Chapter

Complex Space Nature of the Quantum World: Return Causality to Quantum Mechanics

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Abstract

As one chapter, we about to begin a journey with exploring the limitation of the causality that rules the whole universe. Quantum mechanics is established on the basis of the phenomenology and the lack of ontology builds the wall which blocks the causality. It is very difficult to reconcile the probability and the causality in such a platform. A higher dimension consideration may leverage this dilemma by expanding the vision. Information may seem to be discontinuous or even so weird if only be viewed from a part of the degree of freedoms. Based on this premise, we reexamined the microscopic world within a complex space. Significantly, some knowledge beyond the empirical findings is revealed and paves the way for a more detailed exploration of the quantum world. The random quantum motion is essential for atomic particle and exhibits a wave-related property with a bulk of trajectories. It seems we can break down the wall which forbids the causality entering the quantum kingdom and connect quantum mechanics with classical mechanics. The causality returns to the quantum world without any assumption in terms of the quantum random motion under the optimal guidance law in complex space. Thereby hangs a tale, we briefly introduce this new formulation from the fundamental theoretical description to the practical technology applications.

Keywords: random quantum trajectory, optimal guidance law, complex space

1. Introduction

It took scientists nearly two centuries from first observation of flower powder's Brownian motion to propose a mathematical qualitative description [1]. Time is an arrow launched from the past to the future, every event happens for a reason. "The world is woven from billions of lives, every strand crossing every other. What we call premonition is just movement of the web. If you could attenuate to every strand of quivering data, the future would be entirely calculable. As inevitable as mathematics [2]." All physical phenomena are connected to the same web. As long as we can see through the quivering data and cut into the very core, we can glimpse the most elegant beauty of nature. As precise as physics.

It took nearly 30 years for physicists to establish quantum mechanics but nearly 100 years to seek for its essence. Quantum mechanics is the most precise theory to describe the microscopic world but also is the most obscure one among all theories. It collects lots data but not all. Just like what we can observed is the shadow on the ground not the actual object in the air. It is impossible to see the whole appearance of the object by observing its shadow. The development of the quantum era seems started in such circumstances and missed something we call the essence of nature. In this chapter, we hope to recover the missing part by considering a higher dimension to capture the actual appearance of nature. At the end, we will find out that nature dominates the web where we live as well as the theories we develop. Everything should follow the law of the nature, and there is no exception.

Trajectory is a typical classical feature of the macroscopic object solved by the equation of motion. The trajectory of the microscopic particle is supposed to be observed if the law of nature remains consistent all the way down to the atomic scale. However, such an observation cannot be made till 2011. Kocsis and his coworkers propose an observation of the average trajectories of single photons in a two-slit interferometer on the basis of weak measurement [3]. Since then quantum trajectories are observed for many quantum systems, such as superconducting quantum bit, mechanical resonator, and so on [4–6]. Weak measurement provides the weak value which is a measurable quantity definable to any quantum observable under the weak coupling between the system and the measurement apparatus [7]. The significant characteristic of the weak value does not lie within the range of eigenvalues and is complex. It is pointed out that the real part of the complex weak value represents the average quantum value [8], and the imaginary part is related to the rate of variation in the interference observation [9].

The trajectory interpretation of quantum mechanics is developed on the basis of de Broglie's matter wave and Bohm's guidance law. In recent years, the importance of the quantum trajectory in theoretical treatment and experimental test has been discussed in complex space [10–21]. All these research indirectly or directly show that the complex space extension is more than a mathematical tool, it implies a causal essence of the quantum world.

On the other hand, it is found out that the real part of momentum's weak value is the Bohmian momentum representing the average momentum conditioned on a position detection; while its imaginary part is proportional to the osmotic velocity that describes the logarithmic derivative of the probability density for measuring the particular position directed along the flow generated by the momentum [22]. This not only implies the existence of randomness in a quantum system, but also discloses that the random motion occurs in complex space. Numerous studies with the complex initial condition and the random property have been discussed [23–25]. A stochastic interpretation of quantum mechanics is proposed which regards the random motion as a nature property of the quantum world not the interference made by the measurement devices [26, 27]. These investigations suggest that a complex space and the random motion are two important features of the quantum world.

Based on the complex space structure, we propose a new perspective of quantum mechanics that allows one to reexamine quantum phenomena in a classical way. We will see in this chapter how the quantum motion can provide the classical description for the quantum kingdom and is in line with the probability distribution. One thing particular needed to be emphasized is that the stochastic Hamilton Jacobi Bellman equation can reduce to the Schrödinger equation under some specific conditions. In other words, the Schrödinger equation is one special case of all kinds of random motions in complex space. A further discussion of the relationship between the trajectory interpretation and probability interpretation is presented in Section 2. In particular, the solvable nodal issue is put into discussion, and the continuity equation for the complex probability density function is proposed. In Section 3, we demonstrate how the quantum force could play the crucial role in the force balanced condition within the hydrogen atom and how the quantum potential forms the shell structure where the orbits are quantized. A practical application to the Nano-scale is demonstrated in Section 4. We consider the quantum potential relation to the electronic channel in a 2D Nano-structure. In addition, the conductance quantization is realized in terms of the quantum potential which shows that the lower potential region is where the most electrons pass through the channel. And then, concluding remarks are presented in Section 5.

2. Random quantum motion in the complex plane

In the macroscopic world, it is natural to see an object moving along with a specific path which is determined by the resultant optimal action function. However, in the microscopic world, we cannot repetitively carry out this observation since there is no definition of the trajectory for a quantum particle. With the limit on the observation, only a part of trajectory, more precisely, the trajectory in the real part of complex space can be detected. As particle passing or staying in the imaginary part of complex space it disappears from our visible world and becomes untraceable. The particle randomly transits in and out of the real part and imaginary part of complex space, causes a discontinuous trajectory viewed from the observable space. Therefore, it can only be empirically described by the probability in quantum mechanics.

