Chapter

Uncertainty Relations

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Abstract

Uncertainty relations are inequalities representing the impossibility of simultaneous measurement in quantum mechanics. The most well-known uncertainty relations were presented by Heisenberg and Schrödinger. In this chapter, we generalize and extend them to produce several types of uncertainty relations.

Keywords: trace inequality, variance, covariance, skew information, metric adjusted skew information, noncommutativity, observable, operator inequality

1. Introduction

Let $M_n(\mathbb{C})$ (resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = Tr[A^* B]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C}) \subset M_{n,+}(\mathbb{C})$ be the set of strictly positive density matrices, that is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr[\rho] = 1, \rho > 0\}$. If not otherwise specified, hereafter, we address the case of faithful states, that is $\rho > 0$. It is known that the expectation of an observable $A \in M_{n,sa}(\mathbb{C})$ in state $\rho \in M_{n,+,1}(\mathbb{C})$ is defined by

$$E_\rho(A) = Tr[\rho A],$$

and the variance of an observable $A \in M_{n,sa}(\mathbb{C})$ in state $\rho \in M_{n,+,1}(\mathbb{C})$ is defined by

$$V_\rho(A) = Tr\left[\rho\left(A - E_\rho(A)I\right)^2\right] = Tr\left[\rho A^2\right] - E_\rho(A)^2 = Tr\left[\rho A_0^2\right],$$

where $A_0 = A - E_\rho(A)I$.

In Section 2, we introduce the Heisenberg and Schrödinger uncertainty relations. In Section 3, we present uncertainty relations with respect to the Wigner-Yanase and Wigner-Yanase-Dyson skew information. To represent the degree of noncommutativity between $\rho \in M_{n,+,1}(\mathbb{C})$ and $A \in M_{n,sa}(\mathbb{C})$, the Wigner-Yanase skew information $I_\rho(A)$ is defined by

$$I_\rho(A) = \frac{1}{2} Tr\left[\left(i\left[\rho^{1/2}, A\right]\right)^2\right] = Tr\left[\rho A^2\right] - Tr\left[\rho^{1/2} A \rho^{1/2} A\right],$$

where $[X, Y] = XY - YX$. Furthermore, the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(A)$ is defined by

$$I_{\rho,\alpha}(A) = \frac{1}{2}Tr\left[(i[\rho^{\alpha}, A])(i\left[\rho^{1-\alpha}, A\right])\right] = Tr\left[\rho A^2\right] - Tr\left[\rho^{\alpha} A \rho^{1-\alpha} A\right], \quad (\alpha \in [0,1]).$$

The convexity of $I_{\rho,\alpha}(A)$ with respect to ρ was famously demonstrated by Lieb [1], and the relationship between the Wigner-Yanase skew information and the uncertainty relation was originally developed by Luo and Zhang [2]. Subsequently, the relationship between the Wigner-Yanase-Dyson skew information and the uncertainty relation was provided by Kosaki [3] and Yanagi-Furuichi-Kuriyama [4]. In Section 4, we discuss the metric adjusted skew information defined by Hansen [5], which is an extension of the Wigner-Yanase-Dyson skew information. The relationship between metric adjusted skew information and the uncertainty relation was provided by Yanagi [6] and generalized by Yanagi-Furuichi-Kuriyama [7] for generalized metric adjusted skew information and the generalized metric adjusted correlation measure. In Sections 5 and 6, we provide non-Hermitian extensions of Heisenberg-type and Schrödinger-type uncertainty relations related to generalized quasi-metric adjusted skew information and the generalized quasi-metric adjusted correlation measure. As a result, we obtain results for non-Hermitian uncertainty relations provided by Dou and Du as corollaries of our results. Finally, in Section 7, we present the sum types of uncertainty relations.

2. Heisenberg and Schrödinger uncertainty relations

Theorem 1.1 (Heisenberg uncertainty relation). For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$,

$$V_{\rho}(A)V_{\rho}(B) \geq \frac{1}{4}\left|Tr\left[\rho[A,B]\right]\right|^2, \tag{1}$$

where $[A,B] = AB - BA$ is the commutator.

Theorem 1.2 (Schrödinger uncertainty relation). For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$,

$$V_{\rho}(A)V_{\rho}(B) - \left|\,\mathrm{Re}\,\{Tr[\rho A_0 B_0]\}\right|^2 \geq \frac{1}{4}Tr\left[\rho[A,B]\right]\right|^2.$$

Proof of Theorem 1.2. By the Schwarz inequality

$$\begin{aligned} |Tr[\rho A_0 B_0]|^2 &= \left|Tr\left[\left(\rho^{1/2}B_0\right)^{*}\left(\rho^{1/2}A_0\right)\right]\right|^2 \\ &\leq Tr\left[\left(\rho^{1/2}B_0\right)^{*}\left(\rho^{1/2}B_0\right)\right] \cdot Tr\left[\left(\rho^{1/2}A_0\right)^{*}\left(\rho^{1/2}A_0\right)\right] \\ &= Tr\left[\rho A_0^2\right] \cdot Tr\left[\rho B_0^2\right] = V_{\rho}(A) \cdot V_{\rho}(B). \end{aligned}$$

Since

$$\begin{aligned} Tr[\rho[A_0, B_0]] &= Tr[\rho A_0 B_0] - Tr[\rho B_0 A_0] = Tr[\rho A_0 B_0] - \overline{Tr[A_0 B_0 \rho]} \\ &= Tr[\rho A_0 B_0] - \overline{Tr[\rho A_0 B_0]} = 2i\mathrm{Im}\{Tr[\rho A_0 B_0]\}, \end{aligned}$$

we have

$$\begin{aligned} |Tr[\rho A_0 B_0]|^2 &= (\,\mathrm{Re}\,\{Tr[\rho A_0 B_0]\})^2 + (\mathrm{Im}\{Tr[\rho A_0 B_0]\})^2 \\ &= (\,\mathrm{Re}\,\{Tr[\rho A_0 B_0]\})^2 + \frac{1}{4}|Tr[\rho[A_0, B_0]]|^2. \end{aligned}$$

