Chapter

Mathematical Fundamentals of a Diagnostic Method by Long Nonlinear Waves for the Structured Media

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Abstract

We have proven that the long wave with finite amplitude responds to the structure of the medium. The heterogeneity in a medium structure always introduces additional nonlinearity in comparison with the homogeneous medium. At the same time, a question appears on the inverse problem, namely, is there sufficient information in the wave field to reconstruct the structure of the medium? It turns out that the knowledge on the evolution of nonlinear waves enables us to form the theoretical fundamentals of the diagnostic method to define the characteristics of a heterogeneous medium using the long waves of finite amplitudes (inverse problem). The mass contents of the particular components can be denoted with specified accuracy by this diagnostic method.

Keywords: diagnostics, nonlinear waves, inverse problem, asymptotic model, structured medium

1. Introduction

Natural media, in the general case, should not be treated as structureless. The experiments have shown that the intrinsic structure of a medium influences the wave motions [1–8]. Existing inhomogeneities complicate the problem and, at the same time, are fully manifested under the propagation of nonlinear waves. The principal part of the problem is associated with the phenomena caused by the nonlinear behavior of natural media, in particular, more substantial increase of nonlinear effects in structured media than that in homogeneous ones [1–5, 7].

2. The model notion of the medium with structure

The wave processes in heterogeneous media are usually described in terms of more or less complicated models. Under the conditions of local equilibrium, the media are traditionally modelled irrespective of their structure. In the framework of continuum mechanics, the known idealization of a real medium as a homogeneous one has been relatively successful in the description of wave processes (see, e.g., [9–11]). The continuum models are commonly applied to the mixtures whose dispersive dissipative properties are treated with regard to the interactions between the components [12–15]. On this level, the media are modelled in the framework of a homogeneous elastic, viscous elastic, and elastic-plastic media. In this case, the features of the medium structure are taken into account indirectly through the kinetic parameters (relaxation time, viscous coefficients, etc.) [4–6, 10, 12–15].

The model of multi-velocity interpenetratable continua was developed in terms of classical continuum mechanics [16] and statistical physics in order to describe the dynamical behavior of multicomponent media. A fundamental assumption in the theory of mixtures [15] reproduces the assumption in the model of multi-velocity interpenetratable continua [16], namely, that the particles of the existing components occupy each microvolume. The equations of motion for each component involve the terms describing the mass, force, and energy interactions between the components. The problem is complicated by the necessity to employ, in the general case, the experimental data for establishing theoretical relations between the macro parameters at the component interaction level. Moreover, if the component interaction is determined, then these models would be indispensable in the theory of multicomponent media.

In all the models mentioned, the formalism of continuum mechanics is based on the principle of local action as well as on the generalization of the mechanics laws relating the point mass to the continuum [11].

When going from the integral equations to differential balance equations, the existence of a differentially small microvolume dv is assumed. On the one hand, this volume is so small that the mechanics laws of the point can be extended to the whole microvolume. On the other hand, the volume contains so many structural elements of the medium that, in this sense, it can be regarded as macroscopic one despite its smallness as compared to the entire volume occupied by the medium. So, the passage to the differential balance equations is based on the assumption that microstructural scales ε are small as compared to the characteristic macroscopic scale of the λ and the passage should be made to the limiting case $\varepsilon/\lambda \to 0$. Contraction of the volume dv to the point is in the general case correct for continuous functions [11, 15]. This means that all points within the differentially small volume are equivalent. Hence, for the case of a mixture, the equivalence of the points implies that field characteristics should be averaged over dv. Hence, it is assumed that the equations of motion can be written in terms of average density, mass velocity, and pressure of each component. We note that these models do not contain the exact sizes of components.

The application of the models of a homogeneous medium to the description of the dynamical wave processes in a structured natural medium is associated with specific fundamental difficulties [3, 4, 6, 8]. In what follows, we treat the medium structure at the macrolevel. We abandon the assumption that the differentially small volume dv contains all the components of the medium. Nevertheless, we consider the long-wave approach with the wavelength λ much higher than the characteristic length of the medium structure ε. We consider a structured medium (**Figure 1**) in which separated components are considered as a homogeneous medium (the differentially small volume dv is much smaller than the characteristic size of a particular component ε).

Within continuum mechanics [17], the known idealization of a real medium as homogeneous is widely used for modelling their dynamic behavior. In these models, the effect of heterogeneity is taken into account indirectly throughout the kinetic parameters such as a viscous coefficient and relaxation time. The inner processes, in this case, are manifested through dispersive dissipative properties of a medium.

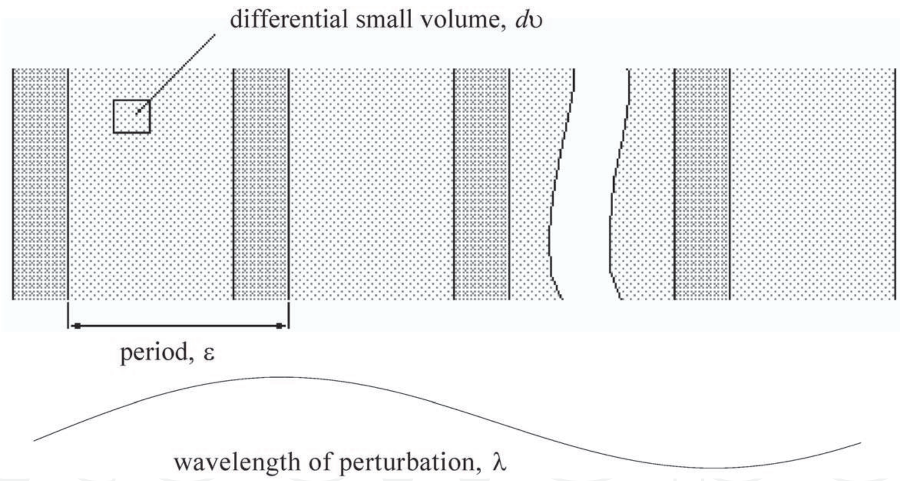

Figure 1.
Model of the layered medium with two homogeneous components in the period.

Traditionally, it was considered that in heterogeneous media with wavelength appreciably exceeding the size of the structural heterogeneities, the perturbations propagate in the same way as inhomogeneous media [9, 10, 15]. However, this statement should be proven, and we shall show that this approximation is not universally correct. In general case, the wave propagation cannot be described in terms of the average characteristics (continuum model).

The properties of a medium deviate from the equilibrium state under the propagation of intensive waves. Moreover, an unperturbed medium can be in one of the unstable stationary states. So, a geophysical medium, within a current physical concept, is an open thermodynamic system, which substantially influences the exchanges of energy and mass. Thus, a description of open systems should take into account the peculiarities of their inner structure, dynamical processes occurring on the level of structural elements. What is more, the state of media under the action of high-frequency wave perturbations departs from equilibrium, and, thus, the behavior of media cannot be described in the framework of equilibrium thermodynamics. Consequently, there is a necessity to develop new mathematical models to take into account the nonlinear wave perturbations and irreversible inner exchange processes.

3. Asymptotic averaged model for structured medium

We describe the wave processes in nonequilibrium heterogeneous media in terms of an asymptotic averaged model [18–22]. The obtained integral differential system of equations cannot be reduced to the average terms (pressure, mass velocity, specific volume) and contains the terms with characteristic sizes of individual components.

3.1 Background and initial equations

The most straightforward heterogeneous media for which the effect of the structure can be analyzed are media with a regular structure. Features of the propagation of long-wave perturbations will be investigated by using, as an example, a periodic medium under conditions of equality of stresses and mass velocities on the boundaries of neighboring components. It is supposed that the microstructure elements of medium dv (see **Figure 1**) are large enough that it is possible to submit to the laws of classical continuum mechanics for each component. At the same time, the inner processes in each component will be considered within a relaxation approach. The notions based on the relaxation nature of a phenomenon are regarded to be promising and fruitful. We consider that the properties of the medium, such as density, sound velocity, and relaxation time, vary periodically (although this assumption is unessential in the final result).

3.1.1 Motion equations for individual component

The analysis of wave motions is based on the hydrodynamic approach. This restriction can be imposed for the modelling of nonlinear waves in water-saturated soils, bubble media, aerosols, etc. [13]. The set of acceptable media could be extended to solid media where the powerful loads are studied in the condition that the strength and plasticity of the material can be neglected [23]. In the hydrodynamic approach, we have considered the media without tangential stresses, while there are equalities of the stresses as well as of mass velocities on boundaries of neighboring components. Also, we assume that the medium is barotropic. The individual components of the medium are considered to be described by the classical equations of hydrodynamics. In the Lagrangian coordinate system (l, t), the equations of one-dimensional motion for each component have the form

$$\begin{gathered} \frac{\partial r^{\nu}}{\partial l^{\nu}} = \frac{V}{V_0}, \quad u = \frac{\partial r}{\partial t}, \\ \frac{\partial u}{\partial t} + V_0 \left(\frac{r}{l}\right)^{\nu-1} \frac{\partial p}{\partial l} = 0. \end{gathered} \tag{1}$$

The equation of continuity can also be used in the alternative form

$$\frac{\partial V}{\partial t} - \nu V_0 \frac{\partial r^{\nu-1} u}{\partial l^{\nu}} = 0. \tag{2}$$

Here $V = \rho^{-1}$ is the specific volume; ν is a parameter of symmetry, where $\nu = 1$ is planar symmetry, $\nu = 2$ is cylindrical one, and $\nu = 3$ is spherical one; and index 0 relates to the initial state. The other notations are those that are generally accepted.

