Chapter

Boundary Integral Equations of no Stationary Boundary Value Problems for the Klein-Gordon Equation

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Abstract

The non-stationary boundary value problems for Klein-Gordon equation with Dirichlet or Neumann conditions on the boundary of the domain of definition are considered; a uniqueness of boundary value problems is proved. Based on the generalized functions method, boundary integral equations method is developed to solve the posed problems in strengths of shock waves. Dynamic analogs of Green's formulas for solutions in the space of generalized functions are obtained and their regular integral representations are constructed in 2D and 3D over space cases. The singular boundary integral equations are obtained which resolve these tasks.

Keywords: boundary value problem, Klein-Gordon equation, generalized functions method, boundary integral equations, shock waves

1. Introduction

Klein-Gordon equation is hyperbolic differential equation in partial derivatives of second order. As is known, a class of solutions of hyperbolic equations contains no differentiable functions that have discontinuous derivatives on characteristic surfaces. By these cause, the construction of solutions for such equations, using smoothness of their differentiability, is impossible. However, this type of solution describes the shock waves, a presence of which is typical for physical processes.

Fundamental solutions of hyperbolic equations are singular generalized functions, the features of which are concentrated on moving surfaces—wave fronts, which propagate at a certain speed (it is called *light* or *sound speed*). This sharply distinguishes them from the fundamental solutions of elliptic and parabolic equations that are singular at a point. Therefore, classical methods of constructing boundary integral equations for solving nonstationary boundary value problems (BVP) based on the methods of potential theory or Green's formulas are unsuitable. To solve nonstationary boundary value problems for hyperbolic equations based on these methods, the Laplace or Fourier transform is usually used, which leads hyperbolic equations to parameterized elliptic type equations in the spaces of transforms. They are solved on the basis of the direct or indirect method of boundary integral equations (BIE). To restore original solutions, various numerical procedures of the inverse Fourier transform, Laplace, and others are used. The class of such problems of mathematical physics began to be considered as early as the 1970s of the last century, which was connected with the advent of computing technology, but so far, the number of works in this direction is very limited. As is known, the inverse transformation procedures are unstable, which are typical for solutions of the Fredholm equations of the first type and require regularization of the corresponding equations to obtain reliable results. Therefore, the problem of developing constructive methods for solving initial-boundary value problems for hyperbolic equations and systems of equations of mathematical physics for studying of various wave processes in bodies and in continuous medium remains relevant to the present time.

An effective method for solving boundary value problems for hyperbolic equations and systems of mixed type is the method of generalized functions (MGF), which allows to move from solving boundary value problems to solving the corresponding differential equations in the space of generalized functions and build integral representations of generalized solutions of initial boundary value problems in the original space-time to investigate the processes accompanied by shock waves.

To solve the Cauchy problem for hyperbolic equations, this method was proposed by Vladimirov [1]. In Refs. [2–6], the MGF was developed to solve nonstationary and stationary boundary value problems of the theory of elasticity, thermoelasticity, and electrodynamics and initial-boundary value problems for hyperbolic equations systems which are typical to the mathematical physics [7].

In the present chapter, this method is used to solve initial-boundary value problems for the Klein-Gordon equation (KG-Eq), a hyperbolic equation of the theory of elementary particles of quantum mechanics [8]. Here, nonstationary boundary value problems for KG-Eq with Dirichlet or Neumann conditions on the boundary of the domain of definition are considered and the uniqueness of the stated boundary value problems taking into account of shock waves is proved. Based on the method of generalized functions, the boundary integral equations method (BIEM) was developed to solve the stated tasks. Dynamic analogs of Green's formulas for their solutions in the space of generalized functions are obtained and regular integral representations in the plane and three-dimensional cases are constructed. Solving singular boundary integral equations are obtained for solving the stated initial-boundary value problems.

2. Klein-Gordon equation: shock waves

Klein-Gordon equation is formulated as:

$$\Box_c u + q(x)u = f(x,t). \tag{1}$$

Here, we denote the wave operator,

$$\Box_c = \Delta - c^{-2}\frac{\partial^2}{\partial t^2},$$

where $\Delta = \sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}$ is Laplace operator, $x \in R^N$, $t \in [0,\infty)$, and scattering potential $q(x) \in L_1(R^N)$. It is a hyperbolic equation. Its characteristic equation has the next form:

$$\nu_t^2 - c^2 \sum_{j=1}^{N} \nu_j^2 = 0, \tag{2}$$

where $\nu(x,t) = (\nu_1, \ldots, \nu_N, \nu_t)$ is the normal vector to the characteristic surface F in R^{N+1}, $x \in R^N$, $(x,t) \in R^{N+1}$. It corresponds to the cone of characteristic normals—the light cone at $\nu_t < 0$. The solution of equation of $F(1)$ and its derivatives can be discontinuous. In R^N, characteristic surface F corresponds to wave front F_t (section of F at fixed t), which moves with speed c:

$$c = -\nu_t / \|\nu\|_N, \quad \|\nu\|_N = \sqrt{\nu_j \nu_j} \tag{3}$$

(here and further throughout in order to reduce the record on repeated indexes in the product, the summation from 1 to N is carried out, which is similar to tensor convolution). Such solutions (1) are called *shock waves*.

If solution of (1) is continuous:

$$[u(x,t)]_{F_t} = 0, \tag{4}$$

then wave fronts of Hadamard's conditions of continuity are satisfied under jumps:

$$\left[\dot{u} n_j + c u_{,j}\right]_{F_t} = 0, \quad j = \overline{1, N}; \tag{5}$$

$$\left[\dot{u} + c n_j u_{,j}\right]_{F_t} = 0, \tag{6}$$

for $x \in F_t$, where $n(x,t)$ is wave vector. It is a unit vector, normal to wave front F_t and directed forward its propagation. It is obvious that,

$$n_i = \nu_i / \|\nu\|_N, \quad i = \overline{1, N}; \tag{7}$$

Hereinafter, for abbreviation of the record, a symbol after comma defines the corresponding partial derivative: $u_{,j} = \frac{\partial u}{\partial x_j}$. The condition (5) is a consequence of the continuity condition (4) and ensures continuity on the F_t of tangent derivatives of u. The condition (5) is the law of conservation of momentum at the shock wave fronts. If before the wave front $u \equiv 0$, then at the wave front:

$$(grad\, u, n) = -c^{-1} \dot{u}, \quad x \in F_t.$$

To study the solutions of the KG-Eq, it is convenient to use the apparatus of the theory of generalized functions, which allows one to investigate shock waves, as well as singular solutions from the class of generalized functions typical for mathematical physics problems.

