Chapter

Integral Inequalities and Differential Equations via Fractional Calculus

Zoubir Dahmani and Meriem Mansouria Belhamiti

Abstract

In this chapter, fractional calculus is used to develop some results on integral inequalities and differential equations. We develop some results related to the Hermite-Hadamard inequality. Then, we establish other integral results related to the Minkowski inequality. We continue to present our results by establishing new classes of fractional integral inequalities using a family of positive functions; these classes of inequalities can be considered as generalizations of order n for some other classical/fractional integral results published recently. As applications on inequalities, we generate new lower bounds estimating the fractional expectations and variances for the beta random variable. Some classical covariance identities, which correspond to the classical case, are generalised for any $\alpha \geq 1, \beta \geq 1$. For the part of differential equations, we present a contribution that allow us to develop a class of fractional chaotic electrical circuit. We prove recent results for the existence and uniqueness of solutions for a class of Langevin-type equation. Then, by establishing some sufficient conditions, another result for the existence of at least one solution is also discussed.

Keywords: fractional calculus, fixed point, Riemann-Liouville integral, Caputo derivative, integral inequality

1. Introduction

During the last few decades, fractional calculus has been extensively developed due to its important applications in many field of research [1–4]. On the other hand, the integral inequalities are very important in probability theory and in applied sciences. For more details, we refer the reader to [5–12] and the references therein. Moreover, the study of integral inequalities using fractional integration theory is also of great importance; we refer to [1, 13–17] for some applications.

Also, boundary value problems of fractional differential equations have occupied an important area in the fractional calculus domain, since these problems appear in several applications of sciences and engineering, like mechanics, chemistry, electricity, chemistry, biology, finance, and control theory. For more details, we refer the reader to [3, 18–20].

In this chapter, we use the Riemann-Liouville integrals to present some results related to Minkowski and Hermite-Hadamard inequalities [21]. We continue to present our results by establishing several classes of fractional integral inequalities

using a family of positive functions; these classes of inequalities can be considered as generalizations for some other fractional and classical integral results published recently [22]. Then, as applications, we generate new lower bounds estimating the fractional expectations and variances for the beta random variable. Some classical covariance identities, which correspond to $\alpha = 1$, are generalized for any $\alpha \geq 1$ and $\beta \geq 1$; see [23].

For the part of differential equations, with my coauthor, we present a contribution that allows us to develop a class of fractional differential equations generalizing the chaotic electrical circuit model. We prove recent results for the existence and uniqueness of solutions for a class of Langevin-type equations. Then, by establishing some sufficient conditions on the data of the problem, another result for the existence of at least one solution is also discussed. The considered class has some relationship with the good paper in [20].

The chapter is structured as follows: In Section 2, we recall some preliminaries on fractional calculus that will be used in the chapter. Section 3 is devoted to the main results on integral inequalities as well as to some estimates on continuous random variables. The Section 4 deals with the class of differential equations of Langevin type: we study the existence and uniqueness of solutions for the considered class by means of Banach contraction principle, and then using Schaefer fixed point theorem, an existence result is discussed. At the end, the Conclusion follows.

2. Preliminaries on fractional calculus

In this section, we present some definitions and lemmas that will be used in this chapter. For more details, we refer the reader to [2, 13, 15, 24].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$ is defined as

$$\begin{aligned} J_a^{\alpha}[f(t)] &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0,\ a < t \leq b, \\ J_a^0[f(t)] &= f(t), \end{aligned} \tag{1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Note that for $\alpha > 0, \beta > 0$, we have

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \tag{2}$$

and

$$J_a^{\alpha} J_a^{\beta}[f(t)] = J_a^{\beta} J_a^{\alpha}[f(t)]. \tag{3}$$

In the rest of this chapter, for short, we note a probability density function by $p.d.f$. So, let us consider a positive continuous function ω defined on $[a, b]$. We recall the $\omega-$concepts:

Definition 1.2. The fractional $\omega-$weighted expectation of order $\alpha > 0$, for a random variable X with a positive $p.d.f.f$ defined on $[a, b]$, is given by

$$E_{\alpha,\omega}(X) := J_a^{\alpha}[t\omega f](b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau\omega(\tau) f(\tau) d\tau, \alpha > 0, a < t \leq b, \tag{4}$$

Definition 1.3. The fractional $\omega-$weighted variance of order $\alpha > 0$ for a random variable X having a $p.d.f.f$ on [a, b] is given by

$$\sigma^2_{\alpha,\omega}(X) = V_{\alpha,\omega}(X) := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1}(\tau - E(X))^2 \omega(\tau) f(\tau) d\tau, \alpha > 0. \tag{5}$$

Definition 1.4. The fractional $\omega-$weighted moment of orders $r > 0, \alpha > 0$ for a continuous random variable X having a $p.d.f.f$ defined on [a, b] is defined by the quantity:

$$E_{\alpha,\omega}(X^r) := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1}\tau^r \omega(\tau) f(\tau) d\tau, \alpha > 0. \tag{6}$$

We introduce the covariance of fractional order as follows.