Conditions for matching are the equality of mass velocities and pressures on the boundaries of the components:

$$[u] = 0, \qquad [p] = 0. \tag{3}$$

3.1.2 Dynamic state equation

Considering the models of a relaxing medium as more general than the equilibrium models for describing the evolution of high-gradient waves, we will take into account the relaxing processes for each component. Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations in a medium. There are processes of the interaction that tend to return the equilibrium. The parameters characterizing this interaction are referred to as the inner variables, unlike the macro parameters such as the pressure p, mass velocity u, and density ρ. In essence, the change of macro parameters caused by the changes of inner parameters is a relaxation process. From the nonequilibrium thermodynamics standpoint, the models of a relaxing medium are more general than the equilibrium models for describing the wave propagation.

An equilibrium state equation of a barotropic medium is one-parameter equation. As a result of relaxation, an additional variable ξ (inner parameter) appears in the state equation. It defines the completeness of the relaxation process:

$$p = p(\rho, \xi). \tag{4}$$

There are two limiting cases:

i. The lack of relaxation (inner interaction processes are frozen) $\xi = 1$,

$$p = p(\rho, 1) = p_f(\rho). \tag{5}$$

ii. The relaxation complete (there is the local thermodynamic equilibrium) $\xi = 0$,

$$p = p(\rho, 0) = p_e(\rho). \tag{6}$$

The equations of state (5) and (6) are considered to be known. These relationships enable us to introduce the sound velocities for fast processes:

$$c_f^2 = dp_f/d\rho \tag{7}$$

and for slow processes

$$c_e^2 = dp_e/d\rho. \tag{8}$$

The slow and fast processes are compared, utilizing relaxation time τ_p. The dynamic state equation is written down in the form of the first-order differential equation:

$$\tau_p \left(\frac{d\rho}{dt} - c_f^{-2} \frac{dp}{dt} \right) + (\rho - \rho_e) = 0. \tag{9}$$

The equilibrium equations of state are considered to be known:

$$\rho_e - \rho_0 = \int_{p_0}^{p} c_e^{-2} dp. \tag{10}$$

Clearly, for the fast processes $(\omega\tau_p \gg 1)$, we have the relation (5), and for the slow ones $(\omega\tau_p \ll 1)$, we obtain (6).

The substantiation of Eq. (9) within the framework of the thermodynamics of irreversible processes has been given in [17, 24–26]. As far as we know, the first work in this field was paper by Mandelshtam and Leontovich (see Section 81 in [17]). We note that the mechanisms of the exchange processes are not explicitly defined when deriving Eq. (9), and the thermodynamic and kinetic parameters appear only in this equation. These characteristics can be found experimentally.

The phenomenological approach for describing the relaxation processes in hydrodynamics has been developed in many publications [13, 17, 26]. The dynamic equation of state was used (a) for describing the propagation of sound waves in a relaxing medium [17], (b) for taking into account the exchange processes within media (gas-solid particles) [26], and (c) for studying wave fields in gas-liquid media and in soil [13]. In most works, the equation of state has been derived from the concept of precise mechanism for the inner process. Within the context of mixture theory, Biot [12] attempted to account for the nonequilibrium in velocities between components directly in the equations of motion in the form of dissipative terms.

We assume that the relaxation time and sound velocities do not depend on time, but they are functions of pressure and the individual properties of the components. This means that in the process of a relaxation interaction, we can take into account the exchange of moment and heat but not that of mass. The dynamic equation of state determines the features of the intrastructural interaction for each component.

The equations of motion (1) have been written in the Lagrangian coordinate system. The necessity of such a description stems from the fact that the dynamic equation of state (9) has been written to the mass element of a medium. Besides, the use of the Lagrangian coordinates is essential for the application of the method of asymptotic averaging, since in these coordinates, the structure is independent of a wave process.

3.2 Asymptotic averaged system of equations

A regularity of structure and a nonlinearity of long-wave processes investigated here specify the choice of mathematical methods. One way of studying this heterogeneous medium is based on a method of asymptotic averaging of equations with high-oscillating coefficients [20, 27–30]. The essence of this method consists in the application of a multiscale method in combination with a space averaging. In accordance with this method, the mass space coordinate $m = l^\nu / V_0$ is divided into two independent coordinates: slow coordinate s and fast one ξ, wherein

$$m = s + \varepsilon\xi, \qquad \frac{\partial}{\partial m} = \frac{\partial}{\partial s} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{11}$$

The slow coordinate s corresponds to a global change of the wave field and s is a constant value during a period, while the fast coordinate ξ traces the variations of a field in the structure period. The dependent functions are presented as a degree series over the structure period ε

$$\begin{aligned}
V(m,t) &= V^{(0)}(s,t,\xi) + \varepsilon V^{(1)}(s,t,\xi) + \varepsilon^2 V^{(2)}(s,t,\xi) + \ \dots \\
p(m,t) &= p^{(0)}(s,t,\xi) + \varepsilon p^{(1)}(s,t,\xi) + \varepsilon^2 p^{(2)}(s,t,\xi) + \ \dots \\
u(m,t) &= u^{(0)}(s,t,\xi) + \varepsilon u^{(1)}(s,t,\xi) + \varepsilon^2 u^{(2)}(s,t,\xi) + \ \dots \quad . \\
r^\nu(m,t) &= (r^\nu)^{(0)}(s,t,\xi) + \varepsilon (r^\nu)^{(1)}(s,t,\xi) + \varepsilon^2 (r^\nu)^{(2)}(s,t,\xi) + \ \dots
\end{aligned} \tag{12}$$

where $p^{(i)}$, $u^{(i)}$, $V^{(i)}$, and $r^{(i)}$ are defined as the one-period functions of ξ. In the Lagrangian mass coordinates the period is a constant which allows the averaging procedure to be performed.

We now will prove that $p^{(0)} = p^{(0)}(s,t)$, $p^{(1)} = p^{(1)}(s,t)$, $u^{(0)} = u^{(0)}(s,t)$, and $(r^\nu)^{(0)} = (r^\nu)^{(0)}(s,t)$ are independent of the fast variable ξ. Indeed, after substitution of Eqs. (11) and (12) into the initial equations of motion, we obtain

$$
\begin{aligned}
&-\varepsilon^{-1}\frac{\partial (r^{\nu})^{(0)}}{\partial \xi}+\varepsilon^{0}\left(\frac{\partial (r^{\nu})^{(0)}}{\partial s}-\frac{\partial (r^{\nu})^{(1)}}{\partial \xi}-V^{(0)}\right)+\dots=0,\\
&\varepsilon^{0}\left(u^{(0)}-\frac{\partial r^{(0)}}{\partial t}\right)+\dots=0,\\
&-\varepsilon^{-1}\nu(r^{\nu-1})^{(0)}\frac{\partial p^{(0)}}{\partial \xi}+\varepsilon^{0}\left(\frac{\partial u^{(0)}}{\partial t}+\nu\left(r^{\nu-1}\right)^{(0)}\frac{\partial p^{(0)}}{\partial s}+\nu\left(r^{\nu-1}\right)^{(1)}\frac{\partial p^{(0)}}{\partial \xi}+\nu\left(r^{\nu-1}\right)^{(0)}\frac{\partial p^{(1)}}{\partial \xi}\right)+\dots=0,\\
&-\varepsilon^{-1}\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(0)}}{\partial \xi}+\varepsilon^{0}\left(\frac{\partial V^{(0)}}{\partial t}+\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(0)}}{\partial s}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(1)}u^{(0)}}{\partial \xi}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(1)}}{\partial \xi}\right)+\dots=0,
\end{aligned}
\tag{13}
$$

According to the general theory of the asymptotic method, the terms of equal powers of ε should vanish independently of each other. Thus, $\partial p^{(0)}/\partial\xi = 0$, $\partial u^{(0)}/\partial\xi = 0$, and $\partial (r^{\nu-1})^{(0)}/\partial\xi = 0$, i.e., $p^{(0)} = p^{(0)}(s,t)$, $u^{(0)} = u^{(0)}(s,t)$, and $r^{(0)} = r^{(0)}(s,t)$, are independent of ξ. Furthermore

$$
\begin{aligned}
&\frac{\partial (r^{\nu})^{(0)}}{\partial s}+\frac{\partial (r^{\nu})^{(1)}}{\partial \xi}=V^{(0)},\\
&u^{(0)}=\frac{\partial r^{(0)}}{\partial t},\\
&\frac{\partial u^{(0)}}{\partial t}+\nu\left(r^{\nu-1}\right)^{(0)}\frac{\partial p^{(0)}}{\partial s}+\nu\left(r^{\nu-1}\right)^{(0)}\frac{\partial p^{(1)}}{\partial \xi}=0,\\
&\frac{\partial V^{(0)}}{\partial t}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(0)}}{\partial s}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(1)}u^{(0)}}{\partial \xi}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(1)}}{\partial \xi}=0.
\end{aligned}
\tag{14}
$$