Since $Tr[\rho[A_0, B_0]] = Tr[\rho[A, B]]$, we obtain

$$V_\rho(A) \cdot V_\rho(B) - \left| \mathrm{Re}\left\{ Tr[\rho A_0 B_0] \right\} \right|^2 \geq \frac{1}{4} Tr\left[\rho[A, B]\right]|^2. \qquad \square$$

3. Uncertainty relation for Wigner-Yanase-Dyson skew information

3.1 Wigner-Yanase skew information

To represent the degree of noncommutativity between $\rho \in M_{n,+,1}(\mathbb{C})$ and $A \in M_{n,sa}(\mathbb{C})$, the Wigner-Yanase skew information $I_\rho(A)$ and related quantity $J_\rho(A)$ are defined as

$$I_\rho(A) = \frac{1}{2} Tr\left[\left(i\left[\rho^{1/2}, A_0\right] \right)^2 \right] = Tr\left[\rho A_0^2\right] - Tr\left[\rho^{1/2} A_0 \rho^{1/2} A_0\right].$$

$$J_\rho(A) = \frac{1}{2} Tr\left[\rho\{A_0, B_0\}^2\right] = Tr\left[\rho A_0^2\right] + Tr\left[\rho^{1/2} A_0 \rho^{1/2} A_0\right],$$

where $\{A, B\} = AB + BA$. The quantity $U_\rho(A)$ representing a quantum uncertainty excluding the classical mixture is defined as

$$U_\rho(A) = \sqrt{I_\rho(A) \cdot J_\rho(A)} = \sqrt{V_\rho(A)^2 - \left(V_\rho(A) - I_\rho(A)\right)^2}.$$

We note the following relation:

$$0 \leq I_\rho(A) \leq U_\rho(A) \leq V_\rho(A). \tag{2}$$

Luo [8] then derived the uncertainty relation of $U_\rho(A)$.
Theorem 1.3. For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$,

$$U_\rho(A) \cdot U_\rho(B) \geq \frac{1}{4} \left| Tr\left[\rho[A, B]\right]|^2. \tag{3}$$

Inequality (3) is a refinement of (1) in terms of (2).

3.2 Wigner-Yanase-Dyson skew information

Here, we introduce a one-parameter inequality extended from (3). For $0 \leq \alpha \leq 1, A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we define the Wigner-Yanase-Dyson skew information as follows:

$$I_{\rho,\alpha}(A) = \frac{1}{2} Tr\left[(i[\rho^\alpha, A_0]) \left(i\left[\rho^{1-\alpha}, A_0\right] \right) \right] = Tr\left[\rho A_0^2\right] - Tr\left[\rho^\alpha A_0 \rho^{1-\alpha} A_0\right].$$

We also define

$$J_{\rho,\alpha}(A) = \frac{1}{2} Tr\left[\{\rho^\alpha, A_0\} \{\rho^{1-\alpha}, A_0\} \right] = Tr[\rho A_2] + Tr\left[\rho^\alpha A_0 \rho^{1-\alpha} A_0\right].$$

We note that

$$\frac{1}{2} Tr[i[\rho^\alpha, A_0]) \left(i\left[\rho^{1-\alpha}, A_0\right] \right)] = \frac{1}{2} Re\left[(i[\rho^\alpha, A]) \left(i\left[\rho^{1-\alpha}, A\right] \right) \right];$$

however, we have

$$\frac{1}{2}Tr\left[\{\rho^{\alpha},A_0\}\{\rho^{1-\alpha},A_0\}\right] \neq \frac{1}{2}Tr\left[\{\rho^{\alpha},A\}\{\rho^{1-\alpha},A\}\right].$$

We then have the following inequalities:

$$I_{\rho,\alpha}(A) \leq I_{\rho}(A) \leq J_{\rho}(A) \leq J_{\rho,\alpha}(A), \tag{4}$$

because $Tr\left[\rho^{1/2}A\rho^{1/2}A\right] \leq Tr[\rho^{\alpha}A\rho^{1-\alpha}A]$. We define

$$U_{\rho,\alpha}(A) = \sqrt{I_{\rho,\alpha}(A)\cdot J_{\rho,\alpha}(A)} = \sqrt{V_{\rho}(A)^2 - \left(V_{\rho}(A) - I_{\rho,\alpha}(A)\right.}. \tag{5}$$

From (2), (4), and (5), we have

$$0 \leq I_{\rho,\alpha}(A) \leq I_{\rho}(A) \leq U_{\rho}(A)$$

and

$$0 \leq I_{\rho,\alpha}(A) \leq U_{\rho,\alpha}(A) \leq U_{\rho}(A).$$

We provide the following uncertainty relation with respect to $U_{\rho,\alpha}(A)$ as a direct generalization of (3).

Theorem 1.4 ([9]). For $A,B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$,

$$U_{\rho,\alpha}(A)\cdot U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)\left|Tr\left[\rho[A,B]\right]\right|^2. \tag{6}$$

Proof of Theorem 1.4. By spectral decomposition, there exists an orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ consisting of eigenvectors of ρ. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues, where $\sum_{i=1}^{n}\lambda_i = 1$ and $\lambda_i \geq 0$. Thus ρ has a spectral representation $\rho = \sum_{i=1}^{n}\lambda_i|\phi_i\rangle\langle\phi_i|$. We can obtain the following representations of $I_{\rho,\alpha}(A)$ and $J_{\rho,\alpha}(A)$:

$$I_{\rho,\alpha}(A) = \sum_{i<j}\left(\lambda_i + \lambda_j - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|A_0\left|\phi_j\right\rangle\right|^2.$$

$$J_{\rho,\alpha}(A) \geq \sum_{i<j}\left(\lambda_i + \lambda_j + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\left|\langle\phi_i|A_0\left|\phi_j\right\rangle\right|^2.$$

Since $(1-2\alpha)^2(t-1)^2 - (t^{\alpha} - t^{1-\alpha})^2 \geq 0$ for any $t > 0$ and $0 \leq \alpha \leq 1$, we define $t = \frac{\lambda_i}{\lambda_j}$ and have

$$(1-2\alpha)^2\left(\frac{\lambda_i}{\lambda_j} - 1\right)^2 - \left(\left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha} - \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha}\right)^2 \geq 0.$$

