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A Survey on Hilbert Spaces and Reproducing Kernels

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Abstract

The main purpose of this chapter is to provide a brief review of Hilbert space with its fundamental features and introduce reproducing kernels of the corresponding spaces. We separate our analysis into two parts. In the first part, the basic facts on the inner product spaces including the notion of norms, pre-Hilbert spaces, and finally Hilbert spaces are presented. The second part is devoted to the reproducing kernels and the related Hilbert spaces which is called the reproducing kernel Hilbert spaces (RKHS) in the complex plane. The operations on reproducing kernels with some important theorems on the Bergman kernel for different domains are analyzed in this part.

Keywords: Hilbert spaces, norm spaces, reproducing kernels, reproducing kernel Hilbert spaces (RKHS), operations on reproducing kernels, sesqui-analytic kernels, analytic functions, Bergman kernel

1. Framework

This chapter consists of introductory concept on the Hilbert space theory and reproducing kernels. We start by presenting basic definitions, propositions, and theorems from functional analysis related to Hilbert spaces. The notion of linear space, norm, inner product, and pre-Hilbert spaces are in the first part. The second part is devoted to the fundamental properties of the reproducing kernels and the related Hilbert spaces. The operations with reproducing kernels, inclusion property, Bergman kernel, and further properties with examples of the reproducing kernels are analyzed in the latter section.

2. Introduction to Hilbert spaces

We start by the definition of a vector space and related topics. Let \mathbb{C} be the complex field. The following preliminaries can be considered as fundamental concepts of the Hilbert spaces.

2.1 Vector spaces and inner product spaces

Vector space. A vector space is a linear space that is closed under vector addition and scalar multiplication. More precisely, if we denote our linear space by \mathcal{H} over the field \mathbb{C} , then it follows that

i. if $x, y, z \in \mathcal{H}$, then

$$x + y = y + x \in \mathcal{H}, \quad x + (y + z) = (x + y) + z \in \mathcal{H};$$

ii. if k is scalar, then $kx \in \mathcal{H}$.

Inner product. Let \mathcal{H} be a linear space over the complex field \mathbb{C} . An *inner product* on \mathcal{H} is a two variable function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \text{ satisfying}$$

$$\text{i. } \langle f, g \rangle = \overline{\langle g, f \rangle} \text{ for } f, g \in \mathcal{H}.$$

$$\text{ii. } \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \text{ and } \langle f, \alpha g + \beta h \rangle = \overline{\alpha} \langle f, g \rangle + \overline{\beta} \langle f, h \rangle \text{ for } \alpha, \beta \in \mathbb{C} \text{ and } f, g, h \in \mathcal{H}.$$

$$\text{iii. } \langle f, f \rangle \geq 0 \text{ for } f \in \mathcal{H} \text{ and } \langle f, f \rangle = 0 \Leftrightarrow f = 0.$$

Pre-Hilbert space. A *pre-Hilbert space* \mathcal{H} is a linear space over the complex field \mathbb{C} with an inner product defined on it.

Norm space or inner product space. A norm on an inner product space \mathcal{H} denoted by $\| \cdot \|$ is defined by

$$\|f\| = \langle f, f \rangle^{1/2} \text{ or } \|f\|_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}^{1/2}$$

where $f \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$ denote the inner product on \mathcal{H} . The corresponding space is called as the inner product space or the norm space.

Properties of norm. For all $f, g \in \mathcal{H}$, and $\lambda \in \mathbb{C}$, we have

- $\|f\| \geq 0$. (Observe that the equality occurs only if $f = 0$).
- $\|\lambda f\| = |\lambda| \|f\|$.

Schwarz inequality. For all $f, g \in \mathcal{H}$, it follows that

$$|\langle f, g \rangle| \leq \|f\| \|g\|. \quad (1)$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Triangle inequality. For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\| \leq \|f\| + \|g\|. \quad (2)$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Polarization identity. For all $f, g \in \mathcal{H}$, it follows that

$$\langle f, g \rangle = \frac{1}{4} (\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - \|f - ig\|^2) \text{ for } f, g \in \mathcal{H}. \quad (3)$$

Parallelogram identity. For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2. \quad (4)$$

Metric. A metric on a set X is a function $d: X \times X \rightarrow \mathbb{R}$ satisfying the properties.

- $d(x, y) \geq 0$ and $d(x, y) = 0$ only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, y) \leq d(x, z) + d(z, y)$;

for all $x, y, z \in X$. Moreover the space (X, d) is the associated metric space. If we rearrange the metric with its properties for the inner product space \mathcal{H} , then it follows that for all $f, g, h \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$, where d satisfies all requirements to be a metric, we have

- $d(f, g) \geq 0$ and equality occurs only if $f = g$.
- $d(f, g) = d(g, f)$.
- $d(f, g) \leq d(f, h) + d(h, g)$.
- $d(f - h, g - h) = d(f, g)$.
- $d(\lambda f, \lambda g) = |\lambda| \cdot d(f, g)$.

Note. The binary function d given in the metric definition above represents the metric topology in \mathcal{H} which is called *strong topology* or *norm topology*. As a result, a sequence $(f_n)_{n \geq 0}$ in the pre-Hilbert space \mathcal{H} converges strongly to f if the condition

$$\|f_n - f\| \rightarrow 0 \text{ whenever } n \rightarrow \infty$$

is satisfied.