Thus, we can average the equations during the period ξ. We define $\langle\cdot\rangle = \int_0^1(\cdot)d\xi$ and perform the normalization $\int_0^1 d\xi = 1$. Since $p^{(1)}$, $u^{(1)}$, and $r^{(1)}$ are periodic, the integrals can be calculated as $\langle \partial p^{(1)}/\partial\xi\rangle = 0$, $\langle \partial u^{(1)}/\partial\xi\rangle = 0$, and $\langle \partial r^{(1)}/\partial\xi\rangle = 0$. Moreover, as $\langle u^{(0)}\rangle = u^{(0)}$, $\langle p^{(0)}\rangle = p^{(0)}$ than $\partial p^{(1)}/\partial\xi = 0$. This means that $p^{(1)}$ does not also depend on ξ. After integrating over the structure period the equations containing the value of zero order of ε, we obtain the averaged system

$$
\begin{aligned}
&\frac{\partial (r^{\nu})^{(0)}}{\partial s}=\left\langle V^{(0)}\right\rangle,\\
&u^{(0)}=\frac{\partial r^{(0)}}{\partial t},\\
&\frac{\partial u^{(0)}}{\partial t}+\nu\left(r^{\nu-1}\right)^{(0)}\frac{\partial p^{(0)}}{\partial s}=0,\\
&\frac{\partial\left\langle V^{(0)}\right\rangle}{\partial t}-\nu\frac{\partial \left(r^{\nu-1}\right)^{(0)}u^{(0)}}{\partial s}=0
\end{aligned}
\tag{15}
$$

with the averaged equation of state

$$
d\left\langle V^{(0)}\right\rangle=-\left\langle\frac{\left(V^{(0)}\right)^{2}}{c_f^{2}}\right\rangle dp-\left\langle\frac{V^{(0)}}{\tau_p V_e\left(p^{(0)}\right)}\left(V^{(0)}-V_e\left(p^{(0)}\right)\right)\right\rangle dt. \tag{16}
$$

Unlike the values $u^{(0)}$, $p^{(0)}$, $p^{(1)}$, and $r^{(0)}$, the specific volume $V^{(0)}$ is a function of ξ. Hereafter, we will consider only the zero approximation of the equations, and, therefore, the upper index 0 is omitted. Choosing the wavelength λ to be large enough, we can always reduce the effect to zero from other approximation terms.

The averaged system of Eqs. (15) and (16) is an integrodifferential one and, in the general case, is not reduced to the averaged variables p, u, and $\langle V^{(0)} \rangle$. The derivation of Eqs. (15) and (16) relates to a rigorous periodic medium. However, it may be shown that Eqs. (15) and (16) are also relevant to media with a quasiperiodic structure. Indeed, the pressure p and the mass velocity u are independent of the fast variable ξ. Hence on a microscale ξ, the action is statically uniform (waveless) over the whole period of the medium structure, while on the slow scale s, the action of perturbation is manifested by the wave motion of the medium. On a microlevel, the behavior of medium adheres only to the thermodynamic laws. There is a mechanical equilibrium. On a macrolevel, the motion of the medium is described by the wave dynamics laws for averaged variables. Mathematically, in the zero-order case of ε, the size of the period is infinitesimal ($\varepsilon \to 0$). This signifies that the location of particular components in the period is irrelevant. Eqs. (15) and (16) do not change their form if the components are broken and/or change their location in an elementary cell. This means that Eqs. (15) and (16) describe the motion of any quasiperiodic (statistical heterogeneous) medium which has a constant mass content of components on the microlevel and the location of these components within the cell is not important.

In the case of nonlinear wave propagation, the individual components suffer different compressions. The structure of the medium is changed, with the result that the averaged specific volume $\langle V \rangle$ is changed. This change differs from the change of the specific volume for homogeneous medium under the same loading. Thus, the structure of medium is manifested in the wave motion, despite the fact that the equations of motion (15) (but not the equation of state) are written down for the averaged values u, p, and $\langle V \rangle$ only.

3.3 System of equations in Eulerian coordinates

In certain cases of theoretical analysis, it is more convenient to use the Eulerian coordinate system. The immediate employment of a method of the asymptotic averaging in Eulerian variables is impossible because of the variability of the microstructure sizes. However, from the zero approximation in the equations of motion (11), which are presented by the averaged values p, u, and $\langle V \rangle$, the equations can be rewritten in the Eulerian system of coordinates (r, t_E) utilizing a transformation from the Lagrangian system (s, t) [18–22]:

$$r = r(s,t), \quad t_E = t. \tag{17}$$

There is an essential presumption that the velocity of the particle in the zero approximation is constant throughout of the structure, and, consequently, we can describe an averaged trajectory for the particle:

$$\left(\frac{\partial r(s,t)}{\partial t} \right)_s = u(s,t). \tag{18}$$

From the physical point of view, it is clear that the position of the particle is unambiguously defined by its coordinate and time:

$$dr^{\nu} = A ds + \nu r^{\nu - 1} u dt, \qquad t_E = t. \tag{19}$$

From the mathematical point of view, this means that in the transformation (19) the value dr^{ν} is a total differential. Therefore, we must have

$$\frac{\partial A}{\partial t} = \frac{\partial \nu r^{\nu-1} u}{\partial s}. \tag{20}$$

This condition is satisfied if $A = \langle V \rangle$, because the equation converts into the continuity Eq. (15). We obtain the following transformation between Lagrangian and Eulerian systems of coordinates:

$$dr^{\nu} = \langle V \rangle ds + \nu r^{\nu-1} u dt, \qquad t_E = t. \tag{21}$$

It is reasonable to define the slow Lagrangian coordinate (non-mass one) as

$$R^{\nu} = s\langle V \rangle. \tag{22}$$

Eq. (15) in the Eulerian system of coordinates then take the form

$$\begin{aligned} &\frac{\partial \langle V \rangle^{-1}}{\partial t_E} + \frac{\partial r^{\nu-1} u \langle V \rangle^{-1}}{\partial r} = 0, \\ &\frac{\partial u}{\partial t_E} + u\frac{\partial u}{\partial r} + \langle V \rangle \frac{\partial p}{\partial r} = 0. \end{aligned} \tag{23}$$

It is convenient to determine the fast Eulerian coordinate ζ as

$$\left(\frac{\partial \zeta}{\partial \xi}\right)_t = \frac{\tilde{\rho}}{\rho(\xi)}. \tag{24}$$

It should be noted that the average density $\tilde{\rho}$ in the Eulerian coordinates is a value usually used for density. A chain of identities

$$\langle V \rangle = \int_0^1 V(\xi) d\xi = \int_0^1 V \frac{\rho}{\tilde{\rho}} d\zeta = \tilde{\rho}^{-1} \tag{25}$$

proves that $\langle V \rangle^{-1}$ is the average density of the medium in the Eulerian coordinates. Note that $\tilde{\rho} \neq \langle \rho \rangle$. The value $\tilde{\rho}$ is a real density. The value $\langle V \rangle$ is the specific volume averaged in units of mass over the period, and it is expressed as the ratio of the volume to the mass inside this volume. This value can be determined experimentally. At the same time, the averaged values p and u coincide in both Lagrangian and Eulerian systems of coordinates. Now the equations of motion (23) can be written in the usual form of the averaged density $\tilde{\rho}$.

The notation of the equations of motion in the averaged values enables us to suggest the method of the computer solution for the system of equations, where the integration step is restricted by the perturbation wavelength and not by the period of the structure [22]. Then the main computational problem associated with the smallness of the integration step can be avoided, and the equations of motion can be solved at a significant distance of wave propagation within a reasonable time.

3.4 Nonlinear waves

We will analyze the propagation of nonlinear waves in a structured medium. To make the results more clear, we will restrict our consideration to a nonrelaxation media $(c = c_f = c_e)$. The averaged equation of state in this case is simplified to the form

$$d\langle V\rangle = -\left\langle \frac{V^2}{c^2}\right\rangle dp, \tag{26}$$

and we can introduce an effective sound velocity by the formula

$$c_{\text{eff}} = \sqrt{\langle V\rangle^2 / \left\langle \frac{V^2}{c^2}\right\rangle}. \tag{27}$$

We obtain a traditional representation of the system of Eqs. (15) and (26).

