aligned} 2J_1(u_1, u_2) = {} & x_0^T E_1(t_0)x_0 + x_0^T e_1(t_0) + d_1(t_0) \\ & + \int_{t_0}^{t_f} \|z_1(t)\|_{R_{11}}^2 dt + \int_{t_0}^{t_f} \|z(t)\|_{P_1}^2 dt, \end{aligned} \tag{9}$$

where $\|z_1\|_{R_{11}}^2 = z_1 R_{11} z_1$ with

$$z_1 = u_1 + R_{11}^{-1} B_1^T \left(E_1 x + \frac{1}{2} e_1 \right)$$

and $\|z\|_{P_1}^2 = z P_1 z$ with

$$z = w + P_1^{-1} C^T \left(E_1 x + \frac{1}{2} e_1 \right)$$

and x a solution of (1).

Proof: The proof is similar to the analogous result for the non-disturbed case Freiling et al. [13].

Theorem 3.2. Let the solution E_1 of (6) exist on $\mathcal{T}$. Then the unique response of the follower to the leader's OL strategy $u_2(t)$ is given by:

$$u_1^* = -R_{11}^{-1}B_1^T\left(E_1x + \frac{1}{2}e_1\right), \tag{10}$$

where the maximum disturbance,

$$w_1^* = -P_1^{-1}C^T\left(E_1x + \frac{1}{2}e_1\right), \tag{11}$$

was considered. E_1 and e_1 are the solutions of (6)–(7) and x is then the solution of

$$\dot{x} = [A - (S_1 + T_1)E_1]x - \frac{1}{2}(S_1 + T_1)e_1 + B_2u_2, \tag{12}$$

$$x(t_0) = x_0. \tag{13}$$

The corresponding minimal costs then are

$$J_{10} = 2J_1(u_1, u_2) = x_0^TE_1(t_0)x_0 + x_0^Te_1(t_0) + d_1(t_0). \tag{14}$$

Proof: We have that the unique OL response of the follower to the leader's announced strategy u_2 (10) under worst-case disturbance (11), that we substitute in the trajectory (1) to obtain:

$$\dot{x} = [A - (S_1 + T_1)E_1]x - \frac{1}{2}(S_1 + T_1)e_1 + B_2u_2.$$

The cost functional minimal value is obtained when we substitute in (9) the minimal control and themaximal disturbance.

Notice that $J_{10}(u_2)$ is not depending on u_1. This, as a matter of fact, is only true if we consider OL information structure, since otherwise u_2 would depend on the trajectory x and hence, via (1), also on u_1. In OL Stackelberg games, the leader tries next to find an optimal OL control u_2 that minimises $J_2(u_1(u_2), u_2)$ while $u_1(u_2)$ is defined by (10).

Theorem 3.3. Let the solution of the Riccati differential Eq. (6) and the solution of

$$\begin{aligned}\dot{E}_2 &= -E_2H - H^TE_2 - Q + E_2(S + T)E_2,\\ E_2(t_f) &= \begin{pmatrix} K_{2f} & 0 \\ 0 & 0 \end{pmatrix},\end{aligned} \tag{15}$$

with $S_{21} := B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^T$, $S_2 := B_2R_{22}^{-1}B_2^T$ and $T_2 := CP_2^{-1}C^T$. Also $H := \begin{pmatrix} A & -S_1 \\ -Q_1 & E_1T_1 - A^T \end{pmatrix}$, $Q := \begin{pmatrix} Q_2 & 0 \\ 0 & S_{21} \end{pmatrix}$, $S := \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} T_2 & T_2E_1 \\ E_1T_2 & E_1T_2E_1 \end{pmatrix}$ exist in $\mathcal{T}$, where $E_2 \in \mathbb{R}^{2n\times 2n}$. Also $B := \begin{pmatrix} B_2 \\ 0_{m_1\times n} \end{pmatrix}$.