Definition 1.5. Let f_1 and f_2 be two continuous on $[a, b]$. We define the fractional $\omega-$weighted covariance of order $\alpha > 0$ for $\left(f_1(X), f_2(X)\right)$ by

$$Cov_{\alpha,\omega}\left(f_1(X), f_2(X)\right) := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1}\left(f_1(\tau) - f_1(\mu)\right)\left(f_2(\tau) - f_2(\mu)\right)\omega(\tau) f(\tau) d\tau, \alpha > 0, \tag{7}$$

where μ is the classical expectation of X.

It is to note that when $\omega(x) = 1, x \in [a, b]$, then we put

$$Var_{\alpha,\omega}(X) := Var_\alpha(X), Cov_{\alpha,\omega}(X) := Cov_\alpha(X), E_{\alpha,\omega}(X) := E_\alpha(X)$$

Definition 1.6. For a function $K \in C^n([a, b], \mathbb{R})$ and $n - 1 < \alpha \leq n$, the Caputo fractional derivative of order α is defined by

$$\begin{aligned} D^\alpha K(t) &= J^{n-\alpha}\frac{d^n}{dt^n}(K(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} K^{(n)}(s) ds. \end{aligned}$$

We recall also the following properties.

Lemma 1.7. Let $n \in \mathbb{N}^*$, and $n - 1 < \alpha < n$. The general solution of $D^\alpha y(t) = 0$, $t \in [a, b]$ is given by

$$y(t) = \sum_{i=0}^{n-1} c_i (t-a)^i, \tag{8}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1$.

Lemma 1.8. Let $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$. Then

$$J^\alpha D^\alpha y(t) = y(t) + \sum_{i=0}^{n-1} c_i (t-a)^i, t \in [a, b], \tag{9}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1$.

3. Some integral inequalities

3.1 On Minkowski and Hermite-Hadamard fractional inequalities

In this subsection, we present some fractional integral results related to Minkowski and Hermite-Hadamard integral inequalities. For more details, we refer the reader to [21].

Theorem 1.9. Let $\alpha > 0, p \geq 1$ and let f, g be two positive functions on $[0, \infty[$, such that for all $t > 0, J^{\alpha} f^{p}(t) < \infty, J^{\alpha} g^{p}(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0, t]$, then we have

$$[J^{\alpha} f^{p}(t)]^{\frac{1}{p}} + [J^{\alpha} g^{p}(t)]^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} [J^{\alpha}(f+g)^{p}(t)]^{\frac{1}{p}}. \tag{10}$$

Proof: We use the hypothesis $\frac{f(\tau)}{g(\tau)} < M, \tau \in [0, t], t > 0$. We can write

$$\begin{aligned} &\frac{(M+1)^{p}}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f^{p}(\tau) d\tau \\ &\leq \frac{M^{p}}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} (f+g)^{p}(\tau) d\tau. \end{aligned} \tag{11}$$

Hence, we have

$$J^{\alpha} f^{p}(t) \leq \frac{M^{p}}{(M+1)^{p}} J^{\alpha}(f+g)^{p}(t). \tag{12}$$

Thus, it yields that

$$[J^{\alpha} f^{p}(t)]^{\frac{1}{p}} \leq \frac{M}{M+1} [J^{\alpha}(f+g)^{p}(t)]^{\frac{1}{p}}. \tag{13}$$

In the same manner, we have

$$\left(1 + \frac{1}{m}\right) g(\tau) \leq \frac{1}{m} (f(\tau) + g(\tau)). \tag{14}$$

And then,

$$[J^{\alpha} g^{p}(t)]^{\frac{1}{p}} \leq \frac{1}{m+1} [J^{\alpha}(f+g)^{p}(t)]^{\frac{1}{p}}. \tag{15}$$

Combining (13) and (15), we achieve the proof.

Remark 1.10. Applying the above theorem for $\alpha = 1$, we obtain Theorem 1.2 of [25] on $[0, t]$.

With the same arguments as before, we present the following theorem.