For this purpose, let us consider KG-Eq (1) on the space of generalized functions $D'(R^{N+1})$, which is the space of linear continuous functionals on the space of basic functions $D(R^{N+1})$, which are finite infinitely differentiable functions [1]. Further, the usual locally integrable function (*regular*) $f(x,t)$ will be marked from the top $\hat{f}(x,t)$, by considering it as a generalized.

If u (x,t) is regular differentiable and has a finite discontinuity on F, then in $D'(R^{N+1})$, as is known [1], its partial derivative is equal to the following:

$$\hat{u}_{,j} = u_{,j} + [u]_F \nu_j \delta_F, \tag{8}$$

where the first term on the right is the classical derivative of x_j, $\delta_F(x,t)$ is a simple layer on F, a singular generalized function [1], $\|\boldsymbol{\nu}\| = 1$. Using (8), it is possible to determine the second derivatives sequentially.

Definition. A solution $u(x,\ t)$ of Eq (1), which is continuous together with derivatives up to the second order almost everywhere, with the exception of a finite or countable number of discontinuity surfaces (wave fronts), on which the conditions (5) and (6) for jumps are satisfied, is called as classical solution.

Lemma 1. *If $u(x,t)$ is classic solution of* (1), *then $\hat{u}(x,t)$ is generalized solution of it.*

Proof. Taking into account these equations and (3), we get

$$(\Box_c + q(x))\hat{u} = f(x,t) + \left\{c^{-1}[u,_t]_{F_t} + \left[n_j u,_j\right]_{F_t}\right\}\|\nu\|_N \delta_F + c^{-1}\partial_t\left\{\|\nu\|_N [u]_{F_t}\delta_F)\right\} + \partial_j\left\{\|\nu\|_N [u]_{F_t} n_j \delta_F\right\}$$

By virtue of (4) and (6), the densities of simple and double layers here are equal to zero on the right, which were required to be proved.

From this lemma, it follows that the conditions on the fronts of shock waves are easy to obtain, considering the classical solutions of hyperbolic equations as generalized. It is enough to equate to zero the density of the corresponding independent singular generalized functions—analogs of simple, double, and other layers arising from the generalized differentiation of solutions. The determination of such conditions on the basis of classical methods is a very time-consuming procedure.

Let us put the energy density of the u-field E and the Lagrange L function:

$$E = 0.5\left(\dot{u}^2 + c^2\sum_{j=1}^{N} u,_j^2\right),$$

$$L = 0.5\left(\dot{u}^2 - c^2\sum_{j=1}^{N} u,_j^2\right).$$

Lemma 2. *If the classical solution of the KG-Eq.* (1), *then the following conditions for energy density jumps and Lagrange functions are satisfied at the shock wave fronts:*

$$[E]_{F_t} = -c\left[\dot{u}\,\frac{\partial u}{\partial n}\right]_{F_t},$$

$$[L(x,t)]_{F_t} = 0.$$

Proof: It is easy to show that for jumps, the equation is fulfilled:

$$[ab] = a^+[b] + b^-[a],$$

where the plus and minus signs indicate the limiting values of the functions a and b on the wave front from the side of the wave vector and opposite. Using this equality and the Hadamard's conditions (4) and (5), we get

$$\left[E + c\dot{u}\,\frac{\partial u}{\partial n}\right] = c^2\left[0.5\left(c^{-1}\dot{u}^2 + cu,_j u,_j\right) + \dot{u}\,\frac{\partial u}{\partial n}\right] = \ldots =$$

$$= 0.5c[\dot{u}]\left(\dot{u}^- + cu,_j^- n_j\right) + 0.5c^2\left[u,_j\right]\left(cu,_j^- + \dot{u}^- n_j\right) =$$

$$= 0.5c^3 u,_j^-\left[u,_j + c^{-1}n_j\dot{u}\right] + 0.5c\dot{u}^-\left[cn_j u,_j + \dot{u}\right] = 0$$

Here, n is the normal to the shock wave front in R^N. This implies the first formula of the lemma.

The condition for a jump in the energy density at the shock wave front can be obtained more easily by considering the corresponding energy equation in $D'(R^{N+1})$, which we get by multiplying the Eq. (1) by $\dot{u}$. After simple transformations in the field of differentiability of solutions, we have the equation:

$$c^{-2}E_{,t} - (\dot{u}\, u_{,j})_{,j} + \dot{u}f = 0 \tag{9}$$

For shock waves in $D'(R^{N+1})$, it has the form:

$$\begin{aligned} c^{-2}\dot{E} - (\dot{u}\, u_{,j})_{,j} + \dot{u}f &= \{c^{-2}[E]\nu_t - [\dot{u}\, u_{,j}]\nu_j\}\delta_F = \\ &= -\|\nu\|_N\{c^{-1}[E] + [\dot{u}\, u_{,j}]n_j\}\delta_{F_t} \end{aligned}$$

For the right side to be turned to zero, it is necessary

$$[E] + c[\dot{u}\, u_{,j}]n_j = 0.$$

The latter coincides with the formula of Lemma 2.

Similarly, we can get the equation for L by multiplying Eq. (1) by u:

$$L + c^{-2}(u\dot{u})_{,t} - (uu_{,j})_{,j} + qu^2 - uf = 0 \tag{10}$$

Taking into account (3) and (4), for shock waves, we get

$$-[L] = [c^{-2}(u\dot{u})_{,t} - (uu_{,j})_{,j}] = u\{c^{-2}[\dot{u}]\nu_t - [u_{,j}]\nu_j\} = 0$$

It means that the Lagrange function is continuous at the shock wave fronts.

3. Statement of non-stationary BVP for Klein-Gordon equation: Energy conservation law

Let us construct the solution $u(x, t)$ of Eq. (1) on a set $S^- \in R^N$, bounded by surface S, by $t \geq 0$. Lets introduce next marks: $n(x)$ is vector of external normal to S; $D = \{S \times R^+\}$ is lateral surface of space-time cylinder $D^- = S^- \times R^+$, $R^+ = (0, +\infty)$; and the derivative of u on normal n at i, $\frac{\partial u}{\partial n} = u,_j n_j$.