2.2 Introduction to linear operators

Linear operator. A map L from a linear space to another linear space is called *linear operator* if

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg$$

is satisfied for all $\alpha, \beta \in \mathbb{C}$ and for all $f, g \in \mathcal{H}$.

Continuous operator. An operator L is said to be continuous if it is continuous at each point of its domain. Notice that the domain and range spaces must be convenient for appropriate topologies.

Lipschitz constant of a linear operator. If L is a linear operator from \mathcal{H} to \mathcal{G} where \mathcal{H} and \mathcal{G} are pre-Hilbert spaces, then the Lipschitz constant for L is its norm $\|L\|$ and it is defined by

$$\|L\| = \sup\{\|Lf\|_{\mathcal{G}}/\|f\|_{\mathcal{H}} : 0 \neq f \in \mathcal{H}\}. \quad (5)$$

Theorem 1. Let L be a linear operator from the pre-Hilbert spaces \mathcal{H} to \mathcal{G} . Then the followings are mutually equivalent:

- L is continuous.
- L is bounded, that is,

$$\sup\{\|Lf\|_{\mathcal{G}} : \|f\|_{\mathcal{H}} \leq k\} < \infty$$

for $0 \leq k < \infty$.

iii. L is Lipschitz continuous, that is,

$$\|Lf - Lg\|_G \leq \lambda \|f - g\|_{\mathcal{H}},$$

where $0 \leq \lambda < \infty$ and $f, g \in \mathcal{H}$.

Some properties of linear operators. Let $B(\mathcal{H}, \mathcal{G})$ be the collection of all continuous linear operators from the pre-Hilbert spaces \mathcal{H} to \mathcal{G} . Then

- $B(\mathcal{H}, \mathcal{G})$ is a linear space with respect to the natural addition and scalar multiplication satisfying

$$(\alpha L + \beta M)f = \alpha Lf + \beta Mf,$$

where L and M are linear operators, $f \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

- Whenever $\mathcal{H} = \mathcal{G}$, then $B(\mathcal{H}, \mathcal{G})$ is denoted by $B(\mathcal{H})$.
- If \mathcal{K} is another pre-Hilbert space, $L \in B(\mathcal{H}, \mathcal{G})$ and $K \in B(\mathcal{G}, \mathcal{K})$. Then the product

$$(KL)f = K(Lf) \text{ for } f \in \mathcal{H} \in B(\mathcal{H}, \mathcal{K}).$$

In addition,

$$\text{i. } K(\xi L + \zeta M) = \xi KL + \zeta KM$$

$$\text{ii. } \|\xi L\| = |\xi| \cdot \|L\|$$

$$\text{iii. } \|L + M\| \leq \|L\| + \|M\| \text{ and}$$

$$\text{iv. } \|KL\| \leq \|K\| \|L\|.$$

are also satisfied.

2.3 Hilbert spaces and linear operators

Linear form (or linear functional). A linear operator from the pre-Hilbert space \mathcal{H} to the scalar field \mathbb{C} is called a *linear form* (or *linear functional*).

Hilbert spaces. A pre-Hilbert space \mathcal{H} is said to be a *Hilbert space* if it is complete in metric. In other words if f_n is a Cauchy sequence in \mathcal{H} , that is, if

$$\|f_n - f_m\| \rightarrow 0 \text{ whenever } n, m \rightarrow \infty,$$

then there is $f \in \mathcal{H}$ such that

$$\|f_n - f\| \rightarrow 0 \text{ whenever } n \rightarrow \infty.$$

Note. Every subspace of a pre-Hilbert space is also a pre-Hilbert space with respect to the induced inner product. However, the reverse is not always true. For a subspace of a Hilbert space to be also a Hilbert space, it must be closed.

Completion. The canonical method for which a pre-Hilbert space \mathcal{H} is embedded as a dense subspace of a Hilbert space $\tilde{\mathcal{H}}$ so that

$$\langle f, g \rangle_{\tilde{\mathcal{H}}} = \langle f, g \rangle_{\mathcal{H}} \text{ for } f, g \in \mathcal{H}$$

is called *completion*.

Note. If L is a continuous linear operator from a dense subspace \mathcal{M} of a Hilbert space \mathcal{H} to a Hilbert space \mathcal{G} , then it can be extended uniquely to a continuous linear operator from \mathcal{H} to \mathcal{G} with preserving norm.

Theorem 2. Let \mathcal{M} and \mathcal{N} be dense subspaces of the Hilbert spaces \mathcal{H} and \mathcal{G} , respectively. For $f \in \mathcal{H}, g \in \mathcal{M}$ and $0 \leq \lambda < \infty$, if a linear operator L from \mathcal{M} to \mathcal{G} satisfies

$$|\langle Lf, g \rangle_{\mathcal{G}}| \leq \lambda \|f\|_{\mathcal{H}} \|g\|_{\mathcal{G}}, \quad (6)$$

then L is uniquely extended to a continuous linear operator from \mathcal{M} to \mathcal{G} with norm $\leq \lambda$ where the norm coincides with the minimum of such λ .