In this section, we briefly introduce how particle's motion can be fully described by the optimal guidance law in the complex plane [28]. Then we will discuss under what condition the statistical distribution of an ensemble of trajectories in the complex plane will be compatible with the quantum mechanical and classical results. In the following, we consider a complex plane for the purpose of simplicity; however, there should be no problem to implement the optimal guidance law in complex space. Let us consider a particle with random motion in the complex plane whose dynamic evolution reads

$$dx = f(t, x, u)dt + g(x, u)dw, \quad x = x_R + ix_I \in \mathbb{C}, \tag{1}$$

where x represents a vector, u is the guidance law needed to be determined, w is Wiener process with properties $\langle dw \rangle = 0$ and $\langle dw^2 \rangle = dt, f(t, x, u)$ is the drift velocity, and $g(x, u)$ is the diffusion velocity. The cost function for $x(t)$ with randomness property reads

$$J(t, x, u) = E_{t,x}\left[\int_t^{t_f} L(\tau, x(\tau), u(\tau))d\tau\right], \tag{2}$$

where $E_{t,x}$ represents the expectation of the cost function over all infinite trajectories launched from the single initial condition, $x(t) = x$ in time interval $[t, t_f]$. To find the minimum cost function, we define the value function,

$$V(t, x) = \min_{u[t, t_f]} J(t, x, u). \tag{3}$$

Instead of using the variational method, we apply the dynamic programming method to Eq. (3) for the random motion. We then have the following expression after having the Taylor expansion:

$$-\frac{\partial V(t, x)}{\partial t} = \min_{u[t, t_f]}\left\{L + \frac{\partial V(t, x)}{\partial x}f + \frac{1}{2}tr\left[g^T(x, u)\frac{\partial^2 V(t, x)}{\partial x^2}g(x, u)\right]\right\}, \tag{4}$$

which is recognized as the Hamilton-Jacobi-Bellman (HJB) equation and $\partial^2 V(t,x)/\partial x^2$ is Jacobi matrix. Finding the minimum of the cost function leads to the momentum for the optimal path,

$$p = \frac{\partial L(t,x,u)}{\partial u} = \frac{\partial L(t,x,\dot{x})}{\partial \dot{x}} = -\nabla V(t,x), \tag{5}$$

and determines the optimal guidance law,

$$u = u(t,x,p)|_{p=-\nabla V}. \tag{6}$$

If one replaces Lagrange L by Hamiltonian $H(t,x,p) = p^T u - L(t,x,u)$, defines the action function as $S(t,x) = -V(t,x)$ and let $g(x,u) = \sqrt{-i\hbar/m}$, Eq. (4) can be transferred to the quantum Hamilton-Jacobi (HJ) equation,

$$\frac{\partial S}{\partial t} + H(t,x,p)|_{p=\nabla S} + \frac{i\hbar}{2m}\nabla^2 S = 0. \tag{7}$$

Please notice that the last term in Eq. (7) is what makes the quantum HJ equation differs from its classical counterpart. It is called the quantum potential,

$$Q = \frac{i\hbar}{2m}\nabla^2 S \tag{8}$$

in dBB theory, Bohmian mechanics, and quantum Hamilton mechanics [29–33]. Even the quantum potential we derive here has the same expression appeared in Bohmian mechanics, its relation to the random motion should be noticed. However, it is not yet suitable to claim that the random motion attributes to the quantum potential or vice versa. It is worthwhile to bring into discussion. Before inspecting this question more deeply, we still can take advantage of the quantum potential to describe or even explain some quantum phenomena.

We can transfer the quantum HJ equation (7) to the Schrödinger equation,

$$\mathrm{i}\hbar \frac{\partial \Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi(t,x) + U\Psi(t,x) \tag{9}$$

via the relation between the action function and wave function,

$$S(t,x) = -i\hbar \ln \Psi(t,x), \tag{10}$$

where U represents the external potential. This simple relation reveals a connection between the trajectory and the wave description. In classical mechanics, a particle follows the principle of least action; while the wave picture took place in quantum mechanics. Eq. (10) implies that if we collect all action functions determined by different initial conditions which satisfy the initial probability distribution, a collection of corresponding wave patterns arise and eventually forms the solution wave function of the Schrödinger equation. This process is the same as what Schrödinger attempted to cope with the observable wave and tried to deduce the suitable wave equation based on the classical wave theory. The only difference is that Schrödinger started his deduction from the wave perspective; however, we start from the particle perspective. Even the wave-particle duality troubles physicists to inspect advanced about the essence of nature, the recent experiment confirms relation (10) by observing an ensemble of quantum trajectories [3].

This becomes a solid evidence to support the deduction that the matter wave is formed by a huge number of trajectories.

To fully understand the property of these trajectories under the influence of the guidance law, we consider a particle experiencing a randomness,

$$dx = u(t, x, p)dt + \sqrt{\frac{-i\hbar}{m}}dw, \tag{11}$$

where we have replaced $f(t, x, u)$ by the optimal guidance law $u(t, x, p)$, and assigned $g(x, u) = \sqrt{-i\hbar/m}$ into Eq. (1). Combining Eqs. (6) and (10), the optimal guidance law can be expressed in terms of the wave function,

$$u(t, x, p) = \frac{-i\hbar}{m}\frac{\nabla\Psi(t, x)}{\Psi(t, x)}. \tag{12}$$

Therefore, Eq. (11) can be recast into the following expression:

$$dx = \frac{-i\hbar}{m}\frac{\nabla\Psi(t, x)}{\Psi(t, x)}dt + \sqrt{\frac{-i\hbar}{m}}dw. \tag{13}$$

Eq. (13) will reduce to the equation of motion given by the quantum HJ equation (7) if we take the average of both sides,

$$\dot{x} = \frac{-i\hbar}{m}\frac{\nabla\Psi(t, x)}{\Psi(t, x)}, \tag{14}$$

since the random motion in Eq. (13) has zero mean. This result shows that the quantum HJ equation represents the mean motion of the particle. The trajectory in the complex plane solved from Eq. (13) is random and will become the mean trajectory solved from Eq. (14) after being averaged out. **Figure 1** illustrates this property by demonstrating the quantum motion of the Gaussian wave packet [28].

The first question we would like to answer by the complex random trajectory (CRT) interpretation is its connection to the probability interpretation. In quantum mechanics, the amplitude square of the wave function gives the probability density of physical quantities as shown in **Figure 2(a)**, in which the solid line stands for the quantum harmonic oscillator in $n = 1$ state. The trajectory interpretation is supported by the excellent agreement of the statistical spatial distribution made by collecting all crossovers on the real axis of an ensemble of CRTs as the dots displayed in **Figure 2(a)**. It shows a good agreement of the statistical spatial distribution and the quantum mechanical probability distribution [36].

In most text book of quantum mechanics, the nodes of the probability of harmonic oscillator either be ignored or be regarded as the quantum characteristic. Only the classical-like curve of the averaged probability has been mentioned. The other significant finding brought out by the CRT interpretation is the nodal vanished condition given by the statistical distribution of the collection of all pointes be projected onto the real axis as **Figure 2(b)** shows. It starts to approach the classical probability distribution for high quantum number as **Figure 2(c)** presents. The leverage of complex space structure deals with the probability nodes, and even further to reach the classical region dominated by Newtonian mechanics (more detail refers to [36]). After the matter wave can be interpreted by an ensemble of trajectories in both theoretical and experimental results [3, 18, 34, 35], the CRT interpretation shows both quantum mechanical and classical compatible