For any given control u_2 of the leader, define functions $e_2 \in \mathbb{R}^{3n}$, $v_1, v_w, x \in \mathbb{R}^n$ and $d_2 \in \mathbb{R}$ in $\mathcal{T}$ by the following initial and terminal value problems:

$$\dot{e}_2 = \left(-H^T + E_2(S + T)\right)e_2, \quad e_2(t_f) = 0 \tag{16}$$

$$\dot{d}_2 = \frac{1}{4} e_2^T (S + T) e_2, \quad d_2(t_f) = 0 \tag{17}$$

$$\dot{v}_1 = -Q_1 x + (E_1 T_1 - A^T) v_1 + E_1 C w, \qquad v_1(t_0) = v_{10} \tag{18}$$

$$\dot{x} = Ax - S_1 v_1 + B_2 u_2 + Cw, \quad x(t_0) = x_0, \tag{19}$$

with $v_1 := \left(E_1 + \frac{1}{2} e_1\right)$.
Then, we obtain

$$u_1^* = -R_{11}^{-1} B_1^T v_1,$$
$$w_1^* = -P_1^{-1} C^T v_1,$$

and the following identity

$$\begin{aligned} 2J_2(u_1^*, u_2^*, w_2^*) = {} & (x_0^T \ \ v_{10}) E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix} \\ & + (x_0^T \quad v_{10}) e_2(t_0) + d_2(t_0) \\ & + \int_{t_0}^{t_f} \|z_2\|_{R_{22}}^2 dt + \int_{t_0}^{t_f} \|z\|_{P_2}^2 dt, \end{aligned}$$

where $y = \begin{pmatrix} x \\ v_1 \end{pmatrix}$, $\|z_2\|_{R_{22}}^2 = z_2 R_{22} z_2$ and

$$z_2 = u_2 + \left(R_{22}^{-1} B_2^T \quad 0_{m_1 \times n}\right) \left(E_2 y + \frac{1}{2} e_2\right)$$

and $0_{m_i \times n}, i = 1, 2$ the $m_i \times n$ dimensional zero matrix and $\|z\|_{P_2}^2 = z P_2 z$ and

$$z = w_2 + P_2^{-1} C_1^T \left(E_2 y + \frac{1}{2} e_2\right).$$

Proof: Consider (10): $u_1^* = -R_{11}^{-1} B_1^T \underbrace{\left(E_1 x + \frac{1}{2} e_1\right)}_{:= v_1}$. Then, differentiate v_1 and substitute the derivatives into the obtained expression using (6), (7) and (8). Also, the optimal control u_1^* and disturbance w_1^* in (11). Hence:

$$\dot{v}_1 = -Q_1 x - A^T v_1,$$
$$\dot{x} = Ax - S_1 v_1 + B_2 u_2 + Cw.$$

Hence defining $H := \begin{pmatrix} A & -S_1 \\ -Q_1 & E_1 T_1 - A^T \end{pmatrix}$, $B := \begin{pmatrix} B_2 \\ O_{n \times m_2} \end{pmatrix}$ and $C_1 := \begin{pmatrix} I \\ E_1 \end{pmatrix} C$. We define $y := \begin{pmatrix} x \\ v_1 \end{pmatrix}$ to write these two equations as: (??) as:

$$\dot{y} = Hy + Bu_2 + C_1 w \tag{20}$$

Next, we consider the following value function

$$\tilde{V}_2(t) = V_2(t, y(t)) = y^T E_2 y + e_2^T y + d_2 \tag{21}$$

for some mappings $E_2 : \mathcal{T} \to \mathbb{R}^{2n\times 2n}, e_2 : \mathcal{T} \to \mathbb{R}^{2n}$, and $d_2 : \mathcal{T} \to \mathbb{R}^2$, where E_2 is symmetric for each $t \in T$.