Initial conditions: At $t = 0$ for $x \in S^-$:

$$u(x, 0) = u_0(x) \text{ for } x \in S^- + S \tag{11}$$

$$\dot{u}(x, 0) = v_0(x) \text{ for } x \in S^- \tag{12}$$

We consider two *boundary value problems* corresponding to the Dirichlet and Neumann conditions:

$$(\text{BVP I}) \quad u(x,t) = u_S(x,t) \text{ for } x \in S \tag{13}$$

$$(\text{BVP II}) \quad \frac{\partial u}{\partial n} = p(x,t) \text{ for } x \in S \tag{14}$$

At the shock wave fronts, the Hadamard conditions (5) and (6) on jumps are satisfied. Note that shock waves always occur if the condition of matching the initial and boundary data on the velocities is not satisfied

$$\dot{u}_S(x,0) = v_0(x), \quad x \in S, \tag{15}$$

which is typical for physical tasks. In this case, at the initial moment of time, a shock front is formed at the boundary S, which propagates with a velocity c in S^-. To construct continuously differentiable solutions, this condition is necessary. Here we will not enter it. Here, we not enter it and suppose that $u_0(x) \in C(S^- + S)$, $v_0(x) \in L_1(S^- + S)$, $p(x,t) \in L_1(D)$, and $u_S(x,t)$ a Holder's function on S: $\forall \beta$, $0 < \beta \le 1$, such that for $\forall x \in S, \ \ y \in S, \ \ t \ge 0$

$$|u_S(x,t) - u_S(y,t)| \le const \|x - y\|^\beta, \tag{16}$$

here $L_1(\ldots)$ is the Lebeg's space of summable on the specified set of functions.

Let us mark as $D = \{S \times R^+\}$, the lateral surface of the space-time cylinder is $D^- = S^- \times R^+$, $R^+ = (0, +\infty)$.

Theorem 1. (*Energy conservation law*). *If $u(x,t)$ is classic solution of edge problem, then*

$$\int_{S^-} (E(x,t) - E_0(x)) \ dV(x) + 0.5c^2 \int_{S^-} q(x)\left(u^2(x,t) - u_0^2(x)\right) \ dV(x)$$
$$= c^2 \int_0^t dt \int_S (\dot{u}_S(x,t) p(x,t)) dS(x) - c^2 \int_0^t dt \int_{S^-} f(x,t) \dot{u}(x,t) \ dV(x)$$

Proof. We integrate the energy Eq. (9) over a field with allowance for the partition of the field of integration by F_k wave fronts. Note that the first two terms can be considered as the divergence of the corresponding vector in space R^{N+1}, which is continuous in the regions between the fronts. Therefore, using the Ostrogradsky-Gauss theorem in R^{N+1}, we get

$$\int_{D^-} \left(c^{-2}E + \frac{1}{2} q(x) u^2\right)_{,t} dV(x,t) - \int_{D^-} (\dot{u} \, u_{,j})_{,j} dV(x,t)$$
$$+ \int_{D^-} \dot{u} f(x,t) dV(x,t) = \int_{D^-} \dot{u} f(x,t) dV(x,t) +$$
$$+ \int_{S^-} \left\{ c^{-2}(E(x,t) - E_0(x)) + \frac{1}{2} q(x) \left(u^2(x,t) - u_0^2(x)\right) \right\} dV(x) -$$
$$- \int_0^t \int_S \left(\dot{u} \frac{\partial u}{\partial n}\right) dS(x) dt + \sum_{F_k} \int_{F_k} \left[c^{-2} E \nu_t - \frac{\partial u}{\partial \nu} \dot{u} \right]_{F_k} dF_k(x,t) = 0$$

Hereinafter, we denote $dV(x) = dx_1 \ldots dx_N$, $\ dV(x,t) = dV(x) dt$; $dF_k(x,t)$ is the differential of the surface area at the corresponding point of the wave front. By virtue of (3) and Lemma 2,

$$\left[c^{-2} E \nu_t - \dot{u} u_{,j} \nu_j\right]_{F_k} = -\|\nu\|_N c^{-1} \left[E + c \dot{u} \frac{\partial u}{\partial n}\right] = 0.$$

Therefore the last integral is zero. Taking into account the notation for the boundary functions, we get the formula of the theorem. From this theorem follows the Theorem 2.

Theorem 2. If $q(x) \geq 0$, *then the classic solution of first (second) BVP for Klein-Gordon equation is unique.*

Proof. Due to the linearity of the problem, it suffices to prove the uniqueness of the zero solution. For him, f = 0, the initial conditions and the corresponding boundary conditions are also zero. Then from Theorem 1, it follows:

$$\int_{S^-} \left\{ E(x,t) + 0,5c^2 q(x) u^2(x,t) \right\} \ dV(x) = 0.$$

Since both terms are nonnegative, therefore, $E = 0$ and $u = 0$. The theorem is proved.

4. The dynamic analogue of Green's formula with constant scattering potential

Consider the case when scattering potential is constant:

$$q(x) = \pm m^2$$

To build the solution of BVP, we move to the space of generalized functions. To do this, we introduce the characteristic function of the solution domain

$$H_D^-(x,t) \equiv H_S^-(x)H(t),$$

where $H_S^-(x)$ is a characteristic function of set S^-, which is equal to 0.5 on its boundary S; and $H(t)$ is Heaviside's function, which is equal to 0.5 at t = 0. H_D^- is a characteristic function of space-time cylinder D^-. It is easy to show that:

$$\frac{\partial H_D^-}{\partial x_j} = -n_j \delta_S(x) H(t), \quad \frac{\partial H_D^-}{\partial t} = -n_j H_S^-(x)\delta(t), \tag{17}$$

where $\delta(t)$ is singular Dirac's function.

To use the methods of the theory of generalized functions, we define the solution by zero outside the domain of the solution of the boundary value problem. For this, we put regular generalized functions:

$$\hat{u} = u(x,t) H_D^-(x,t), \quad \hat{f} = f(x,t) H_D^-(x,t), \tag{18}$$

where $u(x,t)$ is the classical solution of the BVP.

Consider the action of the KG-operator on $\hat{u}$. Since $[u]_S = -u,$ and performing generalized differentiation using (17), we get

$$\begin{aligned} \Box_c \hat{u} \pm m^2 \hat{u} = & -\frac{\partial u}{\partial n} \delta_S(x) H(t) - H(t) \left(u n_j \delta_S(x) \right)_{,j} - c^{-2} H_S^-(x) u_0(x) \dot{\delta}(t) \\ & - c^{-2} H_S^-(x) \dot{u}_0(x) \delta(t) + \hat{f}(x,t), \end{aligned} \tag{19}$$

where $\delta_S(x)H(t)$ is simple layer on lateral surface of a space-time cylinder $D = \{S \times R^+\}$.

Note that the densities of simple and double layers here are determined by the boundary conditions, some of which (depending on the boundary value problem) are known, and the given initial conditions.