The system of the equations is concerned in the hyperbolic type of a system. Now we restrict ourselves to the plane symmetry $(\nu = 1)$. Substituting the equation of state (26) into the equation of the continuity (the last equation in (15)), we get

$$\left\langle \frac{V^2}{c^2}\right\rangle \frac{\partial p}{\partial t} + \frac{\partial u}{\partial s} = 0. \tag{28}$$

The combination of this equation with the third equation in (15) $(\nu = 1)$ leads to the relationships

$$\left(\frac{\partial u}{\partial t} \pm \left\langle \frac{V^2}{c^2}\right\rangle^{1/2} \frac{\partial p}{\partial t}\right) \pm \left\langle \frac{V^2}{c^2}\right\rangle^{-1/2} \left(\frac{\partial u}{\partial s} \pm \left\langle \frac{V^2}{c^2}\right\rangle^{1/2} \frac{\partial p}{\partial s}\right) = 0. \tag{29}$$

From this relationship, it is seen that the averaged system of the equations pertains to the hyperbolic system. The equations for the characteristic in Lagrangian coordinates (mass space coordinate) have the forms

$$\frac{ds}{dt} = \pm \left\langle \frac{V^2}{c^2}\right\rangle^{-1/2}. \tag{30}$$

In characteristic, the relations are the following:

$$I_\pm = u \pm \int \left\langle \frac{V^2}{c^2}\right\rangle^{1/2} dp. \tag{31}$$

Analogously to the homogeneous medium, we call these relations as the Riemann invariants. The value (30) has the physical meaning, namely, it is the averaged velocity of the wave propagation in the Lagrangian coordinates. This velocity depends on pressure and integrally on a structure. Note the particular case. It is known that in vacuum the wave does not propagate. This result also follows formally from Eq. (30). The hyperbolism of a system points up that this system can describe the shock wave. The equations for the characteristic (30) and the Riemann invariants (31) are the integrodifferential equations, since they retain the variable $\langle V^2/c^2\rangle$, which depends on the properties of the structural elements in medium.

It should be noted that c_{eff} is not an averaged value, i.e., $c_{\text{eff}}^2 \neq \langle c^2\rangle$. Evidently, the structure of the medium introduces a certain contribution to the nonlinearity. In fact, even if $c_f \neq f(p)$, then in the general case, the value of c_{eff} is a function of pressure.

DOI: http://dx.doi.org/10.5772/intechopen.91462

The system of Eq. (15) is hyperbolic ones, and this specifies the breaking solutions, which are shock waves. For the analysis of such solutions, it is necessary to present Eq. (15) in the form of integral conservation laws:

$$\oint[\langle V\rangle ds + udt] = 0, \quad \oint[uds - pdt] = 0. \tag{32}$$

Now we can quickly formulate the conditions on the shock front, when there is conservation of the fluxes of mass and impulse through the shock front:

$$(\langle V_1\rangle - \langle V_0\rangle)D + u_1 - u_0 = 0, \quad (u_1 - u_0)D - p_1 + p_0 = 0, \tag{33}$$

where indexes 0 and 1 relate to the parameters of the flow before and after the front, respectively. Hence, the formula for the averaged velocity of the shock front in terms of the Lagrangian variable D (dimension $[D]$ is kg/s) and the mass velocity u follow from the following relations:

$$\begin{aligned} D &= \sqrt{(p_1 - p_0)/(\langle V_0\rangle - \langle V_1\rangle)}, \\ u_1 - u_0 &= \sqrt{(p_1 - p_0)(\langle V_0\rangle - \langle V_1\rangle)}. \end{aligned} \tag{34}$$

These systems are not expressed in the average hydrodynamical terms; hence the dynamical behavior of the medium cannot be modelled by a homogeneous medium even for long waves, if they are nonlinear. The structure of the medium influences the nonlinear wave propagation.

In the next section, it will be proven that the heterogeneity of the medium structure always introduces additional nonlinearity that does not arise in a homogeneous medium. This effect makes it possible to formulate the theoretical grounds of a new diagnostic method that determines the characteristics of a heterogeneous medium with the use of finite-amplitude long waves (inverse problem). This diagnostic method can also be employed to find the mass contents of individual components.

4. Diagnostics of a medium by long nonlinear waves

In the previous section, we proved that the long wave with finite amplitude responds to the structure of medium. At the same time, a question appears, namely, is there sufficient information in the wave field to reconstruct the structure of medium? It turns out that the knowledge on the evolution of nonlinear waves enables one to define with certain accuracy the concentrations of medium components.

4.1 The increase of nonlinearity in medium with structure

In this section, we shall prove the statement that the structure of medium always exalts the nonlinear effects under the propagation of long waves. At first, let us consider the sound velocity in homogeneous c_{hom} and heterogeneous c_{eff} media. Now we will show that in the general case with pressure increase the velocity of the sound becomes higher in a structured medium than in a homogeneous one:

$$c_{\text{eff}} \geq c_{\text{hom}}. \tag{35}$$

For the sake of clarity, we consider a medium in which the sound velocities of individual components are independent of the pressure:

$$c \neq f(p), \quad dc/dp = 0. \tag{36}$$

The equality sign is fulfilled (a) for an initial pressure, by virtue of the normalization, and also (b) for a special structured medium in which the relation $V(\xi)/c^2(\xi)$ is not a function on the fast variable ξ. We must prove which case results in equality and which gives the inequality.

Let us write the relations (36) for homogeneous medium consisting only one component:

$$c_{\text{hom}} \neq f(p), \quad dc_{\text{hom}}/dp = 0. \tag{37}$$

For multicomponent medium, the derivative dc_{eff}/dp is defined from the relationship

$$\frac{dc_{\text{eff}}}{dp} = \frac{2\langle V\rangle}{\langle V^2/c^2\rangle}\left(\langle V\rangle\left\langle\frac{V^3}{c^4}\right\rangle - \left\langle\frac{V^2}{c^2}\right\rangle^2\right) \geq 0. \tag{38}$$

This last inequality follows from the well-known Cauchy-Schwarz inequality (see, e.g., [31]). Therefore, with the increase of pressure, the sound velocity c_{eff} increases. Consequently, we have the inequality (35) at $p \geq p_0$.

Moreover, at $p > p_0$ the shock adiabatic curve for the medium with a structure always lies above that for the homogeneous medium (they touch only at the initial point $p = p_0$):

$$\frac{d^2p}{d\langle V\rangle^2} \geq \left(\frac{d^2p}{dV^2}\right)_{\text{hom}}. \tag{39}$$

Indeed, a ratio of these derivatives is equal to

$$\begin{aligned} \frac{d^2p}{d\langle V\rangle^2} \Big/ \left(\frac{d^2p}{dV^2}\right)_{\text{hom}} &= \frac{\langle V^3/c^4\rangle\langle V^2/c^2\rangle^{-3}}{c^2_{\text{hom}}\langle V\rangle^3} \\ &= \frac{\langle V^3/c^4\rangle\langle V\rangle c^2_{\text{eff}}}{c^2_{\text{hom}}\langle V^2/c^2\rangle^2} \geq \frac{\langle V^3/c^4\rangle\langle V\rangle}{\langle V^2/c^2\rangle^2} \geq 1. \end{aligned} \tag{40}$$

Hence, a long wave with a finite amplitude responds to the structure of the medium, and the nonlinear effects increase as compared with those in the homogeneous medium. The nonlinearity takes place even if individual components are described by the linear evolution equation (i.e., at condition (36)).

The exception, as it was noted already, is a medium with the properties of structure $V(\xi)/c^2(\xi) \neq f(\xi)$. For this medium, only the equality sign is correct in the inequalities (35) and (39). Particular elements of the structure respond to the pressure variations, but the relative structure does not change, i.e., the ratio $V(\xi,p)/V(\xi,p_0)$ does not depend on ξ. In this case, the value $c_{\text{eff}} = \sqrt{\langle c^2\rangle}$ is an averaged characteristic (see Eq. (27)). Therefore, the system of equations may be presented using the averaged variables p, u, $\langle V\rangle$, and $c_{\text{eff}} = \sqrt{\langle c^2\rangle}$. Heterogeneity does not introduce an additional nonlinearity for this medium, and the structure of medium does not affect the wave motion.