We consider (21), where we substitute (20):

$$\begin{aligned}
\frac{d\tilde{V}_2}{dt} &= \frac{d}{dt}\left(y^T E_2 y + e_2^T y + d_2\right)\\
&= \dot{y}^T E_2 y + y^T \dot{E}_2 y + y^T E_2 \dot{y} + \dot{e}_2^T y + e_2^T \dot{y} + \dot{d}_2\\
&+x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2 + w^T P_2 w - \psi_2\\
&= \left(y^T H^T + u_2^T B^T w^T C_1^T\right) E_2 y\\
&+y^T \dot{E}_2 y + y^T E_2 (Hy + Bu_2 + C_1 w)\\
&+\dot{e}_2^T y + e_2^T (Hy + Bu_2 + C_1 w) + \dot{d}_2\\
&+y^T \underbrace{\begin{pmatrix} Q_2 & 0\\ 0 & S_{21}\end{pmatrix}}_{:=Q} y + u_2^T R_{22} u_2 + w^T P_2 w - \psi_2
\end{aligned}$$

Now we associate certain terms

$$\begin{aligned}
&= y^T\left(H^T E_2 + \dot{E}_2 + E_2 H + Q\right) y\\
&+\left(u_2 - y_2\right)^T R_{22}\left(u_2 - y_2\right)\\
&+y_2^T R_{22} u_2 + u_2^T R_{22} y_2 - y_2^T R_{22} y_2\\
&+(w-\alpha)^T P_2 (w-\alpha)\\
&+w^T P_2 \alpha + \alpha^T P_2 w - \alpha^T P_2 \alpha\\
&+u_2^T\left(B^T E_2 y + B^T e_2\right) + \left(y^T E_2 B + \frac{1}{2} e_2^T B\right) u_2\\
&+w^T\left(C_1^T E_2 y + C_1^T e_2\right) + \left(y^T E_2 C_1 + \frac{1}{2} e_2^T C_1\right) w\\
&\left(\dot{e}_2^T + e_2^T H\right) y + \dot{d}_2 - \psi_2
\end{aligned}$$

and furthermore

$$\begin{aligned}
&= y^T\left(H^T E_2 + \dot{E}_2 + E_2 H + Q\right) y - \psi_2\\
&+\left(u_2 - y_2\right)^T R_{22}\left(u_2 - y_2\right) - y_2^T R_{22} y_2\\
&+(w-\alpha)^T P_2 (w-\alpha) - \alpha^T P_2 \alpha\\
&+u_2^T\left(B^T E_2 y + B^T \frac{1}{2} e_2 + R_{22} y_2\right)\\
&+\left(y^T E_2 B + \frac{1}{2} e_2^T B y_2^T R_{22}\right) u_2\\
&+w^T\left(C_1^T E_2 y + C_1^T \frac{1}{2} e_2 + P_2 \alpha\right)\\
&+\left(y^T E_2 C_1 + \frac{1}{2} e_2^T C_1 - \alpha^T P_2\right) w\\
&\left(\dot{e}_2^T + e_2^T H\right) y + \dot{d}_2 - \psi_2
\end{aligned}$$

Consider

$$R_{22}y_2 + \begin{pmatrix} B_2^T & O_{m_2\times n} \end{pmatrix}\left(E_2 y + \frac{1}{2}e_2\right) = 0$$

and also

$$C_1^T\left(E_2 y + \frac{1}{2}e_2\right) + P_2\alpha = 0$$

If $R_{22} > 0$ then $y_2 = -R_{22}^{-1}\begin{pmatrix} B_2^T & O_{m_2\times n} \end{pmatrix}\left(E_2 y + \frac{1}{2}e_2\right)$. If $P_2 > 0$ then $\alpha = -P_2^{-1}C_1\left(E_2 y + \frac{1}{2}e_2\right)$.

Define $S := \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} T_2 & T_2E_1 \\ E_1T_2 & E_1T_2E_1 \end{pmatrix}$. Substitute this y_2 and α in the calculations:

$$\begin{aligned} &= y^T\left(H^TE_2 + \dot{E}_2 + E_2H + Q - E_2(S+T)E_2\right)y \\ &+\left(u_2 - y_2\right)^T R_{22}\left(u_2 - y_2\right) + (w-\alpha)^T P_2(w-\alpha) \\ &+\left(\dot{e}_2^T + e_2^T H - E_2(S+T)\right)y \\ &+\dot{d}_2 - \frac{1}{4}e_2^T(S+T)e_2 - \psi_2 \end{aligned}$$