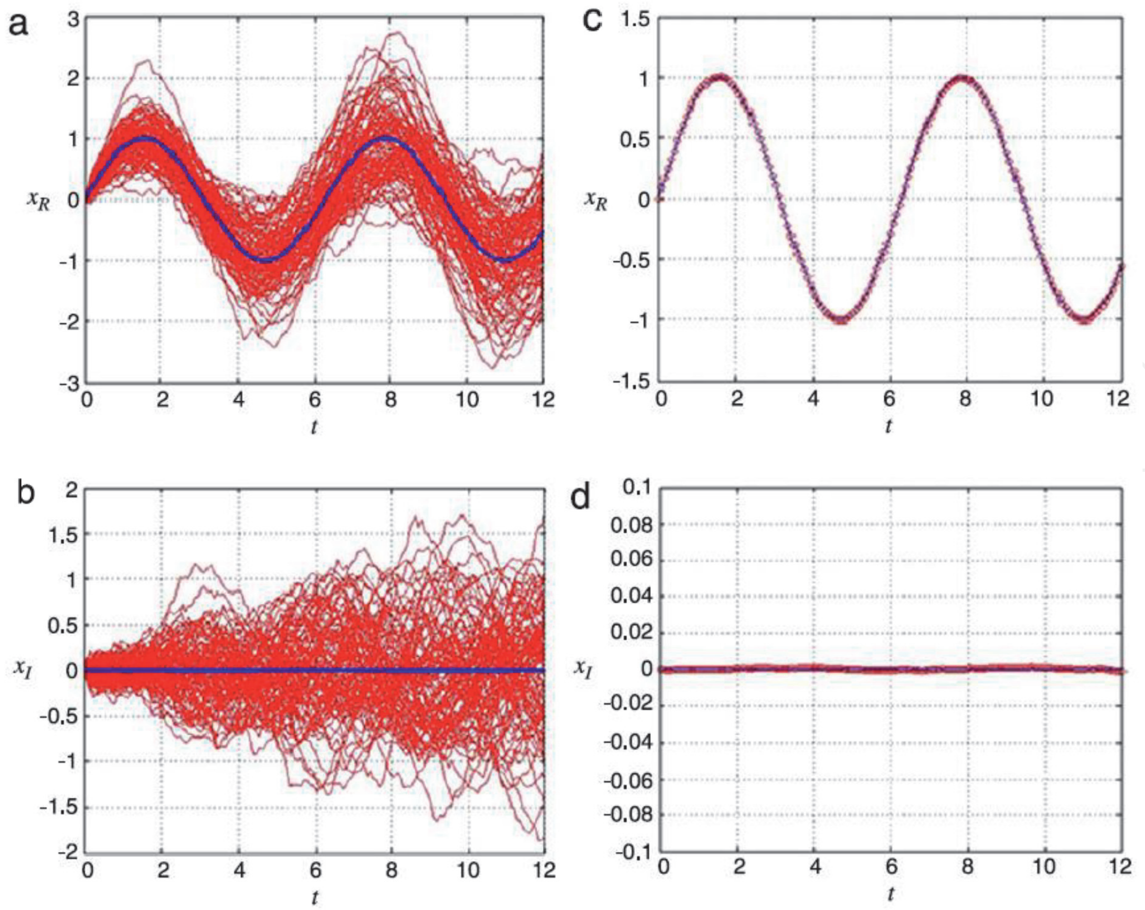

Figure 1.
100,000 trajectories solved from Eq. (13) with the same initial condition of the Gaussian wave packet in the complex plane: (a) the time evolution on the real axis for which the mean is denoted by the blue line; (b) the time evolution on the imaginary axis with zero mean represented by the blue line. The complex trajectory solved from Eq. (14) with one initial condition: (c) the time evolution on the real axis; (d) the time imaginary part of the motion. This figure reveals that the mean of the CRT is the trajectory solved from the quantum Hamilton equations of motion [28].

results under two kinds of point collections. In other words, Bohr's correspondence principle can be interpreted by the CRT interpretation without loss of generality [36].

The second question we try to cope with by means of the CRT interpretation is the conservation of the complex probability. In quantum mechanics, the continuity equation for the probability density function is given by Bohr's law $\rho_{QM} = |\Psi|^2$, and the current density J,

$$\frac{\partial \rho_{QM}}{\partial t} = -\nabla \cdot J. \tag{15}$$

The probability density function of the CRT interpretation satisfies the Fokker-Planck equation,

$$\frac{\partial \rho(t,x)}{\partial t} = -\nabla \cdot \left(\dot{x}(t,x)\rho(t,x)\right) - \frac{i\hbar}{2m}\nabla^2 \rho(t,x), \tag{16}$$

and has the complex value. Multiplying Eq. (16) and its complex conjugate then dividing by 2, we obtain the continuity equation for complex probability density,

$$\frac{\partial \rho(t,\overline{x})}{\partial t} = -\nabla \cdot \left(\dot{\overline{x}}\rho(t,\overline{x})\right), \tag{17}$$