In addition to the analysis of the sound velocity in homogeneous and heterogeneous media, we consider now the evolution equations with the nonlinear term and compare the coefficients of nonlinearity in these media. Let us derive the evolution equation with weak nonlinearity. First of all, we have to note that the mass velocity u is related to the pressure p by means of [22]

$$u = \int_{p_o}^{p} \sqrt{\langle V^2/c^2 \rangle} dp. \tag{41}$$

Functional dependence of an average specific value on the pressure increment $p' = p - p_0$ with the accuracy $O(p'2)$ can be presented as a series:

$$\langle V \rangle (p) = \langle V \rangle_0 + \left. \frac{d\langle V \rangle}{dp} \right|_{p=p_0} p' + \frac{1}{2} \left. \frac{d^2 \langle V \rangle}{dp^2} \right|_{p=p_0} p'^2. \tag{42}$$

In this case, the system of Eq. (23) for planar symmetry $\nu = 1$ can be written as

$$\langle V \rangle_0 \frac{\partial u}{\partial x} + \left\langle \frac{V^2}{c^2} \right\rangle_0 \frac{\partial p'}{\partial t} - \frac{1}{2} \left. \frac{d^2 \langle V \rangle}{dp^2} \right|_{p=p_0} \frac{\partial p'^2}{\partial t} = 0, \tag{43}$$

$$\frac{\partial u}{\partial t} + \langle V \rangle_0 \frac{\partial p'}{\partial x} = 0. \tag{44}$$

The relationship $u \frac{\partial p'}{\partial x} = p' \frac{\partial u}{\partial x}$ follows from Eq. (41) with the assumed accuracy $O\left(p'^2\right)$ and was used for derivation of the first equation. The evolution equation for one variable assumes the form

$$\langle V \rangle_0^2 \frac{\partial^2 p'}{\partial x^2} - \left\langle \frac{V^2}{c^2} \right\rangle_0 \frac{\partial^2 p'}{\partial t^2} + \frac{1}{2} \left. \frac{d^2 \langle V \rangle}{dp^2} \right|_{p=p_0} \frac{\partial^2 p'^2}{\partial t^2} = 0. \tag{45}$$

Now let us consider the waves propagating in one direction, and then with the indicated accuracy, we can write (hereinafter index 0 is omitted):

$$-\frac{\sqrt{\langle V^2/c^2 \rangle}}{\langle V \rangle} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \to 2 \frac{\partial}{\partial x} \tag{46}$$

(see, e.g., Section 93 in Ref. [17]). Thus, after factorization of Eq. (45) we get

$$\frac{\partial p'}{\partial t} + c_{\text{eff}} \frac{\partial p'}{\partial x} + \frac{1}{2} \langle V \rangle \left\langle \frac{V^2}{c^2} \right\rangle^{-3/2} \frac{d^2 \langle V \rangle}{dp^2} p' \frac{\partial p'}{\partial x} = 0. \tag{47}$$

The coefficient of nonlinearity α_p for the structured medium, when the sound velocities in the individual components are independent of the pressure $c \neq f(p)$, can be presented as

$$\alpha_p \equiv \frac{1}{2} \langle V \rangle \left\langle \frac{V^2}{c^2} \right\rangle^{-3/2} \frac{d^2 \langle V \rangle}{dp^2} = \frac{d(u + c_{\text{eff}})}{dp} = \langle V \rangle \left\langle \frac{V^3}{c^4} \right\rangle \left\langle \frac{V^2}{c^2} \right\rangle^{-3/2}. \tag{48}$$

For all cases we take $\alpha_p > 0$. For a homogeneous medium with $dc/dp = 0$, we have $\alpha_{p\,\mathrm{hom}} = V/c$.

In certain media the value V/c^2 does not change within the period. The individual elements of the structure respond to the pressure variations so that a relative structure does not change, i.e., the ratio $V(\xi, p)/V(\xi, p_0)$ does not depend on ξ. In this case, the value $c_{\mathrm{eff}} = \sqrt{\langle c^2 \rangle}$ derived from Eq. (27) is the averaged characteristic. Consequently, the system of equations may be presented in the averaged variables p, u, $\langle V \rangle$, and $c_{\mathrm{eff}} = \sqrt{\langle c^2 \rangle}$. Heterogeneity does not introduce the additional nonlinearity for these media. Such media behave like the homogeneous media under the action of the nonlinear wave perturbations.

For media, when the sound velocity is independent of the pressure ($c \neq f(p)$), it is possible to show that heterogeneity of the medium, in the general case, introduces the additional nonlinearity. Let us consider the ratio of the nonlinearity coefficients for heterogeneous and homogeneous media. In the space of dimensionless normalized variables, this implies that at $p = p_0$ we have $\langle V \rangle_0 = 1$ as well as $\langle V^2/c^2 \rangle_0 = 1$ for the compared media.

Using the conditions (27) we can obtain

$$\frac{\alpha_p}{\alpha_{p\,\mathrm{hom}}} = \langle V \rangle \left\langle \frac{V^3}{c^4} \right\rangle \left\langle \frac{V^2}{c^2} \right\rangle^{-2} \geq 1. \tag{49}$$

This inequality is the well-known Cauchy-Schwarz inequality (see formula (15.2-3) in Ref. [31]). Since $\langle V \rangle \geq 0$ and $\langle V/c^2 \rangle \geq 0$, we prove

$$\begin{aligned} \langle V \rangle \langle V^3/c^4 \rangle \quad &\equiv \int_{-\infty}^{\infty} V d\xi \cdot \int_{-\infty}^{\infty} \frac{V^3}{c^4} d\xi = \int_{-\infty}^{\infty} \frac{V^2}{c^2} \left(\frac{V}{c^2} \right)^{-1} d\xi \cdot \int_{-\infty}^{\infty} \frac{V^2}{c^2} \frac{V}{c^2} d\xi \\ &\geq \left(\int_{-\infty}^{\infty} \sqrt{\frac{V^2}{c^2} \left(\frac{V}{c^2} \right)^{-1}} \cdot \sqrt{\frac{V^2}{c^2} \frac{V}{c^2}} d\xi \right)^2 \\ &= \left(\int_{-\infty}^{\infty} \frac{V^2}{c^2} d\xi \right)^2 \equiv \langle V^2/c^2 \rangle^2. \end{aligned} \tag{50}$$

It only remains to find the condition for the equality sign in (49). For this purpose, we apply the Cauchy-Schwarz inequality in vector form (see formula (15.2-5) in Ref. [31]):

$$\left| \left(\vec{a}, \vec{b} \right) \right|^2 \leq \left(\vec{a}, \vec{a} \right) \left(\vec{b}, \vec{b} \right). \tag{51}$$

However, the equality sign is realized if and only if the vectors $\vec{a}$ and $\vec{b}$ are linearly dependent, i.e., $\vec{a} = k\vec{b}$ ($k = \mathrm{const}$). By designating $\left(\vec{a}, \vec{a} \right) \equiv V/c^2$ and $\left(\vec{b}, \vec{b} \right) \equiv V^2/c^2$, it is easy to notice that the equality sign is realized if and only if

$$\sqrt{\frac{V^2}{c^2} \left(\frac{V}{c^2} \right)^{-1}} \bigg/ \sqrt{\frac{V^2}{c^2} \frac{V}{c^2}} = \mathrm{const}. \tag{52}$$

(see sections 14.2-6 in Ref. [31]), i.e., when the value $V/c^2 = \text{const}$ does not vary within the period $(V(\xi)/(c(\xi))^2 \neq f(\xi))$. This heterogeneous medium has been considered above. For all other heterogeneous media for which the value V/c^2 changes within period, the inequality is realized in Eq. (49). So, in a heterogeneous medium, the value α_p is always greater than $\alpha_{p\,\text{hom}}$ in a homogeneous medium. Thus, it is proved that, in the general case, the heterogeneities in a medium introduce the additional nonlinearity. This effect provides the basis for a new method of diagnostics to define the properties of multicomponent media using the propagation of long nonlinear waves in such media.

4.2 Fundamentals of new diagnostic method

The structure of the medium affects the wave field. There are different methods which allow the detection of gas bubbles and/or cracks in liquid [32], concrete [33], and ice cover [34] employing the nonlinear effects.

In this section, we describe our new diagnostic method for the properties of the medium. The features of the motion of finite-amplitude long waves and the effect of the increase of nonlinearity in the heterogeneous medium in comparison with homogeneous medium form the basis for the development of theoretical fundamentals of the diagnostic method. In this method, the properties of individual components are defined by long waves of finite amplitudes; more specifically, the dependence $V/c^2 = V/c^2(\zeta)$ on the fast Eulerian coordinate ζ (see Eq. (24)) is defined.

Thus, the nonlinear wave evolution allows one to obtain the structure of the medium with an inherent accuracy. As a final result, the mass concentrations of the individual components can be found using this method.

It should be kept in mind that the period of the structure of medium is infinitely small in the long-wave model, so it is not always possible to indicate the location of the structure elements inside the period reliably. Hence, the media with the different structures plotted in **Figure 2**, for example, affect identically on wave fields. These two media are indistinguishable in the framework of the suggested method. Taking into account this indefiniteness, we consider the function $V/c^2 = V/c^2(\zeta)$ that is to be the decreasing, integrable, mutually one-valued function on the interval $\zeta \in [0, 1]$, and equal to zero outside of this interval.