The solution of Eq. (19) is convolution of the right part of the equation with its fundamental solution $\hat{U}(x,t)$, satisfying the conditions:

$$\square_c \hat{U} \pm m^2 \hat{U} = \delta(x)\delta(t), \tag{20}$$

and *radiation conditions*:

$$\hat{U}(x,t) = 0 \ \text{ at } t < 0, \hat{U}(x,t) = 0 \ \text{ at } \|x\| > ct. \tag{21}$$

Let us call it the *Green's function* of Eq. (1).

The solution of (19) will be obtained in the form of the following convolution of right part of (19) and Green's function, which is equal to

$$\begin{aligned} \hat{u} = u(x,t)H_S^-(x)H(t) = -\hat{U} * \frac{\partial u}{\partial n}\delta_S(x)H(t) - \left(\hat{U} * un_j\delta_S(x)H(t)\right)_{,j} - c^{-2}\left(\hat{U} \underset{x}{*} H_S^-(x)u_0(x)\right)_{,t} \\ - c^{-2}\hat{U} \underset{x}{*} H_S^-(x)\dot{u}_0(x) + \hat{f}(x,t) * \hat{U} \end{aligned} \tag{22}$$

Here the symbol "*" means that convolution is taken only by x. Moreover, the solution is unique in the class of functions that allows convolution with U. Hence, it is easy to obtain a solution to the Cauchy problem (in the absence of S, $S^- = R^N$).

Consequence 1. The generalized solution of the Cauchy problem has the form:

$$\hat{u}(x,t) = -c^{-2}\hat{U} \underset{x}{*} \dot{u}_0 - c^{-2}\left(\hat{U} \underset{x}{*} u_0\right)_{,t} + \hat{f} * \hat{U} \tag{23}$$

Consequence 2. At zero initial data and $f = 0$, the generalized solution has the form:

$$\hat{u} = -\hat{U} * \frac{\partial u}{\partial n}\delta_S(x)H(t) - \hat{U}_{,j} * un_j\delta_S(x)H(t) \tag{24}$$

Formulas (22) and (24) express the solution of boundary value problems through the boundary values of the unknown function and its derivative along the normal to the boundary, i.e., they are similar to the Green formula for solutions of elliptic equations [9]. However, due to the singularities of the fundamental solutions of hyperbolic equations on the wave front, the form of which depends on the dimension of space, their integral representation gives divergent integrals containing derivatives of the fundamental solution. To construct regular integral representations, we introduce an antiderivative function:

$$\hat{W} = \hat{U} * \delta(x)H(t) = \hat{U} \underset{t}{*} H(t) \Rightarrow \partial_t \hat{W} = \hat{U} \tag{25}$$

and

$$\hat{H}(x,n,t) = \frac{\partial \hat{W}}{\partial x_j} n_j = \frac{\partial \hat{W}}{\partial n} \tag{26}$$

It is easy to see that $\hat{W}и\hat{H}$ are also solutions (1) at $\hat{f}(x,t) = H(t)\delta(x)$ and $\hat{f}(x,t) = H(t)\frac{\partial\delta(x,t)}{\partial n}$, respectively. The following theorem is true.

Theorem 3. *The generalized solution of boundary value problems has the form: (dynamic analogue of Green's formula)*

$$
\begin{aligned}
\hat{u} = & -\hat{U} * \frac{\partial u}{\partial n}\delta_S(x)H(t) - \hat{W},_j * \dot{u}n_j(x)\delta_S(x)H(t) - \\
& -\hat{W},_j \underset{x}{*} u_0(x)n_j(x)\delta_S(x) - c^{-2}\hat{U} \underset{x}{*} H_S^-(x)\dot{u}_0(x) - \\
& -c^{-2}\left(\hat{U} \underset{x}{*} H_S^-(x)u_0(x)\right),_t + \hat{f} * \hat{U}
\end{aligned}
\tag{27}
$$

Proof. Let us consider the formula (22). It is easy to show, using the definition of the derivative of a generalized function and the continuity of u, that

$$
\left(un_j\delta_S(x)H(t)\right),_t = \dot{u}(x,t)n_j(x)\delta_S(x)H(t) + u(x,0)n_j(x)\delta_S(x)\delta(t)
$$

Using this equality and the convolution differentiation property [1], we have

$$
\begin{aligned}
\left(\hat{U} * un_j\delta_S(x)H(t)\right),_j &= \left(\hat{W},_t * un_j\delta_S(x)H(t)\right),_j \\
&= \hat{W},_j * \dot{u}(x,t)n_j(x)\delta_S(x)H(t) + \hat{W},_j * u(x,0)n_j(x)\delta_S(x)\delta(t)
\end{aligned}
$$

Since

$$
\hat{W},_j * u(x,0)n_j(x)\delta_S(x)\delta(t) = \hat{W},_j \underset{x}{*} u_0(x)n_j(x)\delta_S(x),
$$

putting these ratios in (22), we obtain the formula of the theorem.

From Theorem 3, it is consequent that the solution of the problem is entirely defined by initial data, boundary means of normal derivative of function $u(x,t)$, and its speed $\dot{u} = u,_t = \partial_t u$. By analog with representation of Laplace's equation solution, these formulas may be called *dynamical analog of Green's formula*.

Formula (27) of Theorem 3 allows at once to go to its integral writing without regularization of under integral functions on fronts.

Then let us consider representation of solution of edge problem for Klein-Gordon equations in spaces with dimensions N = 2,3, characterized for mathematical physics problems. To avoid complexity of formulas under building of integral representation of dynamical analog of Green's formula, let us consider consequently solutions of two BV problems:

1. Cauchy problem at $f(x,t) \neq 0$;

2. BVP at zero initial conditions and $f(x,t) = 0$.

By virtue of linearity of equations, solutions of BVPs may be obtained as a sum of solutions of these two problems with correction of boundary conditions for second problem with account of boundary meanings of Cauchy problem solutions. Solution of Cauchy problem for that equation has been early obtained by Vladimirov (see [9]). We get it here for the complete solution of the initial-boundary problem in the notation used here.

5. The generalized solution of the Cauchy problem for the KG-equation for N = 2

Let us consider the Cauchy problem for the KG equation of the below form:

$$
\Box_c \hat{u} \pm m^2\hat{u} = \hat{f}(x,t), x \in R^N, \quad t > 0,
\tag{28}
$$

where $\hat{f}(x,t)$ is a generalized function.

Let us introduce designations $r = \|y - x\|, S_t(x) = \{y \in S,\ r < ct\}$
$S_t^-(x) = \{y \in S^-,\ r < ct\}$ and $S_t(x) = \{y \in S,\ r < ct\}$, which we will use further.