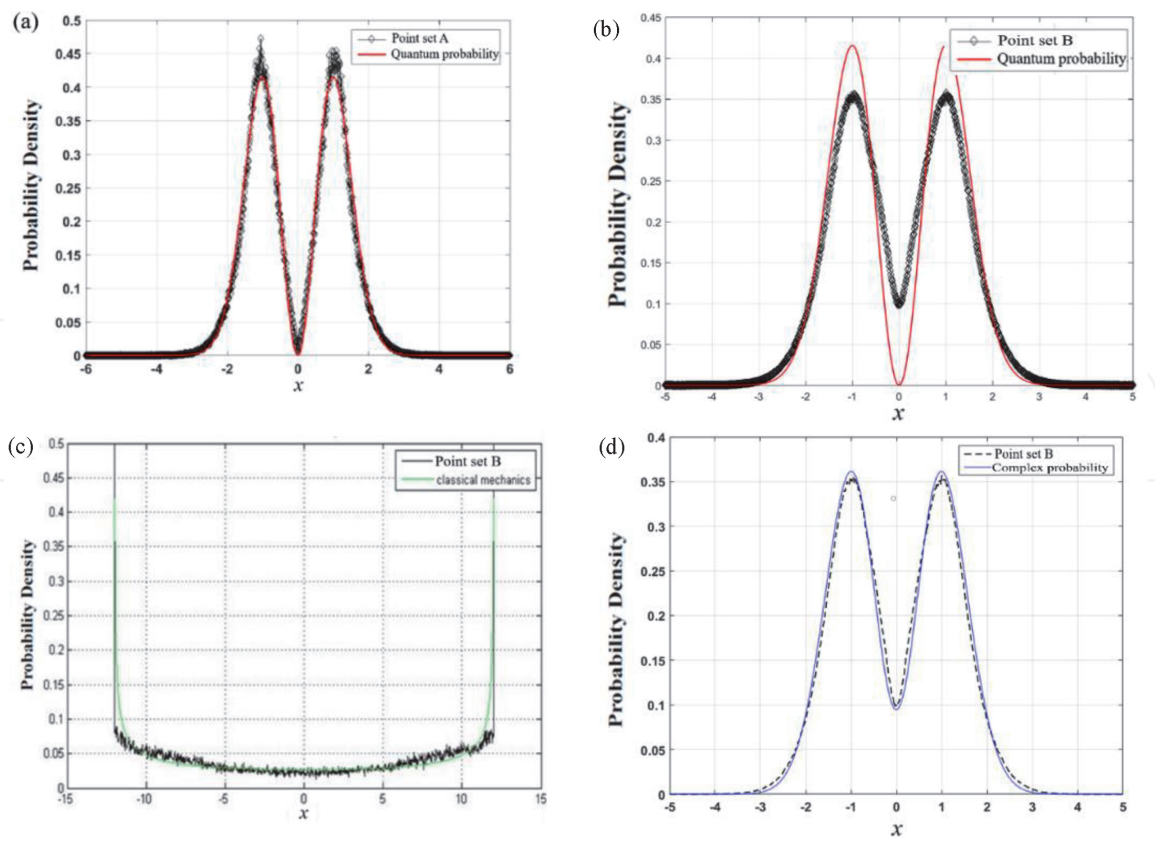

Figure 2.
(a) The quantum mechanical compatible outcome proposed by point collections of an ensemble of CRTs crossing the real axis for quantum harmonic oscillator in $n = 1$ *state with coefficient correlation,* $\Gamma = 0.995$. *(b) The dismissed nodal condition is given by the same trajectory ensemble but is composed of all projected points onto the real axis. (c) The classical-like probability distribution is presented by collecting all projection points on the real axis for* $n = 70$ *state with coefficient correlation,* $\Gamma = 0.9412$. *(d) The analytical solution of the complex probability density function solved from the Fokker-Planck equation shows good agreement with the spatial distribution composed of all projection points on the real axis with coefficient correlation,* $\Gamma = 0.9975$ *[36].*

where $\overline{x}$ denotes the mean of valuable x. From Eq. (17) we can see that the complex probability density is conserved in the complex plane, neither on the real axis nor imaginary axis. **Figure 2(d)** illustrates the good agreement between the solution solved from Eq. (17) (blue dotted line) and the statistical spatial distribution (black solid line) contributed by all points collected by the projections onto the real axis. This result verifies that the analytical solution coheres with the statistical distribution made by CRT. It shows that the same results obtained from two different ways stand from the equal footing of the classical concept.

3. Shell structure in hydrogen atom

In quantum mechanics, the quantized orbits of the electron in the hydrogen atom is determined by solving the Schrödinger equation for different eigen states. There is no further description of these orbits, especially no explanation about the force balanced condition under the influence of the Coulomb force. Less study reports the role that the quantum potential plays in atomic analysis. In this section, a quest for describing the hydrogen atom is stretching underlying the quantum potential in complex space. We show our most equations in dimensionless form for the purposes of simplifying the question.

Let us consider the quantum Hamiltonian with Coulomb potential in complex space [37],

$$H = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{\hbar}{i}\left(\frac{2}{r}\frac{\partial S}{\partial r} + \frac{\partial^2 S}{\partial r^2}\right)\right] + \frac{1}{2mr^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\hbar}{i}\left(\cot\theta\frac{\partial S}{\partial \theta} + \frac{\partial^2 S}{\partial \theta^2}\right)\right.$$
$$\left. + \frac{1}{\sin^2\theta}\left(\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{\hbar}{i}\frac{\partial^2 S}{\partial \phi^2}\right)\right] + \frac{-Ze^2}{4\pi\epsilon_0 r}, \qquad (18)$$

where S is the action function. Hamiltonian (18) is state dependent if we apply the simple relation (9) to it. We can therefore have the dimensionless total potential in terms of the wave function,

$$V_{nlm_l} = -\frac{2}{r} + \left[\frac{1}{4r^2}\left(4 + \cot^2\theta\right) - \frac{d^2 \ln R_{nl}(r)}{dr^2} - \frac{1}{r^2}\frac{d^2 \ln \Theta_{lm_l}(\theta)}{d\theta^2}\right], \qquad (19)$$

where n, l, and m_l denote the principle quantum number, azimuthal quantum number, and magnetic quantum number, respectively. The first term in Eq. (19) is recognized as the Coulomb potential; while the remaining terms are the components of the quantum potential. **Figure 3(a)** illustrates the three potentials varying in radial direction of $(n, l, m_l) = (1, 0, 0)$ state; they are the total potential, Coulomb potential, and quantum potential. The quantum potential yields the opposite spatial distribution to the Coulomb potential, therefore, the total potential performs a neutral situation. When the electron is too close (less than the Bohr radius) to the nucleus, the total potential forms a solid wall that forbids the electron getting closer. The total potential holds an appropriate distribution such that the electron is subject to an attractive force when it is too far away from the nucleus. From the perspective of the electron, it is quantum potential maintains the orbit stable and stop the disaster of crashing on the nucleus.

From Eq. (19) we can obtain the total forces for $(n, l, m_l) = (1, 0, 0)$ state:

$$f_{100}^r = -\frac{2}{r^2} + \frac{1}{2r^3}\left(4 + \cot^2\theta\right), \quad f_{100}^\theta = \frac{1}{2r^2}\frac{\cos\theta}{\sin^3\theta}, \quad f_{100}^\phi = 0. \qquad (20)$$

Under a specific condition $f_{100}^r = f_{100}^\theta = 0$, the electron stays stationary at the equilibrium position $(r, \theta) = (1, \pi/2)$ for which $r = 1$ corresponds to the Bohr radius. The motion of electron at the equilibrium point is determined by

$$f_{100}^r(r, \pi/2) = f_Q^r + f_V^r = \frac{2}{r^3} - \frac{2}{r^2}, \qquad (21)$$

where the first and the second term represent the repulsive quantum force and the attractive Coulomb force with lower label Q and V, respectively. As the distance between the electron and the nucleus changes, the two forces take the lead in turn as **Figure 3(b)** illustrates. It is clear to see that the zero force location happens at $r = 1$ (Bohr radius) owing to the force balancing formed by the Coulomb force and quantum force.

In quantum mechanics, the maximum probability of finding the electron is at the Bohr radius according to

$$\frac{d}{dr}P_{10}(r) = \frac{d}{dr}\left(4\pi r^2 e^{-2r}\right) = 0. \qquad (22)$$

The balanced force and the probability are totally different concepts; however, present the same description of the hydrogen atom. This may reflect the equivalent