Now we represent the theoretical fundamentals for new method of diagnostics of medium by means of the long nonlinear waves. Let us prove the principal relation which enables us to obtain the inverse function $\zeta = \zeta(V/c^2)$ for the desired function $V/c^2 = V/c^2(\zeta)$ through the inverse Fourier transformation [19, 35–37]:

$$\zeta\left(Vc^{-2}\right) = F^{-1}\left[\sum_{n=0}^{\infty} \frac{\left\langle V\left(Vc^{-2}\right)^{n+1}\right\rangle}{(n+1)!\langle V\rangle} \mathrm{i}^n q^n\right]\left(Vc^{-2}\right). \tag{53}$$

It is known from theory of probability that the distribution function $f(x)$ (any one-valued, integrable, positive function) can be expressed by its central moments:

$$\alpha_n = \int_{-\infty}^{\infty} x^n f(x) dx. \tag{54}$$

Indeed, by using the characteristic function

$$\chi(q) = F[f(x)](q), \tag{55}$$

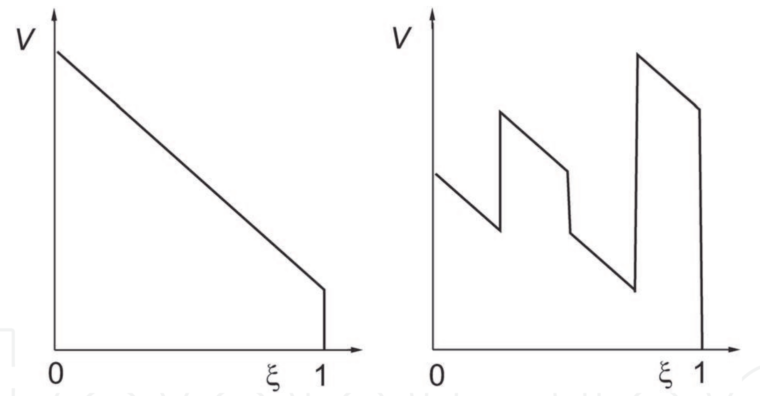

Figure 2.
The equivalent distributions of the specific volume in elementary sell for diagnostic method.

any positive integrable function $f(x)$ can be written as follows:

$$f(x) = F^{-1}[\chi(q)](x), \tag{56}$$

where $F[\cdot]$ is the Fourier transformation and $F^{-1}[\cdot]$ is the inverse Fourier transformation.

We take into account the important fact from the theory of probability: the characteristic function $\chi(q)$ is uniquely determined by the central moments α_n:

$$\chi(q) = \sum_{n=0}^{\infty} \alpha_n \mathrm{i}^n \frac{q^n}{n!}. \tag{57}$$

Hence, the function $f(x)$ can be found by means of the inverse Fourier transform:

$$f(x) = F^{-1}\left[\sum_{n=0}^{\infty} \alpha_n \mathrm{i}^n \frac{q^n}{n!}\right](x), \tag{58}$$

if series $\sum_{n=0}^{\infty} |\alpha_n| (s^n/n!)$ converges absolutely for some value $s > 0$ (see Section 18.3.7 in Ref. [31]).

These facts from the theory of probability are used to prove such a statement: if $V/c^2 = V/c^2(\zeta)$ is a decreasing positive integrable function on the interval $\zeta \in [0, 1]$ and equals to zero outside of it, then the inverse function $\zeta = \zeta(V/c^2)$ for the required function $V/c^2 = V/c^2(\zeta)$ can be written as (53) in the averaged values

$$\left\langle V\left(V/c^2\right)^n \right\rangle \equiv \int_{-\infty}^{\infty} V\left(V/c^2\right)^n d\xi. \tag{59}$$

Indeed, for the monotonic one-valued function $V/c^2 = V/c^2(\zeta)$, we find the integral in (59) by integrating the inverse function $\zeta = \zeta(V/c^2)$, since the transformation Jacobian is not equal to zero. We have the chain of identifies

$$\begin{aligned} \left\langle V\left(V/c^2\right)^n \right\rangle &= \int_0^1 V(\xi)\left(\frac{V}{c^2}\right)^n d\xi = \langle V \rangle \int_0^1 V\left(\frac{V}{c^2}\right)^n \rho d\zeta \\ &= \langle V \rangle \int_{-\infty}^{\infty} \left(\frac{V}{c^2}\right)^n \frac{d\zeta}{d(V/c^2)} d\left(V/c^2\right). \end{aligned} \tag{60}$$

In the geometric sense, this relation signifies that the integral (in our case, it is an area between the curve $V/c^2 = V/c^2(\zeta)$ and axes $O\zeta$ and $O(V/c^2)$) can be calculated either over ζ or over V/c^2 (see **Figure 2**). Whereas, the inequality is realized for the monotonic decreasing function $V/c^2 = V/c^2(\zeta)$.

For a function defined on a finite interval, if this function is positive and bounded above, we have

$$\begin{aligned} \left\langle V\left(V/c^2\right)^n\right\rangle &= \langle V\rangle \int_{-\infty}^{\infty} \left(\frac{V}{c^2}\right)^n \frac{d\zeta}{(dV/c^2)} d\left(V/c^2\right) \\ &= -n\langle V\rangle \int_{-\infty}^{\infty} \left(\frac{V}{c^2}\right)^{n-1} \zeta d\left(V/c^2\right). \end{aligned} \tag{61}$$

This relation provides the connection between the central moment α_n and the value $\left\langle V\left(V/c^2\right)^n\right\rangle$

$$\left\langle V\left(V/c^2\right)^n\right\rangle = -n\langle V\rangle \alpha_{n-1}. \tag{62}$$

Then the characteristic function $\chi(q)$ for the inverse function $\zeta = \zeta(V/c^2)$ is expressed through $\left\langle V\left(V/c^2\right)^n\right\rangle$. By applying the inverse Fourier transformation, finally, we find the required relationship (53).

The physical value Vc^{-2} is bounded by some constant M, hence

$$\alpha_n = \int_{-\infty}^{\infty} \left(Vc^{-2}\right)^{n-1} \zeta d\left(V/c^2\right) \le \int_{0}^{M} \left(Vc^{-2}\right)^n d\left(V/c^2\right) = \frac{M^{n+1}}{n+1}. \tag{63}$$

The series $\sum_{n=0}^{\infty} |\alpha_n| (s^n/n!) \le \sum_{n=0}^{\infty} M^{n+1} s^n/(n+1)!$ converge at $s < M^{-1}$. Consequently, the power series (53) also converges.

The coefficients $\left\langle V\left(V/c^2\right)^n\right\rangle$ $(n = 3, 4, \dots)$ in Eq. (53) can be easily calculated, if we know the functional dependence $\langle V\rangle(p)$ or $\left\langle V^2/c^2\right\rangle(p)$. Indeed, they can be successively defined by the recurrence relation

$$\frac{d\left\langle V\left(Vc^{-2}\right)^n\right\rangle}{dp} = -(n+1)\left\langle V\left(Vc^{-2}\right)^{n+1}\right\rangle, \tag{64}$$

that follows directly from the equation of state. With mentioned accuracy, it is possible to diagnose the structural properties of the medium.

We have proven the principal relation (53) for the method of diagnostics that allows one to find the properties of the individual components in structured media by means of the long nonlinear waves.

4.3 Approximation of diagnosed medium by layer medium

Diagnostics of the structured medium properties by the long nonlinear waves is connected with the definition of values $\left\langle V\left(V/c^2\right)^n\right\rangle$. As indicated above, there is a problem related to the accuracy of the description of the structure by finite series (53).

Now, we shall show that the partial sum of series (53) is a step-function and approximates the desired function $\zeta = \zeta(V/c^2)$ with certain accuracy, namely, the diagnosed medium can be approximated by a layer medium. Let us write down the chain of the identities for any integrable function:

$$2\pi f(-x) = F[F[f(x)](q)](x) = F\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}^n q^n}{n!}\alpha_n\right] \\ = \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\alpha_n}{n!}2\pi(-\mathrm{i})^n\delta^{(n)}(x). \tag{65}$$

Here we used the known relationships for the Fourier transform [31]):

$$F[F[f(x)](q)](x) = 2\pi f(-x), \tag{66}$$

$$F[q^n](x) = 2\pi(-\mathrm{i})^n\delta^{(n)}(x). \tag{67}$$

Hence, any integrable function can be represented by a series:

$$f(-x) = \sum_{n=0}^{\infty}\frac{\alpha_n}{n!}\delta^{(n)}(x). \tag{68}$$

We will prove that the finite series (53) approximates the desired function $f(x)$ by step-function. Consider the step-function $f_1(x)$ consisting of N steps:

$$f_1(x) = \begin{cases} \varphi_1, & 0 < x \le b_1, \\ \varphi_2, & b_1 < x \le b_2, \\ \vdots & \vdots \\ \varphi_N, & b_{N-1} < x \le b_N \end{cases} \tag{69}$$

in order to approximate the desired function $f(x)$. The relation (69) can be written down through the Heavyside functions as follows:

$$\begin{aligned} f_1(x) &= \varphi_1[\Theta(x) - \Theta(x-b_1)] + \varphi_2[\Theta(x-b_1) - \Theta(x-b_2)] + \ldots \\ &\quad +\varphi_N[\Theta(x-b_{N-1}) - \Theta(x-b_N)], \end{aligned} \tag{70}$$