Considering:

$$\begin{aligned} H^TE_2 + \dot{E}_2 + E_2H + Q - E_2(S+T)E_2 &= 0 \\ \dot{e}_2^T + e_2^T H - e_2^T(S+T)E_2 &= 0 \\ \dot{d}_2 - \frac{1}{4}e_2^T(S+T)e_2 &= 0 \end{aligned}$$

that is

$$\begin{aligned} \dot{E}_2 &= -H^TE_2 - E_2H - Q + E_2(S+T)E_2 \\ \dot{e}_2 &= \left(-H^T + E_2(S+T)\right)e_2 \\ \dot{d}_2 &= \frac{1}{4}e_2^T(S+T)e_2 \end{aligned}$$

We end up with

$$\begin{aligned} \frac{d\tilde{V}_2(t)}{dt} &= \left(u_2 - y_2\right)^T R_{22}\left(u_2 - y_2\right) \\ &+(w-\alpha)^T P_2(w-\alpha) - \psi_2 \end{aligned} \tag{22}$$

Integrating yields:

$$\begin{aligned} \tilde{V}_2\left(t_f\right) - \tilde{V}(t) &= \int_t^{t_f}\left[\left(u_2 - y_2\right)^T R_{22}\left(u_2 - y_2\right)\right. \\ &\left.+(w-\alpha)^T P_2(w-\alpha)s\right]d\tau - \int_t^{t_f}\psi_2 d\tau. \end{aligned}$$

Further, we assume the mappings E_2, e_2, d_2 to be chosen in such a way that the following terminal values hold:

$$E_2(t_f) = K_{2_f}$$
$$e_2(t_f) = 0$$
$$d_2(t_f) = 0$$

Then, we obtain $\tilde{V}_2(t_f) = y^T(t_f)K_{y_f}y(t_f)$ and substituting:

$$\tilde{V}_2(t) = y^T(t_f)K_{y_f}y(t_f) - \int_t^{t_f}\Big[(u_2 - y_2)^T R_{22}(u_2 - y_2) + (w - \alpha)^T P_2(w - \alpha)\Big]d\tau + \int_t^{t_f}\psi_2 d\tau \tag{23}$$

Observe that the rhs of (23) does not depend of $u|_{[t_0,t]}$ and the rls of (23) does not depend of $u_2|_{[t,t_f]}$. Then considering now the infimal value, we recall that:

$$V_2(t,y) = \inf_{u_2|_{[t,t_f]}}\int_t^{t_f}\psi_2(\tau,\hat{y}(\tau),u(\tau))d\tau + y^T(t_f)K_{2_f}y(t_f)$$

Now, we substitute this into (23) and consider the infimal values over all possible control functions in $[t,t_f]$:

$$\tilde{V}_2(t) = \underbrace{\inf y^T(t_f)K_{2_f}y(t_f) + \int_t^{t_f}\psi_2 d\tau}_{V_2(t,y)} - \inf\int_t^{t_f}\Big[(u_2 - y_2)^T R_{22}(u_2 - y_2) + (w - \alpha)^T P_2(w - \alpha)\Big]d\tau$$

then we have:

$$V_2(t,y) = \tilde{V}(t) + \inf_{u|_{t,t_f}}\int_t^{t_f}\Big[(u_2 - y_2)^T R_{22}(u_2 - y_2) + (w - \alpha)^T P_2(w - \alpha)\Big]d\tau$$

$V_2(t,y)$ equals $\tilde{V}_2(t)$ if $u_2 - y_2 \equiv 0 \forall t \in \mathcal{T}$ and $w - \alpha = 0$. As the leader chooses his strategy assuming rationality of the follower and worst-case disturbance, the follower should take also the worst-case disturbance into account.