Theorem 3. Let (Ω, μ) denotes a measure space so that Ω is the union of subsets of finite positive measure and $L^2(\Omega, \mu)$ consists of all measurable functions $f(\omega)$ on Ω such that

$$\int_{\Omega} |f(\omega)|^2 d\mu(\omega) < \infty. \quad (7)$$

Then $L^2(\Omega, \mu)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \int_{\Omega} f(\omega) \overline{g(\omega)} d\mu(\omega). \quad (8)$$

Theorem 4 (F. Riesz). For each continuous linear functional φ on a Hilbert space \mathcal{H} , there exists uniquely $g \in \mathcal{H}$ such that

$$\varphi(f) = \langle f, g \rangle \text{ for } f \in \mathcal{H}. \quad (9)$$

Theorem 5. Let \mathcal{M} be a closed subspace of a Hilbert space \mathcal{H} . Then the algebraic direct sum relation

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$$

is satisfied. In other words, $\forall f \in \mathcal{H}$ can be uniquely written by

$$f = f_{\mathcal{M}} + f_{\mathcal{M}^{\perp}} \text{ with } f_{\mathcal{M}} \in \mathcal{M}, f_{\mathcal{M}^{\perp}} \in \mathcal{M}^{\perp}. \quad (10)$$

In addition, $\|f_{\mathcal{M}}\|$ coincides with the distance from f to \mathcal{M}^{\perp}

$$\|f_{\mathcal{M}}\| = \min \{ \|f - g\| : g \in \mathcal{M}^{\perp} \}. \quad (11)$$

Remark. In a Hilbert space, the closed linear span of any subset A of a Hilbert space \mathcal{H} coincides with $(A^{\perp})^{\perp}$.

Total subset of a Hilbert space. A subset \mathcal{A} of a Hilbert space \mathcal{H} is called *total* in \mathcal{H} if 0 is the only element that is orthogonal to all elements of \mathcal{A} . In other words,

$$\mathcal{A}^{\perp} = \{0\}.$$

As a result, \mathcal{A} is total if and only if every element of \mathcal{H} can be approximated by linear combinations of elements of \mathcal{A} .

Orthogonal projection. If \mathcal{M} is a closed subspace of \mathcal{H} , the map $f \mapsto f_{\mathcal{M}}$ gives a linear operator from \mathcal{H} to \mathcal{M} with norm ≤ 1 . We call this operator as the *orthogonal projection* to \mathcal{M} and denote it by $P_{\mathcal{M}}$.

Note. If I is the identity operator on \mathcal{H} , then $I - P_{\mathcal{M}}$ denotes the orthogonal projection to \mathcal{M}^\perp , and the relation

$$\|f\|^2 = \|P_{\mathcal{M}}f\|^2 + \|(I - P_{\mathcal{M}})f\|^2 \quad (12)$$

is satisfied for all $f \in \mathcal{H}$.

Weak topology. The weakest topology that makes continuous all linear functionals of the form $f \mapsto \langle f, g \rangle$ is called *the weak topology* of a Hilbert space \mathcal{H} .

Note. If $f \in \mathcal{H}$, then with respect to the weak topology, a fundamental system of neighborhoods of f is composed of subsets of the form

$$U(f; A, \epsilon) = \{h : |\langle f, g \rangle - \langle h, g \rangle| < \epsilon \text{ for } g \in A\},$$

where A is a finite subset of \mathcal{H} and $\epsilon > 0$. Then a directed net $\{f_\lambda\}$ converges weakly to f if and only if

$$\langle f_\lambda, g \rangle \xrightarrow{\lambda} \langle f, g \rangle \text{ for all } g \in \mathcal{H}.$$

Operator weak topology. The weakest topology that makes continuous all linear functionals of the form

$$L \mapsto \langle Lf, g \rangle \text{ for } f \in \mathcal{H}, g \in \mathcal{G}$$

is called *the operator weak topology* in the space $B(\mathcal{H}, \mathcal{G})$ of continuous linear operators from \mathcal{H} to \mathcal{G} . In addition, a directed net $\{L_\lambda\}$ converges weakly to L if

$$\langle L_\lambda f, g \rangle \xrightarrow{\lambda} \langle Lf, g \rangle.$$

Operator strong topology. The weakest topology that makes continuous all linear operators of the form

$$L \mapsto Lf \text{ for } f \in \mathcal{H}$$

is called *the operator strong topology*. Moreover a directed net $\{L_\lambda\}$ converges strongly to L if

$$\|L_\lambda f - Lf\| \xrightarrow{\lambda} 0 \text{ for all } f \in \mathcal{H}.$$

Theorem 6. Let \mathcal{H} and \mathcal{G} be Hilbert spaces and $B(\mathcal{H}, \mathcal{G})$ be a continuous linear operator from \mathcal{H} to \mathcal{G} . Then

- the closed unit ball $U := \{f : \|f\| \leq 1\}$ of \mathcal{H} is weakly compact;
- the closed unit ball $\{L : \|L\| \leq 1\}$ of $B(\mathcal{H}, \mathcal{G})$ is weakly compact.

Theorem 7. Let \mathcal{H} be a Hilbert space and $A \subseteq \mathcal{H}$. Then if A is weakly bounded in the sense

$$\sup_{f \in A} |\langle f, g \rangle| < \infty \text{ for } g \in \mathcal{H}, \quad (13)$$

then it is strongly bounded, that is, $\sup_{f \in A} \|f\| < \infty$.