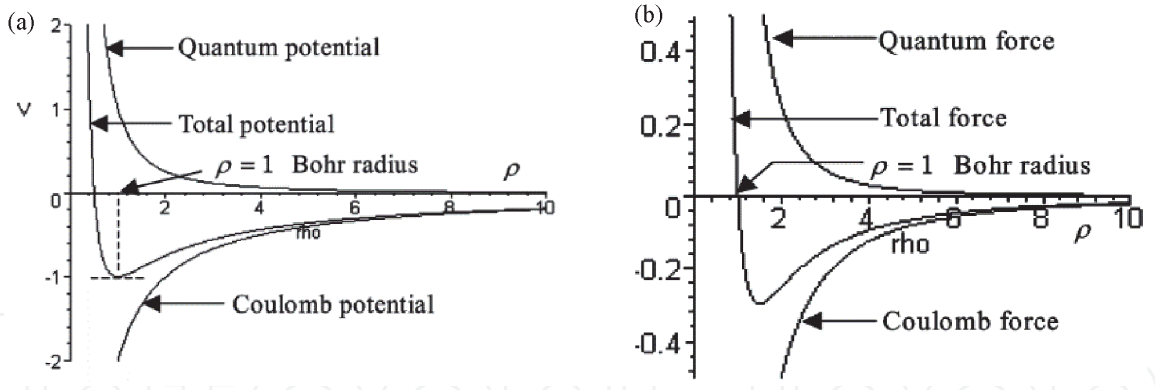

Figure 3.
(a) The variations of three potentials in radial direction for the ground state. (b) The total radial force in the ground state which is composed of the coulomb force and quantum force with zero value at the Bohr radius [37].

meaning between the classical shell layers and the quantum probability. Furthermore, it may help us to realize the probabilistic electron cloud in a classical standpoint.

Let us consider $(n, l, m_l) = (2, 0, 0)$ state, which has the total potential as

$$V_{200} = V + Q = -\frac{2}{r} + \left[\frac{1}{(2-r)^2} + \frac{1}{4r^2}\left(4 + \cot^2\theta\right)\right], \tag{23}$$

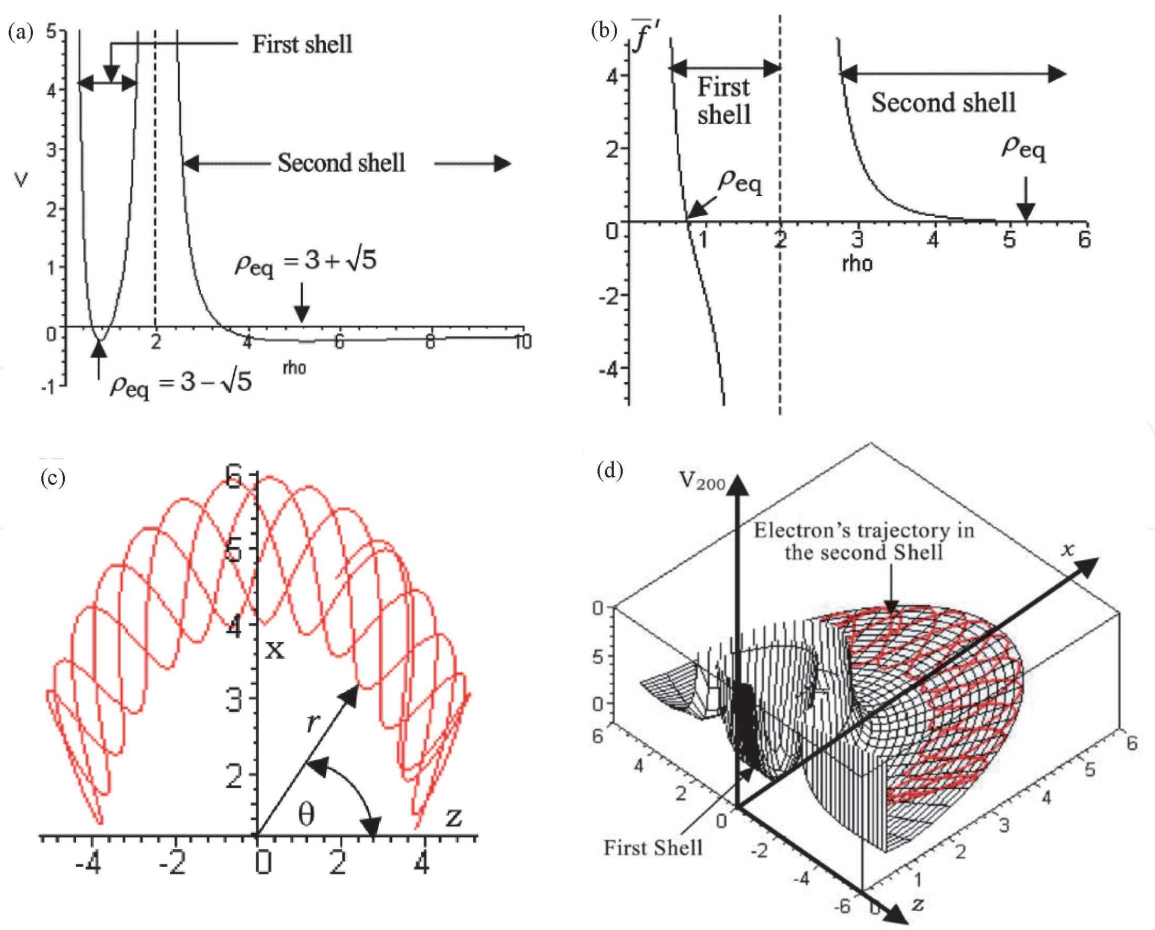

Figure 4.
(a) The shell structure of $(\mathrm{n}, \mathrm{l}, \mathrm{m_l}) = (2, 0, 0)$ *state in radial direction. (b) The dynamic equilibrium points locate where the total force equals to zero. (c) Electron's motion in* $\mathrm{r} - \theta$ *plane, and (d) illustrated in the shell plane [37].*

and the force distributions in three directions:

$$f^r_{200} = -\frac{2}{r^2} + \left[-\frac{1}{(2-r)^3} + \frac{1}{2r^3}\left(4 + \cot^2\theta\right)\right], f^\theta_{200} = \frac{1}{2r^2}\frac{\cos\theta}{\sin^3\theta}, f^\phi_{200} = 0, \quad (24)$$

which indicates the same equilibrium point location $(r_{eq}, \theta_{eq}) = (3 \pm \sqrt{5}, \pi/2)$ given by the equations of motion from Eq. (14):

$$\frac{dr}{dt} = 4i\frac{r^2 - 6r + 4}{r(r-2)}, \frac{d(\cos\theta)}{dt} = i\frac{\cos\theta}{r^2}, \frac{d\phi}{dt} = 0, \quad (25)$$

under the zero resultant force condition and the electron dynamic equilibrium condition. **Figure 4(a)** presents the shell structures in radial direction according to Eq. (24). The range of the layers are constrained by the total potential and divided into two different parts. The two equilibrium points individually correspond to the zero force locations in the two shells as **Figure 4(b)** indicates. Eq. (25) offers how electron move in this state. **Figure 4(c)** illustrates electron's trajectory in the $r - \theta$ plane; while **Figure 4(d)** embodies trajectory in the shell structure.