Evidently, by increasing the number of steps N and choosing the values φ_i and b_i, any integrable function $f(x)$ can be approximated by the step-function $f_1(x)$. It is convenient to use a notation

$$\begin{aligned} f_1(-x) = \varphi_1[\Theta(x+b_1) - \Theta(x)] + \varphi_2[\Theta(x+b_2) - \Theta(x+b_1)] + \ldots \\ +\varphi_N[\Theta(x+b_N) - \Theta(x+b_{N-1})], \end{aligned} \tag{71}$$

that follows immediately from (70) after substitution:

$$\Theta(x) = 1 - \Theta(-x). \tag{72}$$

The Heavyside function $\Theta(x+b)$ can be expanded into a Taylor series in the neighborhood of point x:

$$\Theta(x+b) = \Theta(x) + \sum_{n=1}^{\infty}\frac{b^n}{n!}\Theta^{(n)}(x). \tag{73}$$

It is well-known that the derivative of Heavyside function $\Theta(x)$ is $\delta(x)$-function, then $\Theta^{(n+1)}(x) = \delta^{(n)}(x)$. We equate functions (68) and (71) and consider that the number of steps for function $f_1(x)$ is infinitely larger, and in this case we obtain

$$\begin{gathered}\varphi_1 \sum_{n=0}^{\infty} \frac{b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \varphi_2 \sum_{n=0}^{\infty} \frac{b_2^{n+1} - b_1^{n+1}}{(n+1)!}\delta^{(n)}(x) + \dots + \\ + \varphi_N \sum_{n=0}^{\infty} \frac{b_N^{n+1} - b_{N-1}^{n+1}}{(n+1)!}\delta^{(n)}(x) + \dots = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}\delta^{(n)}(x).\end{gathered} \tag{74}$$

This relationship shows that when we use the partial sum of series on the right-hand side of Eq. (74) $\sum_{n=0}^{2N-1} \frac{\alpha_n}{n!}\delta^{(n)}(x)$ and also the N leading terms on the left-hand side, then the desired function $f(x)$ is approximated by the step-function $f_1(x)$ with N steps. In other words, if it is necessary to restore the structure of medium by means of N periodic repeated layers, then we need to know the $2N-1$ moments α_n, i.e., the values $\left\langle V\left(Vc^{-2}\right)^n\right\rangle$.

For the sake of convenience, we write down the relation (74) in the expanded form. For this purpose, we multiply it by x^n and integrate over x. We obtain the nonlinear system of the equations in the unknowns $b_1, b_2, \dots, b_N, \varphi_2, \varphi_3$, and $\dots, \varphi_N$ (variable $\varphi_1 = 1$ owing to normalization):

$$\begin{aligned}&\varphi_1 b_1 + \varphi_2(b_2 - b_1) + \varphi_3(b_3 - b_2) + \dots + \varphi_N(b_N - b_{N-1}) = \alpha_0, \\ &\varphi_1 b_1^2 + \varphi_2\left(b_2^2 - b_1^2\right) + \varphi_3\left(b_3^2 - b_2^2\right) + \dots + \varphi_N\left(b_N^2 - b_{N-1}^2\right) = 2\alpha_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\varphi_1 b_1^{2N-1} + \varphi_2\left(b_2^{2N-1} - b_1^{2N-1}\right) + \varphi_3\left(b_3^{2N-1} - b_2^{2N-1}\right) + \dots + \\ &\qquad + \varphi_N\left(b_N^{2N-1} - b_{N-1}^{2N-1}\right) = (2N-1)\alpha_{2N-2}.\end{aligned} \tag{75}$$

Now, if b_i implies the partition of $(V/c^2)_i$ and φ_i implies the partition of ζ_i, we can obtain the system of Eq. (75) to define the structure of medium. Solution of these equations gives the information about the component properties of the medium, namely, the value V/c^2 on the structure period $\zeta \in [0, 1]$ is found in the form of the step-function.

Let us note the special case of a periodic medium for which the value V/c^2 is constant within the period. This medium, as we already know, does not differ from a homogeneous one for the propagation of the long nonlinear waves. The same result follows from a system (75). Indeed, for homogeneous media the moments α_n are equal to

$$\alpha_n = \frac{\left\langle V\left(Vc^{-2}\right)^{n+1}\right\rangle}{(n+1)\langle V\rangle} = \frac{b^{n+1}}{n+1}. \tag{76}$$

Here, the conditions of normalization $\left\langle V^2/c^2\right\rangle_0 = \left(V^2/c^2\right)_0 = 1$ and $\langle V\rangle_0 = V_0 = 1$ have been used as before. Therefore, the values in the right-hand side of Eq. (75) are equal to $b \equiv Vc^{-2} = \text{const}$. It is easy to see that the solution of system is $b_1 = b_2 = \dots = b_N = b = 1$ and $\varphi_1 = 1$ (where φ_i is any value for $i \geq 2$). This corresponds to the layer medium, for which $V/c^2 \neq f(\zeta)$, in particular, this medium can be a homogeneous one.

According to the asymptotic averaged model of a structured medium, the period of the structure is infinitely small, and this diagnostic method cannot give the exact

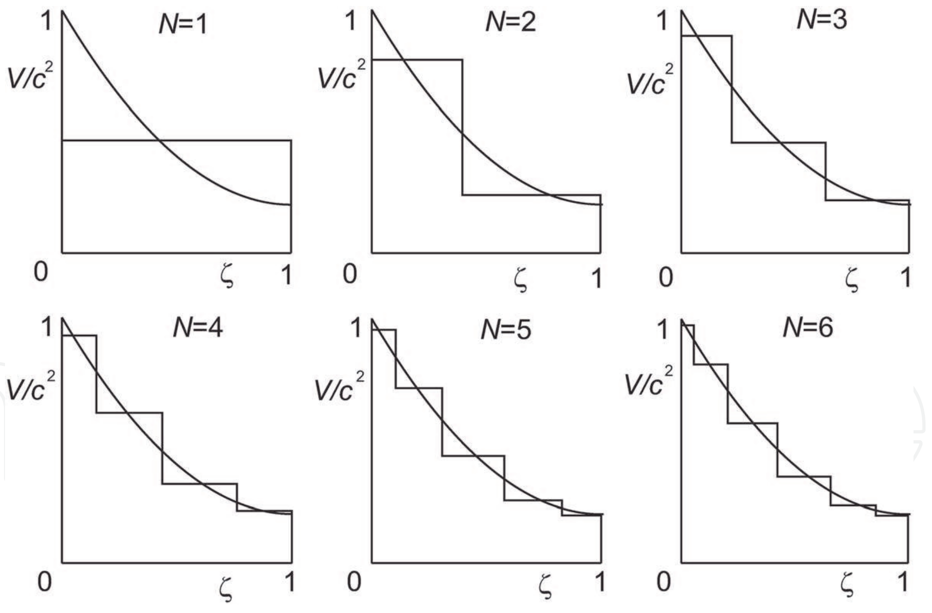

Figure 3.
Approximation of the diagnosed medium $V/c^2 = 0.2 + 0.8(1 - \zeta)^2$ *by* N-*component medium.*

location of the structure elements inside the period. Hence, using this method, only the mass contents of the particular components can be determined.

We present, as an example, the results of the calculation to define the structure of layer media, which can properly approximate the diagnosed medium. The structure of the diagnosed medium is $V/c^2 = 0.2 + 0.8(1 - \zeta)^2$ in **Figure 3**. In order to approximate the diagnosed medium by layer periodic medium, which has N layers within the period, it is necessary to know $2N - 1$ values $\left\langle V\left(Vc^{-2}\right)^n\right\rangle$ for finite series (53). If we regard that the $2N - 1$ averaged characteristics $\left\langle V\left(Vc^{-2}\right)^n\right\rangle$ coincide for the diagnosed medium and the layer medium, these averaged values at $n \le 2N - 1$ can be calculated from the known distributions $V/c^2 = 0.2 + 0.8(1 - \zeta)^2$. At $n > 2N - 1$ the values $\left\langle V\left(Vc^{-2}\right)^n\right\rangle$ for diagnosed medium and for approximated layer medium are different. The distributions of $V/c^2(\zeta)$ within the period for diagnosed medium and for approximated media with N components are shown in **Figure 3**. On the one hand, the calculated distributions for layered media are the best approximation for the medium we test. On the other hand, we have illustrated the accuracy of the approximation of the diagnosed medium by the finite series (53).

Thus, the new method for the diagnostics of the medium characteristics by long nonlinear waves is suggested on the basis of the asymptotic averaged model of the structured medium. The mass contents of the particular components can be denoted by the abovementioned diagnostic method.

5. Conclusion

The asymptotic averaged model is suggested for the description of the wave processes in nonequilibrium heterogeneous media. The obtained integral differential system of equations cannot be reduced to the average terms (pressure, mass velocity, specific volume) and contains the terms with characteristic sizes of individual components.

On the microstructure level of the medium, the dynamical behavior is governed only by the laws of thermodynamics. On the macrolevel, the motion of the medium can be described by the wave-dynamical laws for the averaged variables with the integrodifferential equation of state containing the characteristics of the medium microstructure. A rigorous mathematical proof is given to show that finite-amplitude long waves respond to the structure of the medium in such a way that the homogeneous medium model is insufficient for the description of the behavior of the structured medium. An important result that follows from this model is that, for a finite-amplitude wave, the medium structure (in particular, existence of microcracks) produces nonlinear effects even if the individual components of the medium are described by a linear law. Finding the wave fields in the structured medium is the direct problem, on the one hand.