Theorem 8. If \mathcal{H} and \mathcal{G} are Hilbert spaces and L is a linear operator from \mathcal{H} to \mathcal{G} , then the strong continuity and weak continuity for L are equivalent.

Theorem 9. Let \mathcal{H} and \mathcal{G} be Hilbert spaces. Then the following statements for $\mathbf{L} \subseteq B(\mathcal{H}, \mathcal{G})$ are mutually equivalent:

(i) \mathbf{L} is weakly bounded; that is, for $f \in \mathcal{H}$, $g \in \mathcal{G}$, we have

$$\sup_{L \in \mathbf{L}} |\langle Lf, g \rangle| < \infty$$

(ii) \mathbf{L} is strongly bounded; that is, for $f \in \mathcal{H}$, we have

$$\sup_{L \in \mathbf{L}} \|Lf\| < \infty.$$

(iii) \mathbf{L} is norm bounded (or uniformly bounded); that is,

$$\sup_{L \in \mathbf{L}} \|L\| < \infty.$$

Theorem 10. A linear operator L from the Hilbert spaces \mathcal{H} to \mathcal{G} is said to be *closed* if its *graph*

$$G_L := \{f \oplus Lf : f \in \mathcal{H}\} \quad (14)$$

is a closed subspace of the direct sum space $\mathcal{H} \oplus \mathcal{G}$, that is, whenever $n \rightarrow \infty$,

$$\|f_n - f\| \rightarrow 0 \text{ in } \mathcal{H} \text{ and } \|Lf_n - g\| \rightarrow 0 \text{ in } \mathcal{G} \Rightarrow g = Lf.$$

Theorem 11. If L is a closed linear operator with a domain of a Hilbert space \mathcal{H} to another Hilbert space \mathcal{G} , then it is continuous.

Sesqui-linear form. A function $\Phi : \mathcal{H} \times \mathcal{G} \rightarrow \mathbb{C}$ is a *sesqui-linear form* (or *sesqui-linear function*) if for $f, h \in \mathcal{H}$, $g, k \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$,

$$(i) \quad \Phi(\alpha f + \beta h, g) = \alpha \Phi(f, g) + \beta \Phi(h, g) \quad (15)$$

$$(ii) \quad \Phi(f, \alpha g + \beta k) = \bar{\alpha} \Phi(f, g) + \bar{\beta} \Phi(f, k) \quad (16)$$

are satisfied where \mathcal{H} and \mathcal{G} are Hilbert spaces.

Remark. If $L \in B(\mathcal{H}, \mathcal{G})$, then the sesqui-linear form Φ defined by

$$\Phi(f, g) = \langle Lf, g \rangle_G \quad (17)$$

is bounded in the sense that

$$|\Phi(f, g)| \leq \lambda \|f\|_{\mathcal{H}} \|g\|_{\mathcal{G}} \text{ for } f \in \mathcal{H}, g \in \mathcal{G}, \quad (18)$$

where $\lambda \geq \|L\|$.

Remark. If a sesqui-linear form Φ satisfies the condition (18), then for $f \in \mathcal{H}$, the linear functional

$$g \mapsto \overline{\Phi(f, g)}$$

is continuous on \mathcal{G} . If we apply the Riesz theorem, then there exists uniquely $f' \in \mathcal{G}$ satisfying

$$\|f'\|_{\mathcal{G}} \leq \lambda \|f\|_{\mathcal{H}} \text{ and } \Phi(f, g) = \langle f', g \rangle_{\mathcal{G}} \text{ for } g \in \mathcal{G}.$$

Hence $f \mapsto f'$ becomes linear, and as a result we obtain

$$\Phi(f, g) = \langle f', g \rangle_{\mathcal{G}} = \langle Lf, g \rangle_{\mathcal{G}}.$$

Adjoint operator. If $L \in B(\mathcal{H}, \mathcal{G})$, then the unique operator $L^* \in B(\mathcal{G}, \mathcal{H})$ satisfying

$$\Phi(f, g) = \langle f, L^*g \rangle_{\mathcal{H}} \text{ for } f \in \mathcal{H}, g \in \mathcal{G} \quad (19)$$

is called the *adjoint* of L .

Remark. By the definitions of L and L^* , it follows that

$$\langle Lf, g \rangle_{\mathcal{G}} = \langle f, L^*g \rangle_{\mathcal{H}} \text{ for } f \in \mathcal{H}, g \in \mathcal{G}. \quad (20)$$

Isometric property. The adjoint operation is isometric if

$$\|L\| = \|L^*\| \text{ is satisfied.} \quad (21)$$

Remark. Let \mathcal{H}, \mathcal{G} , and \mathcal{K} be Hilbert spaces and $K \in B(\mathcal{G}, \mathcal{K})$ and $L \in B(\mathcal{H}, \mathcal{G})$ be given. Then

$$KL \in B(\mathcal{H}, \mathcal{K}) \text{ and } (KL)^* = L^*K^* \quad (22)$$

$$\text{Ker}(L) = (\text{Ran}(L^*))^{\perp} \text{ and } (\text{Ker}(L))^{\perp} = \text{Clos}\{\text{Ran}(L)^*\} \quad (23)$$

where $\text{Ker}(L)$ is the kernel of L and $\text{Ran}(L)$ is the range of L .

Theorem 12. If $L, M \in B(\mathcal{H}, \mathcal{G})$, then the following statements are mutually equivalent.