4. Channelized quantum potential and conductance quantization in 2D Nano-channels

The practical technology usage of the proposed formalism is applied to 2D Nano-channels in this section. Instead of the probability density function offered by the conventional quantum mechanics, we stay in line with causalism to perceive what role played by the quantum potential. Consider a 2D straight channel made by GaAs-GaAlAs and is surrounded by infinite potential barrier except the two reservoirs and the channel. The schematic plot of the channel refers to **Figure 5**. The dynamic evolution of the wave function $\psi(x,y)$ in the channel is described by the Schrödinger equation,

$$-\frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) = E\psi(x,y), \quad (26)$$

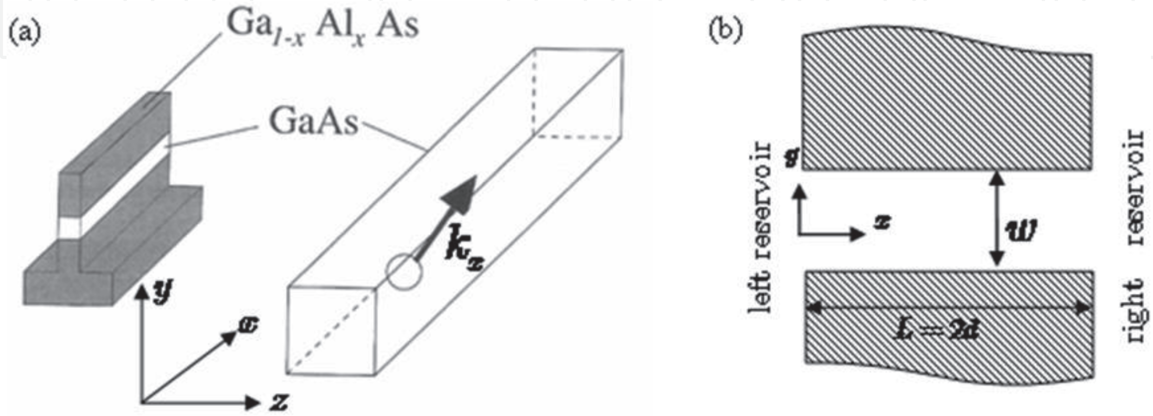

Figure 5.
(a) A single quantum wire and an expanded view showing schematically the single degree of freedom in the x direction. (b) 2D straight channel made up of quantum wire with length 2d *and width* w *connects the left reservoir to the right reservoir.*

where $m^* = 0.067m_e$ is the effective mass of the electron, and E is the total energy of the incident electron. The general solution of Eq. (26) has the form as

$$\psi_k^C(x,y) = \sum_{n=1}^{N} \left(B_n e^{ik_n x} + C_n e^{-ik_n x}\right)\phi_n(y), \phi_n(y) = \sin\left[\frac{n\pi}{w}\left(y + \frac{w}{2}\right)\right], \tag{27}$$

where N is the number of mode, w is the width of the channel, and k_n is the wave number which satisfies the energy conservation law:

$$E_x + E_y = \frac{(k_n\hbar)^2}{2m^*} + E_n = E, \tag{28}$$

in which $E_x = p_x^2/(2m^*) = (k_n\hbar)^2/(2m^*)$ is the free particle energy in the x direction, and $E_y = E_n = n^2\hbar^2\pi^2/(2m^* w^2)$, $n = 1, 2, \cdots$, is quantized energy in the y direction due to the presence of the infinite square well. From Eq. (28), we have the wave number read

$$k_n = \sqrt{2m^*(E - E_n)/\hbar^2}. \tag{29}$$

The function $B_n e^{ik_n x} + C_n e^{-ik_n x}$ in Eq. (27) is the free-particle wave function in the x direction, and $\phi_n(y)$ is an eigen function for the infinite well in the y direction satisfying the boundary condition $\phi_n(y)(w/2) = \phi_n(y)(-w/2) = 0$. The coefficients B_n and C_n are uniquely determined by the incident energy E and incident angle ϕ . (More detail refers to [38].) The quantum potential in the channel can be obtained by combing Eqs. (8), (10) and the wave function (27) (in dimensionless form),

$$Q(x,y) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln \psi_k^C(x,y). \tag{30}$$

The quantum potential provides fully information of electron's motion, its characteristic of inverse proportional to the probability density displays more knowledge in the channel. The inverse proportional relation reads

$$|Q(x,y)| = \frac{1}{P(x,y)}\left[\left(\frac{\partial\psi_k^C}{\partial x}\right)^2 + \left(\frac{\partial\psi_k^C}{\partial y}\right)^2\right], \tag{31}$$

which represents that the high quantum potential region corresponds to the low probability of electrons passing through as **Figure 6** displays; and **Figure 7** illustrates how the quantum potential gradually form the quantized channels as the incident angle increases, which shows the state dependent characteristic of the quantum potential.

The other quantum feature originating from the quantum potential is the quantization of conductance in the channel as **Figure 8** presents. We will show that the high conductance region is where the most electrons gather. To simplify the system, we firstly replace the motion in 2D channel by a motion in 1D square barriers [39]. Therefore, we consider the wave function $\psi_n(x)$ satisfying the following Schrödinger equation,

$$\frac{d^2\psi_n(x)}{dx^2} + \frac{2m^*}{\hbar^2}(E - V_n)\psi_n(x) = 0, \tag{32}$$

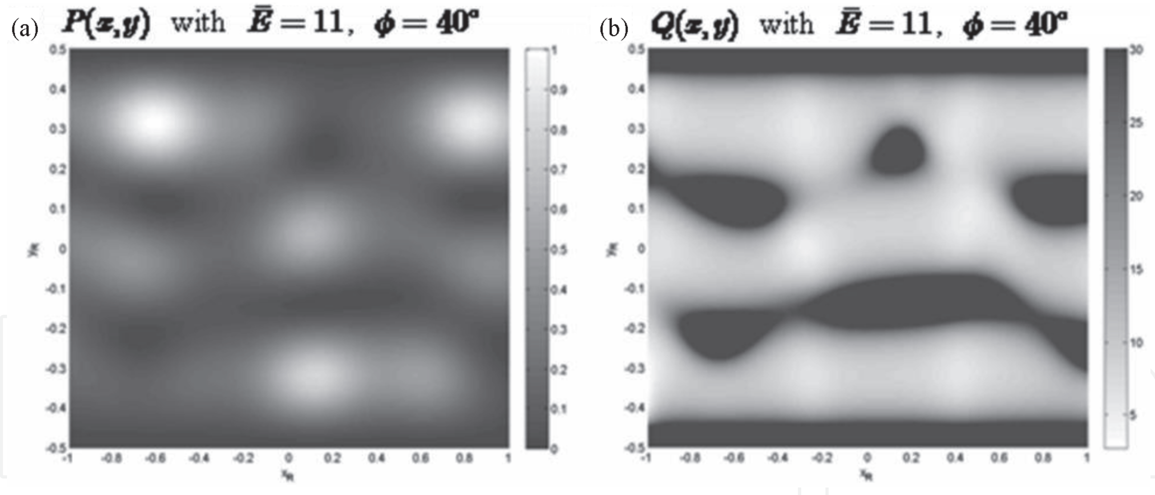

Figure 6.
The incident energy E = 11 *and the incident angle* $\phi = 40^{\circ}$ *for: (a) the probability density function; (b) the corresponding quantum potential of the cross-section in the channel. The bright regions of the quantum potential in (b) represent the lower potential barriers which are in accord with the bright regions in (a) where are the locations with higher probability of finding electrons [38].*