Then,

$$\left(\lambda_i + \lambda_j\right)^2 - \left(\lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)^2 \geq 4\alpha(1-\alpha)\left(\lambda_i - \lambda_j\right)^2. \tag{7}$$

Since

$$\begin{aligned} Tr[\rho[A,B]] &= Tr[\rho[A_0,B_0]] = 2iImTr[\rho A_0 B_0] = 2iIm\sum_{i<j}(\lambda_i-\lambda_j)\left\langle\phi_i|A_0|\phi_j\right\rangle\left\langle\phi_j|B_0|\phi_i\right\rangle \\ &= 2i\sum_{i<j}(\lambda_i-\lambda_j)Im\left\langle\phi_i|A_0|\phi_j\right\rangle\left\langle\phi_j|B_0|\phi_i\right\rangle, \end{aligned}$$

$$\begin{aligned} |Tr[\rho[A,B]]| &= 2|\sum_{i<j}(\lambda_i-\lambda_j)Im\left\langle\phi_i|A_0|\phi_j\right\rangle\left\langle\phi_j|B_0|\phi_i\right\rangle| \le 2\sum_{i<j}|\lambda_i \\ &\quad -\lambda_j||Im\left\langle\phi_i|A_0|\phi_j\right\rangle\left\langle\phi_j|B_0|\phi_i\right\rangle|. \end{aligned}$$

We then have

$$|Tr[\rho[A,B]]|^2 \le 4\left\{\sum_{i<j}|\lambda_i-\lambda_j||Im\left\langle\phi_i|A_0|\phi_j\right\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2.$$

By (7) and the Schwarz inequality,

$$\begin{aligned} \alpha(1-\alpha)|Tr[\rho[A,B]]|^2 &\le 4\alpha(1-\alpha)\left\{\sum_{i<j}|\lambda_i-\lambda_j||Im\left\langle\phi_i|A_0|\phi_j\right\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2 \\ &= \left\{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i-\lambda_j||Im\left\langle\phi_i|A_0|\phi_j\right\rangle\langle\phi_j|B_0|\phi_i\rangle\right\}^2 \\ &\le \left\{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i-\lambda_j||\left\langle\phi_i|A_0|\phi_j\right\rangle\langle\phi_j|B_0|\phi_i\rangle|\right\}^2 \\ &\le \left\{\sum_{i<j}\left\{\left(\lambda_i+\lambda_j\right)^2-\left(\lambda_i^\alpha\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^\alpha\right)^2\right\}^{1/2}|\langle\phi_i|A_0|\phi_j\rangle||\langle\phi_j|B_0|\phi_i\rangle|\right\}^2 \\ &\le \sum_{i<j}\left(\lambda_i+\lambda_j-\lambda_i^\alpha\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^\alpha\right)|\langle\phi_i|A_0\left|\phi_j\right\rangle|^2 \\ &\times\sum_{i<j}\left(\lambda_i+\lambda_j+\lambda_i^\alpha\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^\alpha\right)|\langle\phi_i|B_0\left|\phi_j\right\rangle|^2. \end{aligned}$$

Then, we have

$$I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) \ge \alpha(1-\alpha)|Tr[\rho[A,B]]|^2.$$

We also have

$$I_{\rho,\alpha}(B)J_{\rho,\alpha}(A) \ge \alpha(1-\alpha)\left|Tr[\rho[A,B]]\right|^2.$$

Thus, we have the final result, (6). □

When $\alpha = \frac{1}{2}$, we obtain the result in Theorem 1.3.

4. Metric adjusted skew information and metric adjusted correlation measure

4.1 Operator monotone function

A function $f : (0,+\infty) \to \mathbb{R}$ is considered operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. An operator

monotone function is said to be symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 1 $\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that:

1. $f(1) = 1$.

2. $tf(t^{-1}) = f(t)$.

3. f is operator monotone.

Example 1. Examples of elements of $\mathcal{F}_{op}$ are given by the following:

$$f_{RLD}(x) = \frac{2x}{x+1}, \quad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \quad f_{BKM}(x) = \frac{x-1}{\log x},$$

$$f_{SLD}(x) = \frac{x+1}{2}, \quad f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0,1).$$

Remark 1. Any $f \in \mathcal{F}_{op}$ satisfies

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}, \quad x > 0.$$

For $f \in \mathcal{F}_{op}$, we define $f(0) = \lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}^r_{op} = \{ f \in \mathcal{F}_{op} | f(0) \neq 0 \}, \quad \mathcal{F}^n_{op} \{ f \in \mathcal{F}_{op} | f(0) = 0 \}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}^r_{op} \cup \mathcal{F}^n_{op}$.

Definition 2. For $f \in \mathcal{F}^r_{op}$, we set

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right], \quad x > 0.$$

Theorem 1.5 ([10]). The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}^r_{op}$ and $\mathcal{F}^n_{op}$.

4.2 Metric adjusted skew information

In the Kubo-Ando theory [11] of matrix means, a mean is associated with each operator monotone function $f \in \mathcal{F}_{op}$ by the following formula:

$$m_f(A, B) = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. Using the notion of matrix means, the class of monotone metrics can be defined by the following formula:

$$\langle A, B \rangle_{\rho,f} = Tr\left[A \cdot m_f\left(L_\rho, R_\rho\right)^{-1}(B)\right],$$

where $L_\rho(A) = \rho A, R_\rho(A) = A\rho$.

Definition 3. For $A \in M_{n,sa}(\mathbb{C})$, we define as follows:

$$I_\rho^f(A) = \frac{f(0)}{2}\langle i[\rho,A], i[\rho,A]\rangle_{\rho,f},$$
$$C_\rho^f(A) = Tr\left[m_f(L_\rho,R_\rho)(A)\cdot A\right],$$
$$U_\rho^f(A) = \sqrt{V_\rho(A)^2 - \left(V_\rho(A) - I_\rho^f(A)\right)^2}.$$

Quantity $I_\rho^f(A)$ is referred to as the metric adjusted skew information, and $\langle A,B\rangle_{\rho,f}$ is referred to as the metric adjusted correlation measure.