- i. $\text{Ran}(M) \subseteq \text{Ran}(L)$.
- ii. There exists $K \in B(\mathcal{H})$ such that $M = LK$.
- iii. There exists $0 \leq \lambda < \infty$ such that

$$\|M^*g\| \leq \lambda \|L^*g\| \text{ for } g \in \mathcal{G}.$$

Quadric form. Let \mathcal{H} be a Hilbert space. A function

$$\varphi : \mathcal{H} \rightarrow \mathbb{C}$$

is a *quadratic form* if for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{C}$,

$$\varphi(\zeta f) = |\zeta|^2 \varphi(f) \quad (24)$$

and

$$\varphi(f + g) + \varphi(f - g) = 2\{\varphi(f) + \varphi(g)\} \quad (25)$$

are satisfied.

Note. If $L \in B(\mathcal{H})$, the quadratic form φ on \mathcal{H} is defined by

$$\varphi(f) = \langle Lf, f \rangle \text{ for } f \in \mathcal{H}, \quad (26)$$

and it is bounded

$$|\varphi(f)| \leq \lambda \|f\|^2 \text{ for } f \in \mathcal{H}, \quad (27)$$

where $\lambda \geq \|L\|$.

Remark. The sesqui-linear form Φ associated with L can be recovered from the quadratic form φ by the equation

$$\Phi(f, g) = \frac{1}{4} \{ \varphi(f + g) - \varphi(f - g) \} + \{ \varphi(f + ig) - \varphi(f - ig) \} \quad (28)$$

for all $f, g \in \mathcal{H}$.

Self-adjoint operator. A continuous linear operator L on a Hilbert space \mathcal{H} is said to be *self-adjoint* if $L = L^*$.

Remark. L is self-adjoint if and only if the associated sesqui-linear form Φ is Hermitian.

Remark. If L is self-adjoint, then the norm of L coincides with the minimum of λ given in (27) for the related quadratic form

Theorem 13. If L is a continuous self-adjoint operator, then

$$\|L\| = \sup \{ |\langle Lf, f \rangle| : \|f\| \leq 1 \}. \quad (29)$$

Positive definite operator. A self-adjoint operator $L \in B(\mathcal{H})$ is said to be *positive* (or *positive definite*) if

$$\langle Lf, f \rangle \geq 0 \text{ for all } f \in \mathcal{H}.$$

If $\langle Lf, f \rangle = 0$ only when $f = 0$, then L is said to be *strictly positive* (or, *strictly positive definite*).

Note. For any positive operator $L \in B(\mathcal{H})$, the Schwarz inequality holds in the following sense

$$|\langle Lf, g \rangle|^2 \leq \langle Lf, f \rangle \cdot \langle Lg, g \rangle. \quad (30)$$

Theorem 14. Let L and M be continuous positive operators on \mathcal{H} and \mathcal{G} , respectively. Then a continuous linear operator K from \mathcal{H} to \mathcal{G} satisfies the inequality

$$|\langle Kf, g \rangle_{\mathcal{G}}|^2 \leq \langle Lf, f \rangle_{\mathcal{H}} \langle Mg, g \rangle_{\mathcal{G}} \text{ for } f \in \mathcal{H}, g \in \mathcal{G} \quad (31)$$

if and only if the continuous linear operator

$$\begin{bmatrix} L & K^* \\ K & M \end{bmatrix}$$

on the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{G}$ with

$$f \oplus g \mapsto (Lf + K^*g) \oplus (Kf + Mg)$$

is positive definite.

Theorem 15. Let L be a continuous positive definite operator. Then there exists a unique positive definite operator called the *square root* of L , denoted by $L^{1/2}$, such that $(L^{1/2})^2 = L$.

Modulus operator. The square root of the positive definite operator $L^* L$ is called the *modulus (operator)* of L if L is a continuous linear operator.

Isometry. A linear operator U between Hilbert spaces \mathcal{H} and \mathcal{G} is called *isometric* or an *isometry* if

$$\|Uf\|_{\mathcal{G}} = \|f\|_{\mathcal{H}} \text{ for } f \in \mathcal{H} \quad (32)$$

is satisfied, that is, it preserves the norm.

Note. Eq. (32) implies that a continuous linear operator U is isometric if and only if $U^* U = I_{\mathcal{H}}$; in other words,

$$\langle Uf, Ug \rangle_{\mathcal{G}} = \langle f, g \rangle_{\mathcal{H}} \text{ for } f, g \in \mathcal{H}, \quad (33)$$

that is, U preserves the inner product.

Unitary operator. A surjective isometry linear operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is called a *unitary (operator)*.

Note. Observe that if $U \in B(H)$ is a unitary operator, then $U^* = U^{-1}$.

Partial isometry. A continuous linear operator U between Hilbert spaces \mathcal{H} and \mathcal{G} is called a *partial isometry* if

$$f \in (\text{Ker } U)^{\perp} = \text{Ran}(U^*) \Rightarrow \|Uf\| = \|f\|.$$

The spaces $(\text{Ker } U)^{\perp}$ and $\text{Ran}(U)$ are called the *initial space* of U and the *final space* of U , respectively.

Note. If U is a partial isometry, then its adjoint U^* is also a partial isometry.