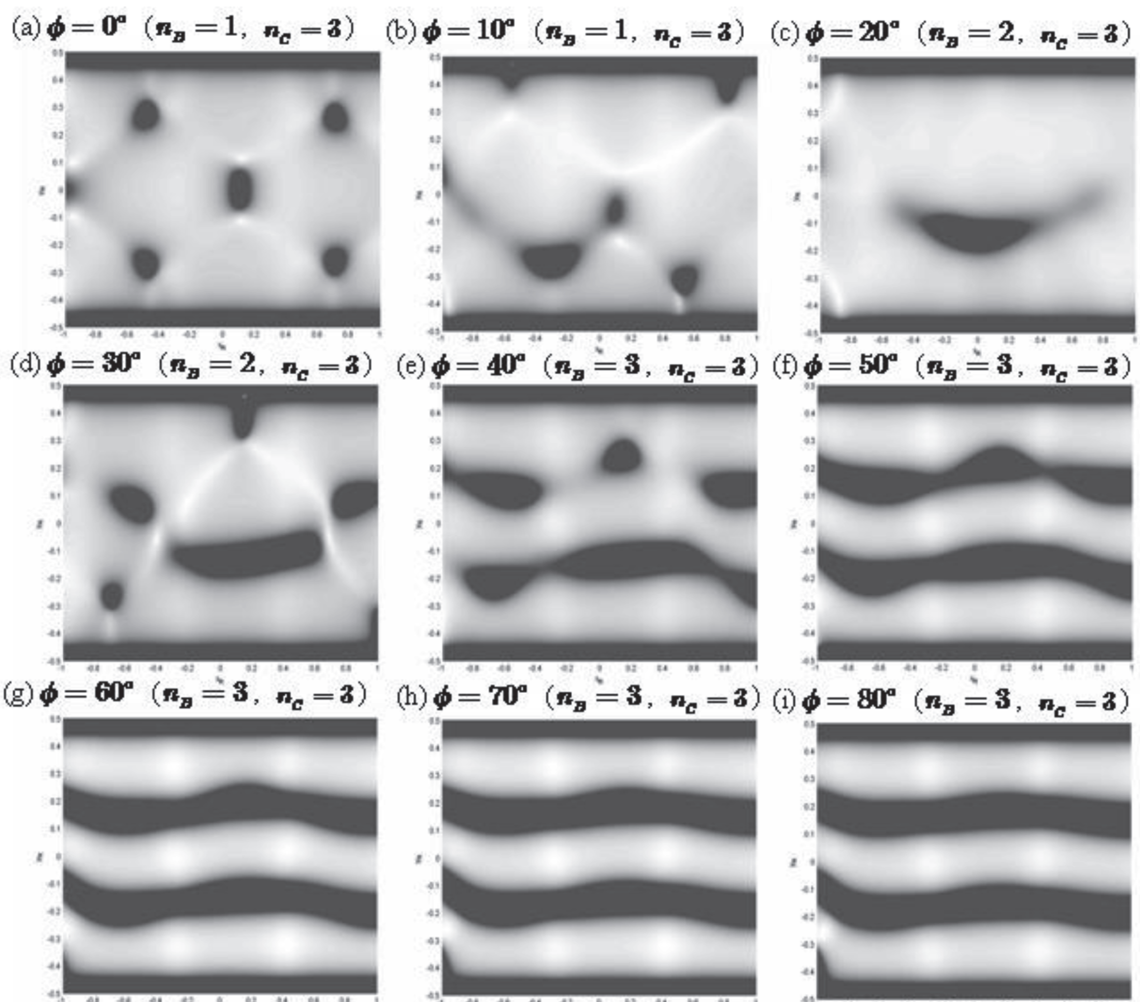

Figure 7.
The variation of the quantum potential with respect to the incident angle ϕ *for a fixed incident energy* E = 11. *It is seen that the channelized structure becomes more and more apparent with the increasing incident angle* ϕ *[38].*

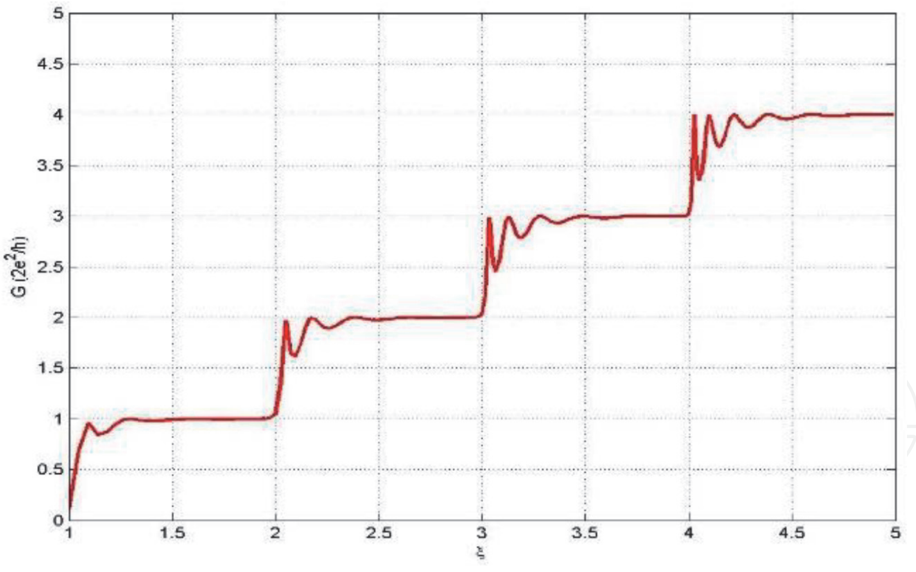

Figure 8.
The conductance G of a narrow channel shows plateaus at integer multiples of $2e^2/h$ *as the electron's energy* $\xi = \sqrt{E}$ *increases [39].*

where V_n is the equivalent square barrier,

$$V_n = \begin{cases} \dfrac{n^2 h^2 \pi^2}{2m^* w^2}, & |x| \le d \\ 0, & |x| > d \end{cases}. \tag{33}$$

Please notice that potential V_n depends on the eigen state, hence, the electron will encounter different heights of the potential barrier in different eigen states. Furthermore, it makes electron with different energy either transmitting or going through the barrier by tunneling. When electrons transmit the channel, the conductance will be changed and is expected to have the quantized value.