On the other hand, the system analyzed here is not expressed in the average hydrodynamical terms; hence the dynamical behavior of the medium cannot be modelled by a homogeneous medium even for long waves, if they are nonlinear. The heterogeneity of the medium structure always introduces additional nonlinearity that does not arise in a homogeneous medium. This effect enabled one to formulate the theoretical grounds of a new diagnostic method that determines the characteristics of a heterogeneous medium with the use of finite-amplitude long waves (inverse problem). This diagnostic method can also be employed to find the mass contents of individual components.

Additional information

Physics and Astronomy Classification Scheme (PACS): 43.25.Zx, 43.25.Ba, 43.25.Ed, 43.25.Gf

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References

[1] Bereznev I, Nikolaev A. Experimental investigations of nonlinear seismic effects. Physics of the Earth and Planetary Interiors. 1988;**50**: 83-87. DOI: 10.1016/0031-9201(88)90096-9

[2] Bonner B, Wanamaker B. Acoustic nonlinearities produced by a single macroscopic fracture in granite. In: Thompson D, Chimenti D, editors. Review of Progress in Quantitative Nondestructive Evaluation. Vol. 10B. Boston, MA: Springer; 1991. pp. 1861-1867. DOI: 10.1007/978-1-4615-3742-7_94

[3] Guyer R, Johnson P. Nonlinear Mesoscopic Elasticity: The Complex Behaviour of Rocks, Soil, Concrete. Weinheim: Wiley-VCH Verlag GmbH and Co. KgaA; 2009. p. 395. ISBN: 978-3-527-40703-3

[4] Johnson P, McCall K. Observation and implications of nonlinear elastic wave response in rock. Geophysical Research Letters. 1994;**21**:165-168. DOI: 10.1029/93GL03162

[5] Johnson P, Shankland T, O'Connell R, Albright J. Nonlinear generation of elastic waves in crystalline rock. Journal of Geophysical Research. 1987;**92**:3597-3602. DOI: 10.1029/JB092iB05p03597

[6] Nesterenko V. Dynamics of Heterogeneous Materials. New York: Springer-Verlag; 2001. p. 510. ISBN: 978-1-4757-3524-6

[7] Ostrovsky L. Wave processes in media with strong acoustic nonlinearity. The Journal of the Acoustical Society of America. 1991;**90**:3332-3337. DOI: 10.1121/1.401444

[8] Sadovsky M, Pisarenko G. Seismic Process in Block Medium. Moscow: Nauka; 1991. p. 95 (in Russian). ISBN: 5-02-000776-5

[9] Achenbach J. Wave Propagation in Plastic Solids. Amsterdam: North-Holland; 1973. p. 440. ISBN: 9780720403251

[10] Aki K, Richards P. Quantitative Seismology: Theory and Methods. Vol. I and II. San Francisco: W.H. Freeman and Co.; 1980. DOI: 10.1002/gj.3350160110

[11] Truesdell C. Rational Thermodynamics. New York: Springer-Verlag; 1984. p. 578. ISBN: 978-1-4612-5206-1

[12] Biot M. Theory of propagation of elastic waves in a fluid-saturated solid. I. Low-frequency range. The Journal of the Acoustical Society of America. 1956; **28**:168-178. DOI: 10.1121/1.1908239

[13] Lyakhov G. Waves in Soils and Porous Multicomponent Media. Moscow: Nauka; 1982. p. 286. Available from: http://elib.pstu.ru/vufind/Record/RUPSTUbooks151687

[14] Raats P. Applications of the theory of mixtures in soil physics. In: Truesdell C, editor. Rational Thermodynamics. New York: Springer-Verlag; 1984. pp. 326-343. ISBN: 978-1-4612-5206-1

[15] Rajagopal K, Tao L. Mechanics of Mixtures. Singapore: World Scientific Publishing; 1995. p. 195. ISBN-10: 9810215851

[16] Nigmatulin R. Dynamics of Multiphase Media. Vols. 1 and 2. Boca Raton, Florida: CRC Press; 1990. p. 878. ISBN-10: 1560322071

[17] Landau L, Lifshitz E. Fluids Mechanics. New York: Pergamon Press; 1987. p. 554. ISBN: 9781483161044

[18] Vakhnenko V, Danilenko V, Michtchenko A. An asymptotic averaged model of nonlinear long waves propagation in media with a regular structure. International Journal of Non-Linear Mechanics. 1999;**34**: 643-654. DOI: 10.1016/S0020-7462(98) 00014-6

[19] Vakhnenko V, Danilenko V, Michtchenko A. Diagnostics of the medium structure by long wave of finite amplitude. International Journal of Non-Linear Mechanics. 2000;**35**: 1105-1113. DOI: 10.1016/S0020-7462 (99)00082-7

[20] Vakhnenko V. Blast waves in multi-component medium with thermal relaxation. Natural Science. 2014;**6**: 1055-1092. DOI: 10.4236/ ns.2014.612096

[21] Vakhnenko V, Danilenko V, Kulich V. Averaged description of shock-wave processes in periodic media. Soviet Journal of Chemical Physics. 1994;**12**(3):534-546. ISSN: 0733-2831

[22] Vakhnenko V, Kulich V. Long-wave processes in periodic media. Journal of Applied Mechanics and Technical Physics. 1992;**32**:814-820. DOI: 10.1007/ BF00851969

[23] Lavrentiev M, Chabat B. Effets hydrodynamiques et modéles mathématiques. Moscow: Éditions Mir; 1980. p. 365. OCLC Number: 301487224

[24] Clarke J. Lectures on plane waves in reacting gases. Annales de Physique. 1984;**9**:211-306. DOI: 10.1051/anphys: 0198400902021100

[25] Danylenko V, Sorokina V, Vladimirov V. On the governing equations in relaxing models and self-similar quasiperiodic solutions. Journal of Physics A: Mathematical and General. 1993;**26**:7125-7135. DOI: 10.1088/ 0305-4470/26/23/047

[26] Yasnikov G, Belousov V. Effective thermodynamic functions of a gas with solid particles. Journal of Engineering Physics. 1978;**34**:734-737. DOI: 10.1007/ BF00860421

[27] Bakhvalov N, Panasenko G. Homogenisation: Averaging Processes in Periodic Media. Boston: Kluwer Academic Publishers; 1989. p. 366. DOI: 10.1007/978-94-009-2247-1. ISBN: 978-0-7923-0049-6

[28] Bakhvalov N, Eglit M. Processes in periodic media not described in terms of averaged characteristics. Doklady Akademii Nauk SSSR (transi, as Soviet Math. Dokl.). 1983;**268**:836-840. Available from: http://mi.mathnet.ru/e ng/dan/v268/i4/p836

[29] Berdichevsky V. Variational Principles of Continuum Mechanics. Berlin-Heidelberg: Springer-Verlag; 2010. p. 900. ISBN: 978-3-540-88465-1

[30] Sanchez-Palencia E. Non-Homogeneous Media and Vibration Theory. Berlin, New York: Springer-Verlag; 1980. p. 398. ISBN: 978-3-540-10000-3

[31] Korn G, Korn T. Mathematical Handbook for Scientists and Engineers. New York, San Francisco, Toronto, London, Sydney: McGraw-Hill Book Company; 1968. p. 720. DOI: 10.1002/ zamm.19690490921

[32] Fenlon F, Wonn J. On the amplification of modulated acoustic waves in gas-liquid mixtures. In: Lauterborn W, editor. Cavitation and Inhomogeneities in Underwater Acoustics. Berlin, Heidelberg: Springer; 1980. pp. 141-150. DOI: 10.1007/ 978-3-642-51070-0. ISBN: 978-3-642-51072-4

[33] Schkolnik I. Nondestructive testing of concretes: New aspects. Nondestructive Testing and Evaluation.

1993;**10**:351-358. DOI: 10.1080/10589759308952807

[34] Ekimov A, Lebedev A, Ostrovskii L, Sutin A. Nonlinear acoustic effects due to cracks in ice cover. Acoustical Physics. 1996;**42**:51-54. ISSN: 10637710

[35] Vakhnenko V, Danilenko V. Nonlinear waves as a technical tool for diagnostics of the medium structure. In: Proceedings of 10th International Congress on Sound and Vibration. Stockholm: International Institute of Acoustics and Vibration; 2003. pp. 3549-3556. Available from: http://www.igph.kiev.ua/vakhnenko/vakhnenko.html (N 31)

[36] Vakhnenko V. Diagnosis of the properties of a structurized medium by long nonlinear waves. Journal of Applied Mechanics and Technical Physics. 1996;**37**(N5):643-649. ISSN: 0021-8944

[37] Vakhnenko V, Danilenko V. An asymptotic model of non-linear waves in natural multi-component media. Geophysical Journal. 1997;**16**:674-685. ISSN: 02759128