Theorem 17. Every continuous linear operator L on \mathcal{H} admits a unique decomposition

$$L = U\tilde{L}, \quad (34)$$

where \tilde{L} is a positive definite operator and U is a partial isometry with initial space the closure of $\text{Ran}(\tilde{L})$.

3. Reproducing kernels and RKHS

We continue our analysis on the abstract theory of reproducing kernels.

3.1 Definition and fundamental properties

Reproducing kernels. Let \mathcal{H} be a Hilbert space of functions on a nonempty set X with the inner product $\langle f, g \rangle$ and norm $\|f\| = \langle f, f \rangle^{1/2}$ for f and $g \in \mathcal{H}$. Then the complex valued function $K(y, x)$ of y and x in X is called a *reproducing kernel* of \mathcal{H} if

i. For all $x \in X$, it follows that $K_x(\cdot) = K(\cdot, x) \in \mathcal{H}$,

ii. For all $x \in X$ and all $f \in \mathcal{H}$,

$$f(x) = \langle f, K_x \rangle, \quad (35)$$

are satisfied.

Note. Let K be a reproducing kernel. Applying (35) to the function K_x at y , we get

$$K_x(y) = K(y, x) = \langle K_y, K_x \rangle, \text{ for } x, y \in X. \quad (36)$$

Then, for any $x \in X$, we obtain

$$\|K_x\| = \langle K_x, K_x \rangle^{1/2} = K(x, x)^{1/2}. \quad (37)$$

Note. Observe that the subset $\{K_x\}_{x \in X}$ is total in \mathcal{H} , that is, its closed linear span coincides with \mathcal{H} . This follows from the fact that, if $f \in \mathcal{H}$ and $f \perp K_x$ for all $x \in X$, then

$$f(x) = \langle f, K_x \rangle = 0 \text{ for all } x \in X,$$

and hence f is the 0 element in \mathcal{H} . As a result, $\{0\}^\perp = \mathcal{H}$.

RKHS. A Hilbert space \mathcal{H} of functions on a set X is called a RKHS if there exists a reproducing kernel K of \mathcal{H} .

Theorem 18. If a Hilbert space \mathcal{H} of functions on a set X admits a reproducing kernel K , then this reproducing kernel K is unique.

Theorem 19. There exists a reproducing kernel K for \mathcal{H} for a Hilbert space \mathcal{H} of functions on X , if and only if for all $x \in X$, the linear functional $\mathcal{H} \ni f \mapsto f(x)$ of evaluation at x is bounded on \mathcal{H} .

Hermitian and positive definite kernel. Let X be an arbitrary set and K be a kernel on X , that is, $K : X \times X \rightarrow \mathbb{C}$. The kernel K is called *Hermitian* if for any finite set of points $\{y_1, \dots, y_n\} \subseteq X$, we have

$$\sum_{i,j=1}^n \bar{\epsilon}_j \epsilon_i K(y_j, y_i) \in \mathbb{R}.$$

It is called *positive definite*, if for any complex numbers $\epsilon_1, \dots, \epsilon_n$, we have

$$\sum_{i,j=1}^n \bar{\epsilon}_j \epsilon_i K(y_j, y_i) \geq 0.$$

Note. From the previous inequality, it follows that for any finitely supported family of complex numbers $\{\epsilon_x\}_{x \in X}$, we have

$$\sum_{x,y \in X} \bar{\epsilon}_y \epsilon_x K(y, x) \geq 0. \quad (38)$$

Theorem 20. The reproducing kernel K of a reproducing kernel Hilbert space \mathcal{H} is a positive definite matrix in the sense of E.H. Moore.

Properties of RKHS. Given a reproducing kernel Hilbert space \mathcal{H} and its kernel $K(y, x)$ on X , then for all $x, y \in X$, we have

$$\text{i. } K(y, y) \geq 0.$$

$$\text{ii. } K(y, x) = \overline{K(x, y)}.$$

$$\text{iii. } |K(y, x)|^2 \leq K(y, y)K(x, x) \text{ (Schwarz inequality).}$$

iv. Let $x_0 \in X$. Then the following statements are equivalent:

- a. $K(x_0, x_0) = 0$.
- b. $K(y, x_0) = 0$ for all $y \in X$.
- c. $f(x_0) = 0$ for all $f \in \mathcal{H}$.

Theorem 21. For any positive definite kernel K on X , there exists a unique Hilbert space \mathcal{H}_K of functions on X with reproducing kernel K .

Theorem 22. Every sequence of functions $(f_n)_{n \geq 1}$ that converges strongly to a function f in $\mathcal{H}_K(X)$ converges also in the pointwise sense, i.e., for any point $x \in X$,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x).$$

In addition, this convergence is uniform on every subset of X on which $x \mapsto K(x, x)$ is bounded.

Theorem 23. A complex valued function g on X belongs to the reproducing kernel Hilbert space $\mathcal{H}_K(X)$ if and only if there exists $0 \leq \lambda < \infty$ such that,

$$[g(y)\overline{g(x)}] \leq \lambda^2 [K(y, x)] \text{ on } X. \quad (39)$$

$\|g\|$ coincides with the minimum of all such λ .