Proposition 1. The following holds:

1. $I_\rho^f(A) = I_\rho^f(A_0) = Tr\left(\rho A_0^2\right) - Tr\left(m_{\tilde{f}}(L_\rho,R_\rho)(A_0)\cdot A_0\right) = V_\rho(A) - C_\rho^{\tilde{f}}(A_0).$

2. $J_\rho^f(A) = Tr\left(\rho A_0^2\right) + Tr\left(m_{\tilde{f}}(L_\rho,R_\rho)(A_0)\cdot A_0\right) = V_\rho(A) + C_\rho^{\tilde{f}}(A_0).$

3. $0 \le I_\rho^f(A) \le U_\rho^f(A) \le V_\rho(A).$

4. $U_\rho^f(A) = \sqrt{I_\rho^f(A)\cdot J_\rho^f(A)}.$

Theorem 1.6 ([6]). For $f \in F_{op}^r$, if

$$\frac{x+1}{2} + \tilde{f}(x) \ge 2f(x), \tag{8}$$

then it holds that

$$U_\rho^f(A)\cdot U_\rho^f(B) \ge f(0)\left|Tr(\rho[A,B])\right|^2, \tag{9}$$

where $A,B \in M_{n,sa}(\mathbb{C})$.

To prove Theorem 1.6, several lemmas are used.

Lemma 1. If (8) holds, then the following inequality is satisfied:

$$\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x,y)^2 \ge f(0)(x-y)^2.$$

Proof of Lemma 1. By (8), we have

$$\frac{x+y}{2} + m_{\tilde{f}}(x,y) \ge 2m_f(x,y). \tag{10}$$

Since

$$m_{\tilde{f}}(x,y) = y\tilde{f}\left(\frac{x}{y}\right) = \frac{y}{2}\left\{\frac{x}{y} + 1 - \left(\frac{x}{y}-1\right)^2\frac{f(0)}{f(x/y)}\right\} = \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x,y)},$$

we have

$$\begin{aligned}\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x,y)^2 &= \left\{\frac{x+y}{2} - m_{\tilde{f}}(x,y)\right\}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x,y)\right\}\\ &= \frac{f(0)(x-y)^2}{2m_f(x,y)}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x,y)\right\} \ge f(0)(x-y)^2. \quad (by\ \ (10))\end{aligned}$$

Lemma 2. Let $\{|\phi_1\rangle, |\phi_2\rangle, \cdots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. We set $a_{jk} = \left\langle \phi_j | A_0 | \phi_k \right\rangle, b_{jk} = \left\langle \phi_j | B_0 | \phi_k \right\rangle$. Then, we have

$$I_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$

$$J_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} + \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$

$$\left(U_\rho^f(A)\right)^2 = \frac{1}{4}\left(\sum_{j,k}(\lambda_j + \lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)|a_{jk}|^2\right)^2.$$

Proof of Theorem 1.6. Since

$$Tr(\rho[A, B]) = Tr(\rho[A_0, B_0]) = \sum_{j,k}(\lambda_j - \lambda_k)a_{jk}b_{kj},$$

we have

$$\begin{aligned} f(0)|Tr(\rho[A, B])|^2 &\leq \left(\sum_{j,k} f(0)^{1/2}|\lambda_j - \lambda_k||a_{jk}||b_{kj}|\right)^2 \\ &\leq \left(\sum_{j,k}\left\{\left(\frac{\lambda_j + \lambda_k}{2}\right)^2 - m_{\tilde{f}}(\lambda_j, \lambda_k)^2\right\}^{1/2}|a_{jk}||b_{kj}|\right)^2 \\ &\leq \left(\sum_{j,k}\left\{\frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k)\right\}|a_{jk}|^2\right) \\ &\times \left(\sum_{j,k}\left\{\frac{\lambda_j + \lambda_k}{2} + m_{\tilde{f}}(\lambda_j, \lambda_k)\right\}|b_{kj}|^2\right) = I_\rho^f(A)J_\rho^f(B). \end{aligned}$$

We also have

$$I_\rho^f(B)J_\rho^f(A) \geq f(0)|Tr(\rho[A, B])|^2.$$

Thus, we have the final result (9). □

5. Generalized metric adjusted skew information

We assume that $f \in \mathcal{F}_{op}^r$ satisfies the following condition (A):

$$g(x) \geq k\frac{(x-1)^2}{f(x)}, \text{for some } k > 0.$$

Let

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in F_{op}.$$

Definition 4. For $A, B \in M_{n,sa}(\mathbb{C}), \rho \in M_{n,+,1}(\mathbb{C})$ we define the following:

$$\begin{aligned} Corr_{\rho}^{(g,f)}(A,B) &= k\langle i[\rho,A_0], i[\rho,B_0]\rangle_f \\ &= Tr\left[A_0 m_g(L_\rho,R_\rho)B_0\right] - Tr\left[A_0 m_{\Delta_g^f}(L_\rho,R_\rho)B_0\right]. \end{aligned}$$

$$I_{\rho}^{(g,f)}(A) = Corr_{\rho}^{(g,f)}(A,A)$$

$$= Tr\left[A_0 m_g(L_\rho,R_\rho)A_0\right] - Tr\left[A_0 m_{\Delta_g^f}(L_\rho,R_\rho)A_0\right] - Tr\left[A_0 m_{\Delta_g^f}(L_\rho,R_\rho)A_0\right].$$

$$J_{\rho}^{(g,f)}(A) = Tr\left[A_0 m_g(L_\rho,R_\rho)A_0\right] - Tr\left[A_0 m_{\Delta_g^f}(L_\rho,R_\rho)A_0\right] + Tr\left[A_0 m_{\Delta_g^f}(L_\rho,R_\rho)A_0\right].$$

$$U_{\rho}^{(g,f)}(A) = \sqrt{I_{\rho}^{(g,f)}(A)\cdot J_{\rho}^{(g,f)}(A)}.$$

$I_{\rho}^{(g,f)}(A)$ is referred to as the generalized metric adjusted skew information, and $Corr_{\rho}^{(g,f)}(A,B)$ is referred to as the generalized metric adjusted correlation measure.