In the flat case (N = 2), the Green's function of Eq. (28) is a regular generalized function of the form [9]:

$$\hat{U} = \frac{H(ct - \|x\|)}{2\pi} \frac{ch\left(m\sqrt{c^2t^2 - \|x\|^2}\right)}{\sqrt{c^2t^2 - \|x\|^2}} \tag{29}$$

with a weak singularity at the front $\|x\| = ct$:

$$\hat{U} \approx \frac{1}{2\pi\sqrt{c^2t^2 - \|x\|^2}} \qquad \text{by } \|x\| \to ct - 0 \tag{30}$$

Its carrier is a light cone: $\|x\| \le ct$.

Theorem 4. *If* $u_0(y) \in L_1(S^- + S)$, $\dot{u}_0(y) \in L_1(S^-)$, *then the solution of the Cauchy problem has the form:*

$$2\pi c^2 u(x,t)H(t) = \int_{S_t^-(x)} \frac{ch\left(m\sqrt{c^2t^2 - r^2}\right)}{\sqrt{c^2t^2 - r^2}} \dot{u}_0(y)dV(y) +$$

$$+\partial_t \int_{S_t^-(x)} \frac{ch\left(m\sqrt{c^2t^2 - r^2}\right)}{\sqrt{c^2t^2 - r^2}} u_0(y)dV(y) - \int_0^t d\tau \int_{S_\tau^-(x)} \frac{ch\left(m\sqrt{c^2\tau^2 - r^2}\right)}{\sqrt{c^2\tau^2 - r^2}} f(y,t-\tau)dV(y)$$

Proof. The integral notation of formula (23) leads to the formula of the theorem. All integrals are proper due to the regularity of integrands. The carrier of the kernel of integrals is a circle expanding over time with the center at the *point x*.

Note that if the initial conditions and the right-hand side of equation (1) (source) belong to the class of singular functions admitting convolution with the Green's function of the equation, to construct a solution to the Cauchy problem, use formulas (23) and (29).

Similarly, we construct a solution to the Cauchy problem in the case. The solution of the problem in this case allows analytic continuation. It can be obtained from the solution in Theorem 4 replacing ***m*** with im.

6. Generalized solution to the Cauchy problem for the KG equation for N = 3

For N = 3, the Green function (28) (for) is a singular generalized function of the form [9]:

$$4\pi\hat{U} = cH(t)\delta(c^2t^2 - r^2) - mcf_0(r,t) \tag{31}$$

where $r = \|x\|$, $H(t)\delta(c^2t^2 - r^2)$ is a simple layer on a light cone $r = ct$ [9].

The function f_0 *is defined by the expression:*

$$f_0(r,t)=\frac{H(ct-r)J_1\left(m\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}} \tag{32}$$

$J_1(\ldots)$ is *Bessel function. Because* [10],

$$J_1(z)\sim 0,5z \quad \text{when} \quad z\to 0 \tag{33}$$

at the front $r=ct$, the second term has a finite jump:

$$[f_0(r,r/c)]=-\frac{m}{2} \tag{34}$$

Theorem 5. *The solution of the Cauchy problem for the KG-Eq.* (28) *for N = 3 has the form:*

$$4\pi cu(x,t)=(2ct)^{-1}\int_{r=ct}\dot{u}_0(y)dS(y)-m\int_{S_t^-(x)}\frac{J_1\left(m\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\dot{u}_0(y)\,dV(y)+$$

$$+\frac{1}{2ct^2}\int_{r=ct}u_0(y)dS(y)+\frac{1}{2ct}\partial_t\int_{r=ct}u_0(y)dS(y)-m\partial_t\left\{\int_{S_t^-(x)}\frac{J_1\left(m\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}u_0(y)\,dV(y)\right\}-$$

$$-mc^2\int_0^t d\tau\int_{S_\tau^-(x)}\frac{f(y,t-\tau)J_1\left(m\sqrt{c^2\tau^2-r^2}\right)}{\sqrt{c^2\tau^2-r^2}}dV(y)+\frac{c}{2}\int_{S_t^-(x)}r^{-1}f(y,t-r/c)dV(y)$$

Proof. It follows from the representation of a generalized solution for the Cauchy problem taking into account the form of the fundamental solution (30). The solution of the Cauchy problem for Eq. (28) in the case $q(x)=-m^2$ also allows analytic continuation by replacing m with im. It has the form:

$$4\pi cu(x,t)=\frac{1}{2t}\int_{r=ct}\dot{u}_0(y)dS(y)-m\int_{S_t^-(x)}\frac{I_1\left(m\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}\dot{u}_0(y)\,dV(y)+$$

$$+\frac{1}{2ct^2}\int_{r=ct}u_0(y)dS(y)+\frac{1}{2ct}\partial_t\int_{r=ct}u_0(y)dS(y)-$$

$$-m\partial_t\left\{\int_{S_t^-(x)}\frac{I_1\left(m\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}}u_0(y)\,dV(y)\right\}+$$

$$+\frac{c}{2}\int_{S_t^-(x)}r^{-1}f(y,t-\frac{r}{c})dV(y)-mc^2\int_0^t d\tau\int_{S_\tau^-(x)}\frac{f(y,t-\tau)I_1\left(m\sqrt{c^2\tau^2-r^2}\right)}{\sqrt{c^2\tau^2-r^2}}dV(y).$$

If the initial functions and the right-hand side of Eq. (1) belong to the class of singular functions admitting convolution with the Green function of Eq. (28), to construct the solution, one should use the formula in ultraprecise form (23). We construct solutions to initial-boundary value problems.

7. Singular boundary integral equations of plane boundary value problems

Let us consider the solutions of the posed boundary value problems in the case $N = 2$. For the integral representation of the dynamic analogue of the Green formula, we also calculate for the Green function (29):

$$\hat{W} = \frac{1}{2\pi} d_0(r,t) * H(t) = \frac{1}{2\pi c} d_1(r,t), \tag{35}$$

$$\hat{H}(x,t,n) = \frac{1}{2\pi c} d_2(r,t) \frac{\partial r}{\partial n}, \quad r = \|x\|, \tag{36}$$

where

$$d_0(r,t) = H(ct-r) \frac{ch\left(m\sqrt{c^2t^2 - r^2}\right)}{\sqrt{c^2t^2 - r^2}},$$

$$d_1(r,t) = H(ct-r) \int_0^{\sqrt{c^2t^2 - r^2}} \frac{ch(mz)}{\sqrt{z^2 + r^2}} dz,$$

$$d_2(r,t) = -H(ct-r)\left\{\frac{1}{r} - \frac{ch\left(m\sqrt{c^2t^2 - r^2}\right)}{ct}\right\} - H(ct-r) r \int_0^{\sqrt{c^2t^2 - r^2}} \frac{ch(mz)}{\left(\sqrt{z^2 + r^2}\right)^3} dz.$$

Consider the values of these functions at the front $r = ct$, $t > 0$. From (29), (27) follows that

$$W\big|_{r=ct} = 0, \quad H\big|_{r=ct} = 0, \tag{37}$$

Consequently, unlike from U, W and H are continuous at the front. When $r \to 0$, we have an asymptotic representation:

$$U = \frac{ch(mct)}{2\pi ct} + O(r), \quad H = -\frac{1}{2\pi cr} \frac{\partial r}{\partial n} + O(1) \tag{38}$$

Now we turn to the integral notation of the dynamic analogue of Green's formula for $N = 2$.