To conclude, consider $t = t_0$ and hence:

$$V_2(t_0,y) = \tilde{V}_2(t_0) + \inf_{u|_{t_0,t_f}}\int_{t_0}^{t_f}\Big[(u_2 - y_2)^T R_{22}(u_2 - y_2) + (w - \alpha)^T P_2(w - \alpha)\Big]d\tau$$

Then from (21):

$$V_2(t_0,y) = y_0^T E_2(t_0) y_0 + e_2^T(t_0) y_0 + d_2(t_0) + \inf_{u|_{t_0,t_f}} \int_{t_0}^{t_f} \Big[(u_2 - y_2)^T R_{22} (u_2 - y_2) + (w-\alpha)^T P_2 (w - \alpha) \Big] d\tau$$

Defining $z_2 := u_2 - y_2 = u_2 + \begin{pmatrix} R_{22}^{-1} B_2^T & 0 \end{pmatrix} \left(E_2 y + \frac{1}{2} e_2 \right)$ and $z := w - \alpha = w + P_2^{-1} C_1 \left(E_2 y + \frac{1}{2} e_2 \right)$, we have:

$$V_2(t_0,y) = y_0^T E_2(t_0) y_0 + e_2^T(t_0) y_0 + d_2(t_0) + \int_{t_0}^{t_f} \|z_2\|_{R_{22}}^2 + \|z\|_{P_2}^2 dt$$

Now, we substitute $y_0 = \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix}$.

The leader may choose its best answer either by accounting directly for its worst-case disturbance or by considering that the follower knows that there is a worst-case disturbance. In this work, the leader takes the worst-case disturbance directly into account.

Notice that in the term

$$J_{20} = \begin{pmatrix} x_0^T & v_{10} \end{pmatrix} E_2(t_0) \begin{pmatrix} x_0 \\ v_{10} \end{pmatrix}, \tag{24}$$

$x_0, E_2(t_0)$, do not depend on the choice of u_1, u_2. Since we shall study the situation for Player-2 when Player-1 applies his optimal response control defined in (10), we have to set $v_1 = E_1 x + \frac{1}{2} e_1$. From (7), we can see that $v_1(t_0) = v_{10}$ depends on $e_1(t_0)$ and hence also on u_2.

In order to derive from Theorems (3.1) and (3.3) sufficient conditions for the existence of a unique worst-case Stackelberg equilibrium, we must get rid of the u_2-dependence on v_{10}. Therefore, we propose to restrict the set of admissible controls to functions representable in linear feedback form. This is what we do next.

Theorem 3.4. Let the solutions $E_1(t) \in \mathbb{R}^{n \times n}, E_2 \in \mathbb{R}^{2n \times 2n}$ of (6) and (15) exist in $\mathcal{T}$, respectively. Let further the coupled system of equations

$$\dot{K}_1 = -Q_1 - K_1 A - A^T K_1 + K_1 (S_1 + T_1) K_1 + K_1 S_2 K_2, \tag{25}$$

$$\dot{K}_2 = -Q_2 - K_2 A - A^T K_2 + Q_1 p + K_2 S_1 K_1 + K_2 (S_2 + T_2) K_2 + K_2 T_2 E_1 p, \tag{26}$$

$$\dot{p} = -pA - S_{21} K_1 + S_1 K_2 + (A - T_1 E_1) p + p S_1 K_1 + p (S_2 + T_2) K_2 + p T_2 E_1 p, \tag{27}$$

admits a solution in $\mathcal{T}$.

Then, there exists a unique open loop disturbed Stackelberg equilibrium in feedback synthesis which is given by

$$u_1^*(t) = -R_{11}^{-1}(t) B_1^T(t) K_1(t) x(t), \tag{28}$$

$$u_2^*(t) = -R_{22}^{-1}(t) B_2^T(t) K_2(t) x(t), \tag{29}$$

considering worst-case disturbamces w_i^* and where $x(t)$ is a solution of the closed loop equation

$$\begin{aligned}\dot{x} &= [A - S_1K_1 - (S_2 + T_2)K_2 - T_2E_1p]x,\\ x(t_0) &= x_0.\end{aligned} \tag{30}$$

The minimal cost for the follower, $J_{10}(u_2^*)$, is as in (14), and for the leader is

$$J_{20}(u_1^*, u_2^*) = \frac{1}{2}\left[x_0^T(I_n, K_1^T(t_0))E_2(t_0)(t_0)\begin{pmatrix} I_n \\ K_1(t_0)\end{pmatrix}x_0 + e_2^T(t_0)\begin{pmatrix} I_n \\ K_1(t_0)\end{pmatrix}x_0 + d_2(t_0)\right]$$

where $e_2(t_0), d_2(t_0)$ are determined by (16) and (17), respectively.