Theorem 1.7 ([7]). Under condition (A), the following holds:

1. (Schrödinger type) For $A, B \in M_{n,sa}(\mathbb{C}), \rho \in M_{n,+,1}(\mathbb{C})$,

$$I_{\rho}^{(g,f)}(A)\cdot I_{\rho}^{(g,f)}(B) \geq \left|Corr_{\rho}^{(g,f)}(A,B)\right|^2.$$

2. (Heisenberg type) For $A, B \in M_{n,sa}(\mathbb{C}), \rho \in M_{n,+,1}(\mathbb{C})$, we assume the following condition (B):

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \text{ for some } \ell > 0.$$

Then,

$$U_{\rho}^{(g,f)}(A)\cdot U_{\rho}^{(g,f)}(B) \geq k\ell\left|Tr\left[\rho[A,B]\right]\right|^2.$$

6. Generalized quasi-metric adjusted skew information

In this section, we present general uncertainty relations for non-Hermitian observables $X, Y \in M_n(\mathbb{C})$.

Definition 5. For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$ we define the following:

$$\begin{aligned} \Gamma_{A,B}^{(g,f)}(X,Y) &= k\langle (L_A - R_B)X, (L_A - R_B)Y\rangle_f \\ &= kTr\left[X^*(L_A - R_B)m_f(L_A,R_B)^{-1}(L_A - R_B)Y\right] \\ &= Tr\left[X^* m_g(L_A,R_B)Y\right] - Tr\left[X^* m_{\Delta_b^f}(L_A,R_B)Y\right], \end{aligned}$$

$$\Psi_{A,B}^{(g,f)}(X,Y) = Tr\left[X^* m_g(L_A,R_B)Y\right] + Tr\left[X^* m_{\Delta_g^f}(L_A,R_B)Y\right],$$

$$I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X),\ J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X),\ U_{A,B}^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X)\cdot J_{A,B}^{(g,f)}(X)}.$$

$I_{A,B}^{(g,f)}(X)$ is referred to as the generalized quasi-metric adjusted skew information, and $\Gamma_{A,B}^{(g,f)}(X,Y)$ is referred to as the generalized quasi-metric adjusted correlation measure.

Theorem 1.8 ([12]). Under condition (A), the following holds:

1. (Schrödinger type) For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$,

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq \left|\Gamma_{A,B}^{(g,f)}(X,Y)\right|^2 \geq \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2.$$

2. (Heisenberg type) For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, we assume condition (B). Then,

$$U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr\left[X^* |L_A - R_B| Y]\right|^2.$$

In particular,

$$\begin{aligned} k\ell |Tr[X^* |L_A - R_B| X]|^2 \leq Tr\left[X^*\left(m_g(L_A,R_B) - m_{\Delta_g^f}(L_A,R_B)\right)X\right] \\ \times Tr\left[X^*\left(m_g(L_A,R_B) + m_{\Delta_g^f}(L_A,R_B)\right)X\right], \end{aligned} \tag{11}$$

where $X \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.
Proof of 1 in Theorem 1.8. By the Schwarz inequality, we have

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) = \Gamma_{A,B}^{(g,f)}(X,X) \cdot \Gamma_{A,B}^{(g,f)}(Y,Y) \geq \left|\Gamma_{A,B}^{(g,f)}(X,Y)\right|^2.$$

Now, we prove the second inequality. Since

$$\begin{aligned} I_{A,B}^{(g,f)}(X+Y) &= Tr\left[(X^* + Y^*)m_g(L_A,R_B)(X+Y)\right] \\ &\quad - Tr\left[(X^* + Y^*)m_{\Delta_g^f}(L_A,R_B)(X+Y)\right], \\ I_{A,B}^{(g,f)}(X-Y) &= Tr\left[(X^* - Y^*)m_g(L_A,R_B)(X-Y)\right] \\ &\quad - Tr\left[(X^* - Y^*)m_{\Delta_g^f}(L_A,R_B)(X-Y)\right], \end{aligned}$$

we have

$$\begin{aligned} &I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y) \\ &= 2Tr\left[X^* m_g(L_A,R_B)Y\right] + 2TrY^* m_g(L_A,R_B)X] - 2Tr\left[X^* m_{\Delta_g^f}(L_A,R_B)Y\right] \\ &-2Tr\left[Y^* m_{\Delta_g^f}(L_A,R_B)X\right] = 2\Gamma_{A,B}^{(g,f)}(X,Y) + 2\Gamma_{A,B}^{(g,f)}(Y,X) = 4\,Re\left\{\Gamma_{A,B}^{(g,f)}(X,Y)\right\}. \end{aligned}$$

Similarly, we have

$$I_{A,B}^{(g,f)}(X+Y) + I_{A,B}^{(g,f)}(X-Y) = 2\left(I_{A,B}^{(g,f)}(X) + I_{A,B}^{(g,f)}(Y)\right).$$

Then,

$$\Gamma_{A,B}^{g,f)}(X,Y) = Re\left\{\Gamma_{A,B}^{(g,f)}(X,Y)\right\} + iIm\left\{\Gamma_{A,B}^{(g,f)}(X,Y)\right\}$$

$$= \frac{1}{4}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right) + iIm\{\Gamma_{A,B}^{(g,f)}(X,Y).$$

Thus,

$$\left|\Gamma_{A,B}^{(g,f)}(X,Y)\right|^2 = \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2 + \left(Im\left\{\Gamma_{A,B}^{(g,f)}(X,Y)\right\}\right)^2$$
$$\geq \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2.$$

□

We use the following lemma to prove 2:
Lemma 3

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2.$$

Proof of Lemma 3. By conditions (A) and (B), we have

$$m_{\Delta_g^f}(x,y) = m_g(x,y) - k\frac{(x-y)^2}{m_f(x,y)},$$
$$m_g(x,y) + m_{\Delta_g f}(x,y) \geq \ell m_f(x,y).$$