Theorem 6. *The solution of the initial-boundary value problem for the KG-equation in the flat case is representable: for* $x \notin S$ *in form of*

$$2\pi\, u(x,t) H(t) = \int_{S_t(x)} H(ct-r) dS(y) \int_{r/c}^{t} \left\{ d_2(r,\tau) \frac{\partial r}{c\partial n(y)} \dot{u}(y,t-\tau) \right\} d\tau -$$

$$- \int_{S_t(x)} H(ct-r) dS(y) \int_{r/c}^{t} d_0(r,\tau) p(y,t-\tau) d\tau, \quad r = \|y - x\|;$$

for $x \in S$ *in form of*

$$\pi c u(x,t) H(t) = V.P. \int_{S_t(x)} H(ct-r)\frac{\partial r}{\partial n(y)} dS(y) \int_{r/c}^{t} d_2(r,\tau)\dot{u}(y,t-\tau)d\tau -$$

$$-c \int_{S_t(x)} H(ct-r) dS(y) \int_{r/c}^{t} d_0(r,\tau) p(y,t-\tau)d\tau.$$

Proof. In the flat case, the formula of Theorem 3, taking into account the carriers of the cores, can be written in the integral form:

$$2\pi \hat{u}(x,t) = \int_{S_t(x)} H(ct-r) dS(y) \int_{r/c}^{t} H(x,y,\tau,n(y))\dot{u}(y,t-\tau)d\tau -$$

$$-H(t) \int_{S_t(x)} H(ct-r) dS(y) \int_{r/c}^{t} U(x,y,\tau)\frac{\partial u(y,t-\tau)}{\partial n(y)} d\tau.$$

Substituting the form of the cores (29), (6.2), we obtain the first formula of the theorem. For $x \notin S$, all integrals are convergent, because $r \neq 0$, and U has an integrable singularity at the front (30). Let us prove the second formula for $x^* \in S$.

We write a dynamic analogue of Green's formula for a region with a puncture ε-vicinity at point x^*. We denote $\Gamma_\varepsilon^-(x) = \{y \in S^- : \|y-x^*\| = \varepsilon\}$, $O_\varepsilon(x) = \{y \in S : \|y-x^*\| < \varepsilon\}$, $\Omega_\varepsilon(x^*) = \{S_t(x^*) - O_\varepsilon(x^*)\} \cup \Gamma_\varepsilon^-(x^*)$. Because the x^* outside the area bounded by a compound path $\Omega_\varepsilon(x^*)$:

$$\Sigma_{1\varepsilon} + \Sigma_{2\varepsilon} == \int_{\Omega_\varepsilon(x^*)} H(ct-r)\frac{\partial r}{\partial n(y)} dS(y) \int_{r^*/c}^{t} d_2(r,\tau)\dot{u}(y,t-\tau)\frac{d\tau}{c} -$$

$$-\int_{\Omega_\varepsilon(x^*)} H(ct-r) dS(y) \int_{r/c}^{t} d_0(r,\tau) p(y,t-\tau)d\tau = 0$$

In this equality, we pass to the limit for $\varepsilon \to +0$. For $\Gamma_\varepsilon^-(x^*)$ $r = \varepsilon$, arc length differential in polar coordinates: $dS(y) = \varepsilon d\varphi$, therefore

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon^-(x^*)} H(ct-\varepsilon) dS(y) \int_{\varepsilon/c}^{t} d_0(\varepsilon,\tau) p(y,t-\tau)d\tau =$$

$$= \lim_{\varepsilon \to 0} \varepsilon\pi \int_{\varepsilon/c}^{t} \left\{\frac{ch(mc\tau)}{c\tau} p(x^*,t-\tau)\right\} d\tau = \pi c \lim_{\varepsilon \to 0} \varepsilon \int_{1}^{ct/\varepsilon} \left\{\frac{ch(m\varepsilon\tau)}{\tau} p\left(x^*,t-\frac{\varepsilon}{c}\tau\right)\right\} d\tau = 0$$

We have the last equality due to the inequality:

$$\left| \int_{1}^{ct/\varepsilon} \left\{\frac{ch(m\varepsilon\tau)}{\tau} p\left(x^*,t-\frac{\varepsilon}{c}\tau\right)\right\} d\tau \right| \leq ch(mct)\int_{0}^{\infty} \left|p(x^*,\tau)\right| d\tau$$

Consider the limit of the first integral:

$$\lim_{\varepsilon\to 0}\Sigma_{1\varepsilon}=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon(x^*)} H(ct-r)\frac{\partial r}{\partial n(y)}dS(y)\int_{r^*/c}^{t} d_2(r,\tau)\dot{u}(y,t-\tau)\frac{d\tau}{c}=$$

$$=V.P.\int_{S_t(x^*)} H(ct-r)\frac{\partial r}{\partial n(y)}dS(y)\int_{r^*/c}^{t} d_2(r,\tau)\dot{u}(y,t-\tau)\frac{d\tau}{c}- \tag{39}$$

$$-\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^-(x^*)} H(ct-r)\frac{\partial r}{\partial n(y)}dS(y)\int_{r/c}^{t} d_2(r,\tau)\dot{u}(y,t-\tau)\frac{d\tau}{c}=I_S+\lim_{\varepsilon\to 0}I_\varepsilon$$

We calculate the last limit on the right side.