Theorem 24. If $K^{(1)}(y, x)$ and $K^{(2)}(y, x)$ are two positive definite kernels on X , then the following statements are mutually equivalent:

- i. $\mathcal{H}_{K^{(1)}}(X) \subseteq \mathcal{H}_{K^{(2)}}(X)$.
- ii. There exists $0 \leq \lambda < \infty$ such that

$$[K^{(1)}(y, x)] \leq \lambda^2 [K^{(2)}(y, x)].$$

Note. For any map φ from a set X to a Hilbert space \mathcal{H} , with the notation $x \mapsto \varphi_x$, a kernel K can be defined by

$$K(y, x) = \langle \varphi_x, \varphi_y \rangle \text{ for } x, y \in X. \quad (40)$$

Theorem 25. Let $\varphi : X \mapsto \mathcal{H}$ be an arbitrary map and for $x, y \in X$ let K be defined as

$$K(y, x) = \langle \varphi_x, \varphi_y \rangle.$$

Then K is a positive definite kernel.

Theorem 26. Let T be the linear operator from \mathcal{H} to the space of functions on X , defined by

$$(Tf)(x) = \langle f, \varphi_x \rangle \text{ for } x \in X, f \in \mathcal{H}.$$

Then $\text{Ran}(T)$ coincides with $\mathcal{H}_K(X)$ and

$$\|Tf\|_K = \|P_{\mathcal{M}}f\| \text{ for } f \in \mathcal{H},$$

where \mathcal{M} is the orthogonal complement of $\text{Ker}(T)$, $P_{\mathcal{M}}$ is the orthogonal projection onto \mathcal{M} , and $\|\cdot\|_K$ denotes the norm in $\mathcal{H}_K(X)$.

Kolmogorov decomposition. Let $K(y, x)$ be a positive definite kernel on an abstract set X . Then there exists a Hilbert space \mathcal{H} and a function $\varphi : X \rightarrow \mathcal{H}$ such that

$$K(y, x) = \langle \varphi_x, \varphi_y \rangle \text{ for } x, y \in X.$$

3.2 Operations with RKHSs

Theorem 27. Let $K^{(0)}$ be the restriction of the positive definite kernel K to a nonempty subset X_0 of X and let $\mathcal{H}_{K^{(0)}}(X)$ and $\mathcal{H}_K(X)$ be the RKHS corresponding to $K^{(0)}$ and K , respectively. Then

$$\mathcal{H}_{K^{(0)}}(X_0) = \{f|_{X_0} : f \in \mathcal{H}_K(X)\} \quad (41)$$

and

$$\|h\|_{K^{(0)}} = \min \left\{ \|f\|_K : f|_{X_0} = h \right\} \text{ for all } h \in \mathcal{H}_{K^{(0)}}(X_0). \quad (42)$$

Remark. If $K^{(1)}(y, x)$ and $K^{(2)}(y, x)$ are two positive definite kernels, then

$$K(y, x) = K^{(1)}(y, x) + K^{(2)}(y, x)$$

is also a positive definite kernel.

Remark. Let $\mathcal{H}_{K^{(1)}}$, $\mathcal{H}_{K^{(2)}}$, and \mathcal{H}_K be RKHSs with reproducing kernels $K^{(1)}(y, x)$, $K^{(2)}(y, x)$, and $K(y, x)$, respectively, and let $K = K^{(1)} + K^{(2)}$. Then

$$\mathcal{H}_K(X) = \mathcal{H}_{K^{(1)}}(X) + \mathcal{H}_{K^{(2)}}(X),$$

and for $f \in \mathcal{H}_{K^{(1)}}(X)$ and $g \in \mathcal{H}_{K^{(2)}}(X)$, it follows that

$$\|f + g\|_K^2 = \min \left\{ \|f + h\|_{K^{(1)}}^2 + \|g - h\|_{K^{(2)}}^2 : h \in \mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X) \right\}. \quad (43)$$

Theorem 28. The intersection $\mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X)$ of Hilbert spaces $\mathcal{H}_{K^{(1)}}(X)$ and $\mathcal{H}_{K^{(2)}}(X)$ is again a Hilbert space of functions on X with respect to the norm

$$\|f\|^2 := \|f\|_{K^{(1)}}^2 + \|f\|_{K^{(2)}}^2.$$

In addition the intersection Hilbert space is a RKHS.

Theorem 29. The reproducing kernel of the space

$$\mathcal{H}_K(X) = \mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X)$$

is determined, as a quadratic form, by

$$\begin{aligned} \sum_{x,y} \overline{\epsilon_y} \epsilon_x K(y, x) &= \inf \left\{ \sum_{x,y} \overline{\eta_y} \eta_x K^{(1)}(y, x) + \sum_{x,y} \overline{\zeta_y} \zeta_x K^{(2)}(y, x) : [\epsilon_x] \right. \\ &= [\eta_x] + [\zeta_x] \left. \right\}, \end{aligned}$$

where $[\epsilon_x]$, $[\eta_x]$, $[\zeta_x]$ are an arbitrary complex valued function on X with finite support.

Theorem 30. The tensor product Hilbert space

$$\mathcal{H}_{K^{(1)}}(X) \otimes \mathcal{H}_{K^{(2)}}(X)$$

is a RKHS on $X \times X$.