We then have

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 = \left\{m_g(x,y) - m_{\Delta_g^f}(x,y)\right\}\left\{m_g(x,y) + m_{\Delta_g^f}(x,y)\right\}$$
$$\geq \frac{k(x-y)^2}{m_f(x,y)}\ell m_f(x,y) = k\ell(x-y)^2.$$

□

Proof of 2 in Theorem 1.8. Let

$$A = \sum_{i=1}^n \lambda_i |\phi_i\rangle\Big\langle \phi_i|, \quad B = \sum_{i=1}^n \mu_i |\psi_i\Big\rangle\langle\psi_i|$$

be the spectral decompositions of A and B, respectively. Then, we have

$$I_{A,B}^{(g,f)}(X) = \sum_{i,j}\left\{m_g\left(\lambda_i,\mu_j\right) - m_{\Delta_g^f}\left(\lambda_i,\mu_j\right)\right\}\left|\langle\phi_i|X\left|\psi_j\right\rangle\right|^2,$$

$$J_{A,B}^{(g,f)}(Y) = \sum_{i,j}\left\{m_g\left(\lambda_i,\mu_j\right) + m_{\Delta_g^f}\left(\lambda_i,\mu_j\right)\right\}\left|\langle\phi_i|Y\left|\psi_j\right\rangle\right|^2,$$

Since

$$|L_A - R_B| = \sum_{i=1}^n\sum_{j=1}^n |\lambda_i - \mu_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|},$$

we have

$$Tr[X^*|L_A - R_B|Y] = \sum_{i=1}^{n}\sum_{j=1}^{n}|\lambda_i - \mu_j|\overline{\left\langle \phi_i|X|\psi_j\right\rangle}\left\langle \phi_i|Y|\psi_j\right\rangle.$$

Then, by Lemma 3, we have

$$\begin{aligned} k\ell|Tr[X^*|L_A - R_B|Y]|^2 &\le \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{k\ell}|\lambda_i - \mu_j|\|\left\langle \phi_i|X|\psi_j\right\rangle\|\langle\phi_j|Y|\psi_i\rangle|\right\}^2 \\ &\le \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}(m_g\left(\lambda_i,\mu_j\right)^2 - m_{\Delta_g^f}(\lambda_i,\mu_j)^2)|\langle\phi_i|X|\psi_j\rangle\|\langle\phi_j|Y|\psi_i\rangle|\right\}^2 \\ &\le \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}(m_g\left(\lambda_i,\mu_j\right) - m_{\Delta_g^f}(\lambda_i,\mu_j))|\langle\phi_i|X\left|\phi_j\right\rangle\right|^2\right\} \\ &\quad \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}(m_g\left(\lambda_i,\mu_j\right) + m_{\Delta_g^f}(\lambda_i,\mu_j))|\langle\phi_j|Y|\psi_i\rangle|^2\right\} \\ &= I_{A,B}^{(g,f)}(X)\cdot J_{A,B}^{(g,f)}(Y). \end{aligned}$$

Similarly, we have $k\ell|Tr[X^*|L_A - R_B|Y]|^2 \le I_{A,B}^{(g,f)}(Y)\cdot J_{A,B}^{(g,f)}(X)$. Therefore,

$$U_{A,B}^{(g,f)}(X)\cdot U_{A,B}^{(g,f)}(Y) \ge k\ell|Tr\left[X^*|L_A - R_B|Y\right]|^2. \qquad \square$$

When $A = B = \rho \in M_{n,+,1}(\mathbb{C}), X = A \in M_n(\mathbb{C}))$, and $Y = B \in M_n(\mathbb{C})$, we obtain the result in Theorem 1.7.

We assume that

$$g(x) = \frac{x+1}{2},\ f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)},\ k = \frac{f(0)}{2},\ \ell = 2.$$

We then obtain the following trace inequality by substituting $X = I$ in (11).

$$\alpha(1-\alpha)(Tr[|L_A - R_B|I])^2 \le \left(\frac{1}{2}Tr[A+B]\right)^2 - \left(\frac{1}{2}Tr\left[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha\right]\right)^2. \quad (12)$$

This is a generalization of the trace inequality provided in [13]. In addition, we produce the following new inequality by combining a Chernoff-type inequality with Theorem 1.8.

Theorem 1.9 ([14]). We have the following:

$$\begin{aligned} &\frac{1}{2}Tr[A+B-|L_A - R_B|I] \le \inf_{0\le\alpha\le 1} Tr\left[A^{1-\alpha}B^\alpha\right] \le Tr\left[A^{1/2}B^{1/2}\right] \\ &\le \frac{1}{2}Tr\left[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha\right] \le \sqrt{\left(\frac{1}{2}Tr[A+B]\right)^2 - \alpha(1-\alpha)(Tr[|L_A - R_B|I)^2}. \end{aligned}$$

The following lemma is necessary to prove Theorem 1.9.

Lemma 4. Let $f(s) = Tr\left[A^{1-s}B^s\right]$ for $A, B \in M_n(\mathbb{C})$ and $0 \le s \le 1$. Then $f(s)$ is convex in s.