Theorem 1.11. Let $\alpha > 0, p \geq 1$ and let f, g be two positive functions on $[0, \infty[$, such that for all $t > 0, J^{\alpha} f^{p}(t) < \infty, J^{\alpha} g^{p}(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0, t]$, then we have

$$[J^{\alpha}f^{p}(t)]^{\frac{1}{p}} + [J^{\alpha}g^{p}(t)]^{\frac{1}{p}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}[J^{\alpha}(f+g)^{p}(t)]^{\frac{1}{p}}. \tag{16}$$

Remark 1.12. Taking $\alpha = 1$ in this second theorem, we obtain Theorem 2.2 in [26] on $[0,t]$.

Using the notions of concave and $L^{p}-$functions, we present to the reader the following result.

Theorem 1.13. Suppose that $\alpha > 0, p > 1, q > 1$ and let f, g be two positive functions on $[0,\infty[$. If f^{p}, g^{q} are two concave functions on $[0,\infty[$, then we have

$$\begin{aligned} &2^{-p-q}(f(0)+f(t))^{p}(g(0)+g(t))^{q}(J^{\alpha}(t^{\alpha-1}))^{2} \\ &\leq J^{\alpha}(t^{\alpha-1}f^{p}(t))J^{\alpha}(t^{\alpha-1}g^{q}(t)). \end{aligned} \tag{17}$$

The proof of this theorem is based on the following auxiliary result.

Lemma 1.14. Let h be a concave function on $[a,b]$. Then for any $x \in [a,b]$, we have

$$h(a)+h(b) \leq h(b+a-x)+h(x) \leq 2h\left(\frac{a+b}{2}\right). \tag{18}$$

3.2 A family of fractional integral inequalities

We present to the reader some integral results for a family of functions [22]. These results generalize some integral inequalities of [27]. We have

Theorem 1.15. Suppose that $(f_{i})_{i=1,\dots n}$ are n positive, continuous, and decreasing functions on $[a,b]$. Then, the following inequality

$$\frac{J^{\alpha}\left[\prod_{i\neq p}^{n} f_{i}^{\gamma_i} f_{p}^{\beta}(t)\right]}{J^{\alpha}\left[\prod_{i=1}^{n} f_{i}^{\gamma_i}(t)\right]} \geq \frac{J^{\alpha}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_{i}^{\gamma_i} f_{p}^{\beta}(t)\right]}{J^{\alpha}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_{i}^{\gamma_i}(t)\right]} \tag{19}$$

holds for any $a < t \leq b, \alpha > 0, \delta > 0, \beta \geq \gamma_{p} > 0,$ where p is a fixed integer in $\{1, 2, \dots, n\}$.

Proof: It is clear that

$$\left((\rho-a)^{\delta}-(\tau-a)^{\delta}\right)\left(f_{p}^{\beta-\gamma_p}(\tau)-f_{p}^{\beta-\gamma_p}(\rho)\right) \geq 0, \tag{20}$$

for any fixed $p \in \{1, \dots n\}$ and for any $\beta \geq \gamma_{p} > 0, \delta > 0, \tau, \rho \in [a,t]; a < t \leq b$.

Taking

$$K_{p}(\tau,\rho) := \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\prod_{i=1}^{n} f_{i}^{\gamma_i}(\tau)\left((\rho-a)^{\delta}-(\tau-a)^{\delta}\right)\left(f_{p}^{\beta-\gamma_p}(\tau)-f_{p}^{\beta-\gamma_p}(\rho)\right), \tag{21}$$

we observe that

$$K_{p}(\tau,\rho) \geq 0. \tag{22}$$

Also, we have

$$0 \le \int_a^t K_p(\tau,\rho)d\tau = (\rho-a)^{\delta} J^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] + f_p^{\beta-\gamma_p}(\rho) J^{\alpha}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] - J^{\alpha}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] - (\rho-a)^{\delta} f^{\beta-\gamma_p}(\rho) J^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]. \tag{23}$$

Hence, we get

$$J^{\alpha}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] J^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] \ge J^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] J^{\alpha}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right]. \tag{24}$$

The proof is thus achieved.

Remark 1.16. Applying Theorem 1.15 for $\alpha = 1, t = b, n = 1,$ we obtain Theorem 3 in [27].

Using other sufficient conditions, we prove the following generalization.

Theorem 1.17. Suppose that $(f_i)_{i=1,\dots n}$ are positive, continuous, and decreasing functions on $[a,b]$. Then for any fixed p in $\{1, 2, \dots, n\}$ and for any $a < t \le b, \alpha > 0, \omega > 0, \delta > 0, \beta \ge \gamma_p > 0,$ we have

$$\frac{J^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\omega}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + J^{\omega}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\alpha}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{J^{\alpha}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\omega}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + J^{\omega}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \ge 1. \tag{25}$$

Proof: Multiplying both sides of (23) by $\frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)}\prod_{i=1}^{n} f_i^{\gamma_i}(\rho), \omega > 0,$ then integrating the resulting inequality with respect to ρ over $(a,t), a < t \le b$ and using Fubini's theorem, we obtain the desired inequality.