Since on $\Gamma_\varepsilon^-(x^*)$: $\dfrac{\partial r}{\partial n}=-1\Rightarrow$

$$H=\frac{1}{2\pi c\varepsilon}+O(1),$$

Therefore

$$\lim_{\varepsilon\to 0}I_\varepsilon=-\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^-(x^*)}\frac{\partial r}{\partial n(y)}dS(y)\int_{\varepsilon/c}^{t} d_2(\varepsilon,\tau)\dot{u}(x^*,t-\tau)\frac{d\tau}{c}=$$

$$=\lim_{\varepsilon\to 0}\pi\varepsilon\int_{\varepsilon/c}^{t}\frac{\dot{u}(x^*,t-\tau)}{\varepsilon}d\tau=\pi\lim_{\varepsilon\to 0}\int_0^t\dot{u}(x^*,t-\tau)\,d\tau=$$

$$=\pi\big(u(x^*,0)-u(x^*,t)\big)=-\pi u(x^*,t)$$

Adding, taking into account the last equality, we obtain the second formula of the theorem. The theorem has been proved.

In the case of the first boundary value problem, the left side of the equation of Theorem 6.1 and the first integral on the right are known, determined by the boundary conditions. Solving it, we determine the normal derivative of the desired function on the boundary, after which the formula of the theorem allows us to calculate the solution at any point in the domain of definition. In the case of the second boundary-value problem, we have a BIE to determine the unknown boundary values of the unknown function u from the boundary values of its normal derivative. Solving it, we determine its values at the border, after which we determine the solution.

8. Dynamic analog of Green's formula for solutions of the KG-equation ($N = 3$)

To construct a dynamic analogue of Green's formula in integral form, we define $\hat{W}$ и $\hat{H}$. By computing formulas (25)and (27), we obtain

$$4\pi c^2\hat{W}=\frac{H(ct-r)}{r}-m\,f_1(r,t),\qquad r=\|x\| \tag{40}$$

$$\hat{H}(x,n,t) = -\frac{(x,n)}{2r}\left(\frac{\delta(ct-r)}{r} + \frac{H(ct-r)}{r^2} + 2mf_2(r,t)\right),$$

where $f_1(r,t) = H(ct-r)\int\limits_0^{\sqrt{c^2t^2-r^2}} \frac{J_1\left(\frac{m}{c}z\right)}{\sqrt{z^2+r^2}}dz,$

$$f_2(r,t) = rH(ct-r)\int\limits_0^{\sqrt{c^2t^2-r^2}} \frac{J_1\left(\frac{m}{c}z\right)}{\left(\sqrt{z^2+r^2}\right)^3}dz + \frac{rH(ct-r)}{ct\sqrt{c^2t^2-r^2}}J_1\left(\frac{m}{c}\sqrt{c^2t^2-r^2}\right) \tag{41}$$

The values of these functions at the front $r = ct$:

$$f_1(r,r/c) = 0\,,\quad f_2(r,r/c) = \frac{m}{2c}$$

Consequently, W is continuous at the wave front; the last term in the representation H has a discontinuity of the first kind at the front. When $r \to 0$, $t > 0$, the following asymptotics are true:

$$\hat{W}(x,t) = \frac{H(t)}{8\pi c^2 r} + O(1) \tag{42}$$

$$\hat{H}(x,n,t) = \frac{(e_x,n)}{8\pi c^2 r^2}H(t) + O(r^{-1}) \tag{43}$$

Let us impose the notation for the shift functions:

$$U(x,y,t) = \hat{U}(x-y,t),\ W(x,y,t) = \hat{W}(x-y,t),$$
$$H(x,y,t,n) = \hat{H}(x-y,t,n)$$

Theorem 7. *The generalized solution of boundary value problems for a homogeneous KG-equation satisfying zero initial conditions* $(u(x,0)=0, \dot{u}(x,0)=0)$, *is representable in the form:*

$$4\pi\hat{u}(x,t) = -H(t)\int\limits_{S_t(x)} \frac{1}{r}\frac{\partial u\left(y,t-\frac{r}{c}\right)}{\partial n(y)}dS(y) - c^{-1}H(t)\int\limits_{S_t(x)} \dot{u}\left(y,t-\frac{r}{c}\right)\frac{(y-x,n(y))}{r^2}dS(y) +$$

$$+mc^{-1}H(t)\int\limits_{S_t^-(x)} f_0(r,r/c)\frac{\partial u(y,t-\tau)}{\partial n(y)}dV(y) + mH(t)\int\limits_0^t d\tau \int\limits_{S_\tau(x)} f_2(r,\tau)\dot{u}(y,t-\tau)\ \ dS(y) +$$

$$+H(t)\int\limits_{S_t(x)} u\left(y,t-\frac{r}{c}\right)\frac{(y-x,n(y))}{r^3}dS(y),$$

For $x \in S$, *the last integral is singular, taken in the sense of the principal value.*

Proof. Using the conditions of Theorem and (30), we write in this case a dynamic analogue of Green's formula (22). We compute convolution sequentially:

$$4\pi \hat{U}(x,t) * \frac{\partial u}{\partial n}\delta_S(x)H(t) = \int_{S_t(x)} \frac{1}{r}\frac{\partial u(y,t-r/c)}{\partial n(y)}dS(y) - mc\int_0^t d\tau \int_{S_\tau(x)} f_0(r,\tau)\frac{\partial u(y,t-\tau)}{\partial n(y)}dS(y) =$$

$$\int_{S_t(x)} \frac{1}{r}\frac{\partial u(y,t-r/c)}{\partial n(y)}dS(y) - m\int_{S_t^-(x)} f_0(r,r/c)\frac{\partial u(y,t-\tau)}{\partial n(y)}dV(y);$$

$$-4\pi \hat{W}_{,j} * \dot{u}(x,t)n_j(x)\delta_S(x)H(t) = -\int_{S_t(x)} \dot{u}(y,t-r/c)\frac{(y-x,n(y))}{cr^2}dS(y) -$$

$$-\int_0^t d\tau \int_{S_\tau(x)} \frac{(y-x,n(y))}{r^3}\dot{u}(y,t-\tau)\,dS(y) + \int_0^t d\tau \int_{S_t(x)} \frac{(y-x,n(y))}{c^2 r} f_2(r,\tau)\,\dot{u}(y,t-\tau)dS(y).$$

When $r = \|y-x\| \neq 0$, you can change the order of integration, therefore:

$$\int_0^t d\tau \int_{S_\tau(x)} \frac{(y-x,n(y))}{r^3}\dot{u}(y,t-\tau)\,dS(y) = -\int_{S_t(x)} u(y,t-r/c)\frac{(y-x,n(y))}{r^3}\,dS(y)$$

Summing up, we obtain the formula of the theorem.

Note that for points $x \in S$ when $t > 0$, all cores have no singularities and the integrals on the right exist and define functions that are regular on a given set. Since regular functions are on left and right and they are equal on this set as generalized, by virtue of the du Bois-Reymond lemma [1], they are equal in the usual sense, like numerical functions.