Proof of Lemma 4. $f'(s) = Tr\left[-A^{1-s}\log AB^s + A^{1-s}B^s\log B\right]$. And then

$$\begin{aligned} f''(s) &= Tr\left[A^{1-s}(\log A)^2 B^s - A^{1-s}\log AB^s \log B\right] - Tr\left[A^{1s}\log AB^s \log B - A^{1-s}B^s(\log B)^2\right] \\ &= Tr\left[A^{1-s}(\log A)^2 B^s\right] - Tr\left[A^{1-s}\log A \log BB^s\right] - Tr\left[\log B \log AA^{1-s}B^s\right] + Tr\left[A^{1-s}(\log B)^2 B^s\right] \\ &= Tr\left[A^{1-s}\log A(\log A - \log B)B^s\right] - Tr\left[A^{1-s}(\log A - \log B)\log BB^s\right] \\ &= Tr\left[A^{1-s}(\log A - \log B)B^s \log A\right] - Tr\left[A^{1-s}(\log A - \log B)\log BB^s\right] \\ &= Tr\left[A^{1-s}(\log A - \log B)B^s(\log A - \log B)\right] \\ &= Tr\left[A^{(1-s)/2}(\log A - \log B)B^s(\log A - \log B)A^{(1-s)/2}\right] \geq 0. \end{aligned}$$

$f(s)$ is convex in s. □

Proof of Theorem 1.9. The third and fourth inequalities follow from Lemma 4 and (12), respectively. Thus, we only prove the following inequality:

$$Tr[A + B - |L_A - R_B|I] \leq 2Tr\left[A^{1-\alpha}B^{\alpha}\right] \quad (0 \leq \alpha \leq 1).$$

Let

$$A = \sum_i \lambda_i |\phi_i\rangle\left\langle \phi_i| = \sum_{i,j} \lambda_i |\phi_i\right\rangle\left\langle \phi_i|\psi_j\right\rangle\langle\psi_j|,$$

$$B = \sum_j \mu_j |\psi_j\rangle\left\langle \psi_j| = \sum_{i,j} \mu_j |\phi_i\right\rangle\left\langle \phi_i|\psi_j\right\rangle\langle\psi_j|.$$

Then, we have

$$Tr[A] = \sum_{i,j} \lambda_i |\langle\phi_i|\psi_j\rangle|^2, \quad Tr[B] = \sum_{i,j} \mu_j |\langle\phi_i|\psi_j\rangle|^2.$$

And since

$$|L_A - R_B| = \sum_{i,j} |\lambda_i - \mu_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|},$$

we have

$$|L_A - R_B|I = \sum_{i,j} |\lambda_i - \mu_j| |\phi_i\rangle\left\langle \phi_i|\psi_j\right\rangle\langle\psi_j|.$$

Then, we have

$$Tr[|L_A - R_B|I] = \sum_{i,j} |\lambda_i - \mu_j| |\langle\phi_i|\psi_j\rangle|^2.$$

Therefore,

$$Tr[A + B - |L_A - R_B|I] = \sum_{i,j} \left(\lambda_i + \mu_j - |\lambda_i - \mu_j|\right) |\langle\phi_i|\psi_j\rangle|^2.$$

However, since we have

$$A^{\alpha} = \sum_i \lambda_i^{\alpha} |\phi_i\rangle\Big\langle \phi_i| = \sum_{i,j} \lambda_i^{\alpha} |\phi_i\Big\rangle\Big\langle \phi_i|\psi_j\Big\rangle\langle\psi_j|,$$

$$B^{1-\alpha} = \sum_j \mu_j^{1-\alpha} |\psi_j\rangle\Big\langle \psi_j| = \sum_{i,j} \mu_j^{1-\alpha} |\phi_i\Big\rangle\Big\langle \phi_i|\psi_j\Big\rangle\langle\psi_j|,$$

$$A^{\alpha}B^{1-\alpha} = \sum_{i,j} \lambda_i^{\alpha}\mu_j^{1-\alpha} |\phi_i\rangle\Big\langle \phi_i|\psi_j\Big\rangle\langle\psi_j|.$$

Then,

$$Tr\left[A^{\alpha}B^{1-\alpha}\right] = \sum_{i,j} \lambda_i^{\alpha}\mu_j^{1-\alpha} \left|\langle\phi_i\Big|\psi_j\rangle\right|^2.$$

Thus,

$$2Tr\left[A^{\alpha}B^{1-\alpha}\right] - Tr[A+B-|L_A - R_B|I] = \sum_{i,j}\left\{2\lambda_i^{\alpha}\mu_j^{1-\alpha} - \left(\lambda_i + \mu_j - |\lambda_i - \mu_j|\right)\right\} \left|\langle\phi_i\Big|\psi_j\rangle\right|^2.$$

Since $2x^{\alpha}y^{1-\alpha} - (x+y-|x-y|) \geq 0$ for $x, y > 0$, $0 \leq \alpha \leq 1$ in general, we can obtain Theorem 1.9. □

Remark 2. We note the following 1, 2:

1. $\frac{1}{2}Tr[A+B-|A-B|] \leq \inf_{0\leq\alpha\leq 1} Tr\left[A^{1-\alpha}B^{\alpha}\right] \leq Tr\left[A^{1/2}B^{1/2}\right]$

$$\leq \sqrt{\left(\frac{1}{2}Tr[A+B]\right)^2 - \frac{1}{4}(Tr[|A-B|])^2}.$$

2. There is no relationship between $Tr[|L_A - R_B|I]$ and $Tr[|A-B|]$. When

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $Tr[|L_A - R_B|I] = 3, \quad Tr[|A-B|] = \sqrt{10}$. When

$$A = \begin{pmatrix} \frac{13}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{13}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix},$$

we have $Tr[|L_A - R_B|I] = 8, \quad Tr[|A-B|] = \sqrt{58}$.

7. Sum type of uncertainty relations

Let $A, B \in M_{n,sa}(\mathbb{C})$ have the following spectral decompositions:

$$A = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\Big\langle \phi_i|, \quad B = \sum_{i=1}^{n} \mu_i |\psi_i\Big\rangle\langle\psi_i|.$$

For any quantum state $|\phi\rangle$, we define the two probability distributions

$$P = (p_1, p_2, \cdots, p_n), \quad Q = (q_i, q_2, \cdots, q_n),$$

where $p_i = |\langle \phi_i | \phi \rangle|^2$, $q_j = |\langle \psi_j | \phi \rangle|^2$. Let