Let us express the transmission coefficient in dimensionless form as

$$T_n(\xi) = \left[1 + \frac{n^4 \sin^2\left(\pi \bar{d}\sqrt{\xi^2 - n^2}\right)}{4\xi^2\left(\xi^2 - n^2\right)}\right]^{-1}, \tag{34}$$

where $\xi = \sqrt{E}$, $\bar{d} = 2d/w$ is the aspect ratio of the channel. To display the quantization of the conductance, we conduct a combination consisting of all transmission coefficients which represents all electrons transmitting through all potential barriers. This combination is expressed in terms of the total transmission coefficients,

$$T_{Total}^{(N)}(\xi) = \sum_{n=1}^{N} T_n(\xi) = \sum_{n=1}^{N} \left[1 + \frac{n^4 \sin^2\left(\pi \bar{d}\sqrt{\xi^2 - n^2}\right)}{4\xi^2\left(\xi^2 - n^2\right)}\right]^{-1}. \tag{35}$$

Figure 9 illustrates the quantization of the total transmission coefficient. Take $N = 2$ as an example, $T_{Total}^{(N)}(\xi)$ is composed of $T_1(\xi)$ and$T_2(\xi)$:

$$T_{Total}^{(2)}(\xi) \approx \begin{cases} 0, & \xi < 1 \\ 1, & 1 \le \xi < 2 \\ 2, & \xi \ge 2 \end{cases}, \tag{36}$$

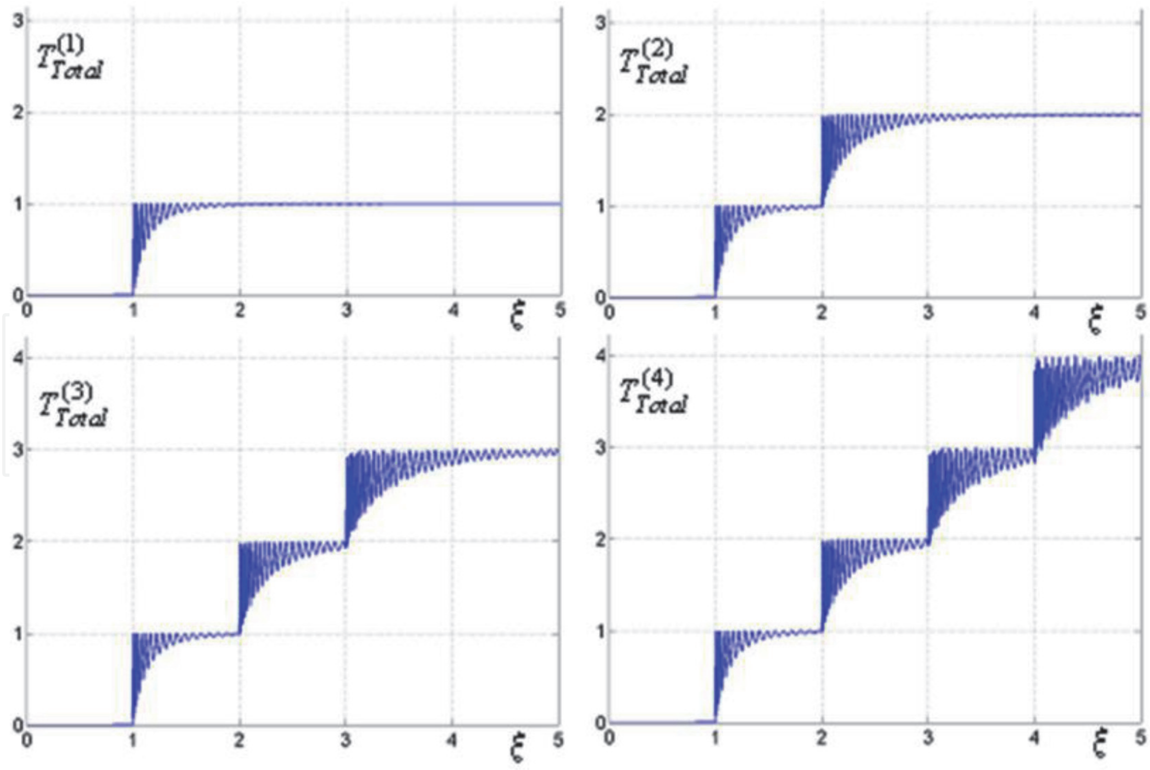

Figure 9.
The total transmission coefficients $T^{(N)}_{Total}(\xi)$ *display the step shape with the increasing of incident energy* ξ *for* $N = 1, 2, 3, 4$ *with* $\overline{d} = 10$. *[39].*

where we have ignored the rapid oscillations parts in the transmission coefficient (more detail refers to [39]). Eq. (36) shows the step structure illustrated in **Figure 9**, which has the same steps shape of the conductance shown in **Figure 8**. We have demonstrated that the total transmission coefficient is proportional to the total number of electrons passing the channel and it is relevant to the conductance in the channel.

5. Concluding remarks

Looking for the unifying theory of quantum and classical mechanics lasts for decades. Several approaches have been proposed, they share some viewpoints and contributions. We have learned that the quantum potential plays a switch role between the quantum and classical world. When the mass is getting larger and larger, the quantum potential will become smaller and smaller, and eventually becomes ignorable. Causality exists everywhere in the universe but hides itself in the microscopic world. What makes physicists miss the link that connects the two scale worlds is the statistical expression of the quantum world. It is impossible to extract the fundamental law from the probability interpretation. As the higher dimension is demanded, there are more evidences of causality emerging from the backbone of quantum mechanics. The complex weak measurement proposes the solid evidence of the complex space structure nature of the quantum world, and evokes the ontology return to the quantum kingdom. All quantum motions happen in complex space. All we can observe is a part of the whole appearance.

In Bohmian mechanics, the quantum potential is a product given by the transformation process which starts from the Schrödinger equation to the quantum HJ equation. In optimal guidance quantum motion formulation, the quantum potential naturally arises in the process of finding the minimum cost function. From the view point of the space geometry, the quantum potential exposits the geometric variation for the particle to lead its motion. This is what makes the quantum world quite different to the classical world as many quantum phenomena reveal. The quantum potential is so charming and plays the most important part that bridges the gap between the quantum and classical world.

Probability is a prescription to deal with the empirical data not to represent the essence of nature in such a small scale. We have demonstrated how to emerge the trajectory from the probability by expanding the dimensions to complex space. As meanwhile, we have pointed out how to reach the classical limit with increasing quantum numbers from the same ensemble of trajectories by adopting different statistical collection method. Take the advantage of the quantum potential, we are allowed to explain the force balanced condition in the hydrogen atom, moreover, we illustrate the formation of the shell structures which cohere with the shape of the electron clouds. The channels in 2D Nano-structure are shown to be related to the quantum potential and so does the conductance. We confirm that the quantized conductance is originated from the electron's transmission behavior. The ontology renders the reality of the identity to the quantum object. It cannot be done without the complex space structure. Complex space is essential for the quantum world and becomes the most crucial part of solving the quantum puzzle. It may proper to say that the causality returns to the quantum world and throughout the whole universe.

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