Theorem 31. The RKHS $\mathcal{H}_K(X)$ of the kernel $K(y, x) = K^{(1)}(y, x) \cdot K^{(2)}(y, x)$ consists of all functions f on X for which there are sequences $(g_n)_{n \geq 0}$ of functions in $\mathcal{H}_{K^{(1)}}(X)$ and $(h_n)_{n \geq 0}$ of functions in $\mathcal{H}_{K^{(2)}}(X)$ so that

$$\sum_1^\infty \|g_n\|_{K^{(1)}}^2 \|h_n\|_{K^{(2)}}^2 < \infty, \quad \sum_1^\infty g_n(x) h_n(x) = f(x), \quad x \in X, \quad (44)$$

and the norm is given by

$$\|f\|_K^2 = \min \left\{ \sum_1^\infty \|g_n\|_{K^{(1)}}^2 \|h_n\|_{K^{(2)}}^2 \right\},$$

where the minimum is taken over the set of all sequences $(g_n)_{n \geq 0}$ and $(h_n)_{n \geq 0}$ satisfying (44).

3.3 Examples of RKHS. Bergman and Hardy spaces

Bergman space. The space of all analytic functions f on Ω for which

$$\iint_{\Omega} |f(z)|^2 dx dy < \infty, \quad (z = x + iy)$$

is satisfied is called the *Bergman space* on Ω and denoted by $A^2(\Omega)$.

Remark. $A^2(\Omega)$ is a RKHS with respect to the inner product

$$\langle f, g \rangle \equiv \langle f, g \rangle_{\Omega} := \iint_{\Omega} f(z) \overline{g(z)} dx dy,$$

and its kernel is called the *Bergman kernel* on Ω and denoted by $B^{(\Omega)}(w, z)$.

Bergman kernel for the unit disc. The Bergman kernel for the open unit disc \mathbb{D} is given by

$$B^{(\mathbb{D})}(w, z) = \frac{1}{\pi} \frac{1}{(1 - w\bar{z})^2} \quad \text{for } w, z \in \mathbb{D}. \quad (45)$$

Bergman kernel of a simply connected domain. The Bergman kernel of a simply connected domain $\Omega (\neq \mathbb{C})$ is given by

$$B^{(\Omega)}(w, z) = \frac{1}{\pi} \frac{\varphi'(w) \overline{\varphi'(z)}}{(1 - \varphi(w) \overline{\varphi(z)})^2} \quad \text{for } w, z \in \Omega, \quad (46)$$

where φ is any conformal mapping function from Ω onto \mathbb{D} .

Theorem 32. A conformal mapping from Ω to \mathbb{D} can be recovered from the Bergman kernel of Ω .

Jordan curve. A *Jordan curve* is a continuous 1 – 1 image of $\{|\xi| = 1\}$ in \mathbb{C} .

Green function. A Green function $G(w, z)$ of Ω is a function harmonic in Ω except at z , where it has logarithmic singularity, and continuous in the closure $\overline{\Omega}$, with boundary values $G(w, z) = 0$ for all $w \in \partial\Omega$, where Ω is a finitely connected domain of the complex plane.

Theorem 33. Let Ω be a finitely connected domain bounded by analytic Jordan curves, and let $G(w, z)$ be the Green's function of Ω . Then the Bergman kernel function is

$$B^{(\Omega)}(w, z) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial w \partial \bar{z}}(w, z), \quad w \neq z. \quad (47)$$

Hardy space. The closed linear span of $\{\varphi_n : n = 0, 1, \dots\}$ in $L^2(\mathbb{T})$ is called the (Hilbert type) *Hardy space* on \mathbb{T} and is denoted by $H^2(\mathbb{T})$. Here $\varphi_n(\xi) = \xi^n$.

Remark. $f \in L^2(\mathbb{T})$ belongs to the Hardy space $H^2(\mathbb{T})$ if and only if it is orthogonal to all φ_n ($n < 0$), that is, all Fourier coefficients of f with negative indices vanish. Then we have

$$\langle f, g \rangle_{L^2} = \sum_{n=0}^{\infty} a_n \bar{b}_n \quad \text{for } f, g \in H^2(\mathbb{T}), \quad (48)$$

where

$$a_n = \langle f, \varphi_n \rangle_{L^2} \quad \text{and} \quad b_n = \langle g, \varphi_n \rangle_{L^2} \quad (n = 0, 1, \dots).$$

Szegő kernel. The kernel $S(\xi, z) := \frac{1}{1 - \xi \bar{z}}$ for $\xi \in \mathbb{T}, z \in \mathbb{D}$, or its analytic extension $\tilde{S}(w, z) := \frac{1}{1 - w \bar{z}}$ for $w, z \in \mathbb{D}$ is called the *Szegő kernel*.

Notes

This chapter intends to offer a sample survey for the fundamental concepts of Hilbert spaces and provide an introductory theory of reproducing kernels. We present the basic properties with important theorems and sometimes with punctual notes and remarks to support the subject. However, due to the limit of content and pages, we skipped the proofs of the theorems. The proofs of the first part can be found in [1, 2] and in most of the basic functional analysis books. Besides, the proofs of the second part (related with the reproducing kernels) can easily be found in [3]. The Hilbert space and functional analysis parts of this chapter are based on the books by J.B. Conway [1] and R.G. Douglas [2]. On the other hand, the reproducing kernel part is based on the lecture notes of T. Ando [4] and N. Aronszajn [5], the book of S. Saitoh and Y. Sawano [6], and the book of B. Okutmustur and A. Gheondea [3]. Moreover, the details of Bergman and Hardy spaces are widely explained in the books [7–9].

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