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \quad H(Q) = -\sum_{j=1}^{n} q_j \log q_j$$

be the Shannon entropies of P and Q, respectively.
Theorem 1.10. The following uncertainty relation holds:

$$H(P) + H(Q) \geq -2\log c,$$

where $c = \max_{i,j} |\left\langle \phi_i | \psi_j \right\rangle|$.
For details, see [15, 16].
Definition 6. The Fourier transformation of $\psi \in L^2(\mathbb{R})$ is defined as

$$\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-2\pi i \omega t} dt.$$

We also define

$$Q(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt < \infty \right\}.$$

Proposition 2. If $\psi \in L^2(\mathbb{R})$, $\|\psi\|^2 = 1$ satisfies $\psi, \hat{\psi} \in Q(\mathbb{R})$, then

$$S(\psi) + S(\hat{\psi}) \geq \log \frac{e}{2},$$

where

$$S(\psi) = -\int_{-\infty}^{\infty} |\psi(t)|^2 \log |\psi(t)|^2 dt, \quad S(\hat{\psi}) = -\int_{-\infty}^{\infty} |\hat{\psi}(t)|^2 \log |\hat{\psi}(t)|^2 dt.$$

For details, see [17].
Theorem 1.11 ([18]). For any $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, the following holds:

1. $I_{A,B}^{(g,f)}(X, Y) + I_{A,B}^{(g,f)}(Y) \geq \frac{1}{2} \max \left\{ I_{A,B}^{(g,f)}(X + Y), I_{A,B}^{(g,f)}(X - Y) \right\}$.

2. $\sqrt{I_{A,B}^{(g,f)}(X)} + \sqrt{I_{A,B}^{(g,f)}(Y)} \geq \max \left\{ \sqrt{I_{A,B}^{(g,f)}(X + Y)}, \sqrt{I_{A,B}^{(g,f)}(X - Y)} \right\}$.

3. $\sqrt{I_{A,B}^{(g,f)}(X)} + \sqrt{I_{A,B}^{(g,f)}(Y)} \leq 2 \max \left\{ \sqrt{I_{A,B}^{(g,f)}(X + Y)}, \sqrt{I_{A.B}^{(g,f)}(X - Y)} \right\}$.

Proof 1. The Hilbert-Schmidt norm $\| \cdot \|$ satisfies

$$\|X\|^2 + \|Y\|^2 = \frac{1}{2} \left(\|X + Y\|^2 + \|X - Y\|^2 \right) \geq \frac{1}{2} \max \left\{ \|X + Y\|^2, \|X - Y\|^2 \right\}. \quad (13)$$

Since $I_{A,B}^{(g,f)}(X,X)$ is the second power of the Hilbert-Schmidt norm, $\|X\| = \sqrt{I_{A,B}^{(g,f)}(X)}$. We then obtain the result by substituting (13),

2. By the triangle inequality of a general norm, we apply the triangle inequality for $\|X\| = \sqrt{I_{A,B}^{(g,f)}(X)}$.

3. We prove the following norm inequality:

$$\|X\| + \|Y\| \leq \|X+Y\| + \|X-Y\|. \tag{14}$$

Since

$$\|X\| = \|\frac{1}{2}(X+Y) + \frac{1}{2}(X-Y)\| \leq \frac{1}{2}\|X+Y\| + \frac{1}{2}\|X-Y\|$$

and

$$\|Y\| = \|\frac{1}{2}(Y+X) + \frac{1}{2}(Y-X)\| \leq \frac{1}{2}\|Y+X\| + \frac{1}{2}\|Y-X\|,$$

we add two inequalities and obtain (14). □

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References

[1] Lieb EH. Convex trace functions and the Wigner-Yanase-Dyson conjecture. Advances in Mathematics. 1973;**11**: 267-288

[2] Luo S, Zhang Q. On skew information. IEEE Transactions of Information Theory. 2004;**50**:1778-1782, and Correction to "On skew information", IEEE Transactions of Information Theory. 2005;**51**:4432

[3] Kosaki H. Matrix trace inequality related to uncertainty principle. International Journal of Mathematics. 2005;**16**:629-646

[4] Yanagi K, Furuichi S, Kuriyama K. A generalized skew information and uncertainty relations. IEEE Transactions of Information Theory. 2005;**IT-51**(12): 4401-4404

[5] Hansen F. Metric adjusted skew information. Proceedings of the National Academy of Sciences of the United States of America. 2008;**105**: 9909-9916

[6] Yanagi K. Metric adjusted skew information and uncertainty relation. Journal of Mathematical Analysis and Applications. 2011;**380**(2):888-892

[7] Yanagi K, Furuichi S, Kuriyama K. Uncertainty relations for generalized metric adjusted skew information and generalized metric adjusted correlation measure. Journal of Uncertainty Analysis and Applications. 2013;**1**(12): 1-14

[8] Luo S. Heisenberg uncertainty relation for mixed states. Physical Review A. 2005;**72**:042110

[9] Yanagi K. Uncertainty relation on Wigner-Yanase-Dyson skew information. Journal of Mathematical Analysis and Applications. 2010;**365**: 12-18

[10] Gibilisco P, Hansen F, Isola T. On a correspondence between regular and non-regular operator monotone functions. Linear Algebra and its Applications. 2009;**430**:2225-2232

[11] Kubo F, Ando T. Means of positive linear operators. Mathematische Annalen. 1980;**246**:205-224

[12] Yanagi K. On the trace inequalities related to left-right multiplication operators and their applications. Linear and Nonlinear Analysis. 2018;**4**(3): 361-370

[13] Audenaert KMR, Calsamiglia J, Masancs LI, Munnoz-Tapia R, Acin A, Bagan E, et al. The quantum Chernoff bound. Review Letters. 2007;**98**: 160501-1-160501-4

[14] Yanagi K. Generalized trace inequalities related to fidelity and trace distance. Linear and Nonlinear Analysis. 2016;**2**(2):263-270

[15] Maassen H, Uffink JBM. Generalized entropic uncertainty relations. Physical Review Letters. 1988; **60**(12):1103-1106

[16] Nielsen MA, Chuang IL. Quantum Computation and Quantum Information. Cambridge University Press; 2000

[17] Hirschman II Jr. A note on entropy. American Journal of Mathematics. 1957; **79**:152-156

[18] Yanagi K. Sum types of uncertainty relations for generalized quasi-metric adjusted skew informations. International Journal of Mathematical Analysis and Applications. 2018;**4**(4): 85-94