Proof: The proof is similar to the analogous result for the non-disturbed case [13].

From the convexity assumptions, it follows that S_1, S, Q_1, Q and $E_1(t_f), E_2(t_f)$ are all semidefinite. Therefore, as far as the convexity conditions hold, the standard Riccati matrix Eqs. (6) and (15) are globally solvable in $(-\infty, t_f]$ [15].

It still remains the following questions to be answered (i) direct criteria for solvability of these equations if the convexity assumption is guaranteed as well as (ii) solvability of the coupled system of Eqs. (25)–(27).

Actually, this system of equations can also be written as a single, nonsymmetric Riccati matrix differential equation. Hence:

$$\begin{aligned}\begin{pmatrix}\dot{K}_1 \\ \dot{K}_2 \\ \dot{p}\end{pmatrix} &= \begin{pmatrix} Q_1 \\ Q_2 \\ 0\end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \\ p\end{pmatrix}A \\ &\quad + \begin{pmatrix} -A^T & 0 & 0 \\ 0 & -A^T & Q_1 \\ -S_{21} & S_1 & A - T_1E_1\end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ p\end{pmatrix} \\ &\quad + \begin{pmatrix} K_1 \\ K_2 \\ p\end{pmatrix}(S_1 + T_1, \quad S_2, \quad 0)\begin{pmatrix} K_1 \\ K_2 \\ p\end{pmatrix} \\ \begin{pmatrix} K_1 \\ K_2 \\ p\end{pmatrix}(t_f) &= \begin{pmatrix} K_{1f} \\ K_{2f} \\ 0\end{pmatrix}.\end{aligned} \tag{31}$$

As it can be easily observed, all these Riccati equations are of nonsymmetric type:

$$\begin{aligned}\dot{W} &= B_{21} - WB_{11} + B_{22}W + WB_{12}W,\\ W(t_f) &= W_f,\end{aligned} \tag{32}$$

where W is a matrix of order $k \times n$ whose coefficients are of adequate size. See AbouKandil et al. [16] for results on the existence of solution of Riccati equations.

4. Discussion and conclusions

High dimension problems appeal to the use of hierarchic and decentralised models as differential games. One example of these problems is large networks, as for instance the management and control of high pressure gas networks. Since this is a large dimension and geographically dispersed problem, a decentralised formulation captures the non-cooperative nature, and sometimes even antagonistic, of the different stake-holders in the network.

The network controllable elements can be seen as players that seek their best settings and then interact among themselves to check for network feasibility. The equilibrium sought by the players depends on the way the players are organised among themselves. It makes some sense to have some autonomous elements that run the network and others follow, as is the case of a main inlet point of a country, as it happens with the inlet of Sines in the portuguese network. The ultimate goal of the network is to meet customers' demand at the lowest cost. As the main variation of the problem is due to the off-takes, these may be seen as perturbations to nominal consumption levels of a deterministic model.

Therefore, it makes some sense to view the gas transportation and distribution system as a disturbed Stackelberg game where the players play against a worst-case disturbance, that means a sudden change in weather conditions from one period of operation to the other. Neverthless, the theory is not ready, and also having in mind the development of algorithms, direct solution methods, and explicit solution representations need to be further investigated. In this work, we have obtained sufficient conditions for the existence of the solution of a 2-player game. However, direct criteria for solvability of this problem needs more work. Also, the solvability of the coupled system of Eqs. (25)–(27) has to be further investigated. Also, we would like to solve the same problem using an operator approach.

Similarly to what we have done in the past for Nash games, we would like to study this problem considering the underlying dynamics as a repetitive process, that seems to be adequate to capture the behaviour seemingly periodic of the network. Also, the boundary control of the network depends on the type of strategy sought by the players. The structure of these versions of the problems need to be examined.

The obtained results, in every stage of the work, should be applied to a single pipe and ideally using some operational data. Furthermore, we expect to apply the work to a simple network, which is not exactly a straightforward extension.

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