Let us assume $x^* \in S$. We write a dynamic analogue of Green's formula for a region with a puncture $\varepsilon-$vicinity point $x^* \in S, \varepsilon < ct$:

$$\begin{aligned}
0 = &-\int_{S_t(x)-O_\varepsilon(x)} \frac{1}{r}\frac{\partial u\left(y,t-\frac{r}{c}\right)}{\partial n(y)}dS(y) - \int_{S_t(x)-O_\varepsilon(x)} \dot{u}\left(y,t-\frac{r}{c}\right)\frac{(y-x,n(y))}{cr^2}dS(y) + \\
&+mc\int_0^t d\tau \int_{S_\tau(x)-O_\varepsilon(x)} f_0(r,\tau)\frac{(y-x,n(y))}{r}\frac{\partial u(y,t-\tau)}{\partial n(y)}dS(y) + \int_0^t d\tau \int_{S_\tau(x)-O_\varepsilon(x)} f_2(r,\tau)\dot{u}(y,t-\tau)dS(y) - \\
&-\int_{\Gamma_\varepsilon^-(x,t)} \frac{1}{r}\frac{\partial u(y,t-r/c)}{\partial n(y)}dS(y) - \int_{\Gamma_\varepsilon^-(x,t)} \dot{u}(y,t-r/c)\frac{(y-x,n(y))}{cr^2}dS(y) + \\
&+mc\int_0^t d\tau \int_{\Gamma_\varepsilon^-(x,\tau)} f_0(r,\tau)\frac{(y-x,n(y))}{r}\frac{\partial u(y,t-\tau)}{\partial n(y)}dS(y) + \int_0^t d\tau \int_{\Gamma_\varepsilon^-(x,\tau)} f_2(r,\tau)\,\dot{u}(y,t-\tau)\,dS(y) + \\
&+\int_{S_t(x)-O_\varepsilon(x)} u(y,t-r/c)\frac{(y-x,n(y))}{r^3}dS(y) + \int_{\Gamma_\varepsilon^-(x,t)} u(y,t-r/c)\frac{(y-x,n(y))}{r^3}dS(y),
\end{aligned} \tag{44}$$

Now let us move on to the limit $\varepsilon \to 0$. In the first integral, the integrand has a weak integrable singularity when $r = 0$. In the second integral, it does not have a singularity when $r = 0$. By virtue of this, the integrals over Γ_ε^- from these functions in the third and fourth term tend to zero. It is obvious that

$$\lim_{\varepsilon \to 0} \int_{S_t(x)-O_\varepsilon(x)} u(y,t-r/c)\frac{(y-x,n(y))}{2r^3}dS(y) = V.P.\int_{S_t(x)} u(y,t-r/c)\frac{(y-x^*,n(y))}{2r^3}dS(y)$$

Moreover, the main value of the integral exists, because the integrand has a singularity of order $1/r^2$ on a two-dimensional surface *S*, function u is continuous on *S*, and the characteristic $(y-x^*, n(x^*))$ antisymmetric in opposite relative x^* points.

Let us consider the last limit. For $\Gamma_\varepsilon^-(x)$, we have $\|y-x\|=\varepsilon$, $n(y)=(x-y)/\varepsilon$, therefore

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon^-(x^*,t)} u(y,t-r/c)\frac{(y-x^*,n(y))}{r^3}dS(y)\right\}=$$

$$=\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon^-(x^*,t)}\left(u(y,t-r/c)-u(x^*,t)\right)\frac{(-n(y),n(y))}{\varepsilon^2}dS(y)\right\}-u(x^*,t)\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon^-(x^*,t)}\frac{dS(y)}{\varepsilon^2}\right\}=-2\pi u\left(x^*,t\right)$$

Passing in (44) to the limit in $\varepsilon\to 0$ and transferring the last term to the right side, we obtain the formula of the theorem. The theorem has been proved.

Formula ($x\in S$) gives a singular boundary integral equation for solving the second initial-boundary value problem. For the first boundary value problem, the unknown normal derivative falls under the sign of the surface integral with a weakly polar core. The remaining terms are known.

9. Conclusion

Note that the constructed delayed singular BIE have a nonclassical type; since in addition to the boundary values $u,_t$ of the function and its normal derivative, the BIE includes a velocity that is unknown for the Dirichlet problem and is known for the Neumann problem. In addition, the integration region at the boundary depends on time, which also distinguishes these equations from the SEI for elliptic and parabolic problems. Solving the Dirichlet problem on the basis of the method of successive approximations, like elliptic problems, is impossible, since it requires the determination of boundary values of velocity. However, differentiation of generalized solutions on the boundary leads to hypersingular relations. This is a new class of BIE in delayed potentials, which requires a special study by the methods of functional analysis. However, to solve the resolving singular BIE that solve the boundary value problems, numerical boundary element method can be used.

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References

[1] Vladimirov VS. Generalized Functions in Mathematical Physics. Moscow: Mir; 1978. p. 282

[2] Alekseeva LA, Dildabaev SA, Zhanbyrbaev AB, Zakiryanova GK. Boundary integral equation method in two & three dimensional problems of elastodynamics. International Journal of Computational Mechanics. 1996;**18**(2): 147-157

[3] Alekseeva LA. Boundary element method of boundary value problems of elastodynamics by stationary running loads. International Journal of Engineering Analysis with Boundary Element. 1998;(11):37-44

[4] Alekseeva LA. Singular boundary integral equations of boundary value problems of elastodynamics in the case of subsonic traveling loads. Differential Equations. 2010;**46**(4):512-519

[5] Alexeyeva LA. Generalized solutions of non-stationary boundary value problems for Maxwell's equations. Journal of Computational Mathematics and Mathematical Physics. 2002; **42**(1):76-88

[6] Alexeyeva LA, Kupesova BN. The method of generalized functions in boundary value problems of coupled thermoelastodynamics. Applied Mathematics and Mechanics. 2001; **65**(2):334-345

[7] Alexeyeva LA, Zakiryanova GK. Generalized solutions of initial-boundary value problems for second-order hyperbolic systems. Journal of Computational Mathematics and Mathematical Physics. 2011;**51**(7): 1280-1293

[8] Bogolyubov NN, Logunov AA, Oksak AI, Todorov IT. General Principles of Quantum Field Theory. M: Science; 1987

[9] Polyanin AD. Handbook of Linear Equations of Mathematical Physics. M: FIZMATLIT; 2001. 576 p. ISBN: 5-9221-0093-9

[10] Abramovits M, Stegana I. Handbook of Special Functions. M: Science; 1979