Remark 1.18.

i. Applying Theorem 1.17 for $\alpha = \omega,$ we obtain Theorem 1.15.

ii. Applying Theorem 1.17 for $\alpha = \omega = 1, t = b, n = 1,$ we obtain Theorem 3 of [27].

Introducing a positive increasing function g to the family $(f_i)_{i=1,\dots n}$, we establish the following theorem.

Theorem 1.19. Let $(f_i)_{i=1,\dots n}$ and g be positive continuous functions on $[a,b]$, such that g is increasing and $(f_i)_{i=1,\dots n}$ are decreasing on $[a,b]$. Then, the following inequality

$$\frac{J^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{J^{\alpha}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right] J^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \ge 1 \tag{26}$$

holds for any $a < t \le b, \alpha > 0, \delta > 0, \beta \ge \gamma_p > 0,$ where p is a fixed integer in $\{1, 2, \dots, n\}$.

Remark 1.20. Applying Theorem 1.19 for $\alpha = 1, t = b, n = 1,$ we obtain Theorem 4 of [27].

3.3 Some estimations on random variables

3.3.1 Bounds for fractional moments of beta distribution

In what follows, we present some fractional results on the beta distribution [23]. So let us prove the following $\alpha-$version.

Theorem 1.21. Let $X, Y, U,$ and V be four random variables, such that $X \sim B(p,q), Y \sim B(m,n), U \sim B(p,n),$ and $V \sim B(m,q)$. If $(p-m)(q-n) \leq 0$, then

$$\frac{E_\alpha(X^r)E_\alpha(Y^r)}{E_\alpha(U^r)E_\alpha(V^r)} \geq \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, \alpha \geq 1.$$

For the proof of this result, we can apply a weighted version of the fractional Chebyshev inequality as is mentioned in [1].

Remark 1.22. The above theorem generalizes Theorem 3.1 of [7].

We propose also the following $(\alpha,\beta)-$version that generalizes the above result. We have

Theorem 1.23. Let $X, Y, U,$ and V be four random variables, such that $X \sim B(p,q), Y \sim B(m,n), U \sim B(p,n),$ and $V \sim B(m,q)$. If $(p-m)(q-n) \leq 0$, then

$$\frac{E_\alpha(X^r)E_\beta(Y^r) + E_\beta(X^r)E_\alpha(Y^r)}{E_\alpha(U^r)E_\beta(V^r) + E_\beta(U^r)E_\alpha(V^r)} \geq \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, \alpha, \beta \geq 1.$$

Remark 1.24. If $\alpha = \beta = 1$, then the above theorem reduces to Theorem 3.1 of [7].

3.3.2 Identities and lower bounds

In the following theorem, the fractional covariance of X and $g(X)$ is expressed with the derivative of $g(X)$. It can be considered as a generalization of a covariance identity established by the authors of [28]. So, we prove the result:

Theorem 1.25. Let X be a random variable having a $p.d.f$ defined on $[a,b]$; $\mu = E(X)$. Then, we have

$$Cov_\alpha(X, g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b g'(x)dx \int_a^x (b-t)^{\alpha-1}(\mu - t)f(t)dt, \alpha \geq 1. \tag{27}$$

We can prove this result by the application of the covariance definition in the case where $\omega(x) = 1$.

The following theorem establishes a lower bound for $Var_\alpha(g(X))$ of any function $g \in C^1([a,b])$. We have

Theorem 1.26. Let X be a random variable having a $p.d.f$ defined on $[a,b]$, such that $\mu = E(X)$. Then, we have

$$Var_\alpha(g(X)) \geq \frac{1}{Var_{X,\alpha}} \left(\frac{1}{\Gamma(\alpha)} \int_a^b g'(x)dx \int_a^x (b-t)^{\alpha-1}(\mu - t)f(t)dt \right)^2, \tag{28}$$

for any $g \in C^1([a,b])$.

To prove this result, we use fractional Cauchy-Schwarz inequality established in [29].

Remark 1.27. Let us consider $\Omega \in C([a,b])$ that satisfies $\int_a^x (b-t)^{\alpha-1}(\mu-t) f(t)dt = (b-x)^{\alpha-1}\sigma^2\Omega(x)f(x)$. Then, we present the following result.

Theorem 1.28. Let X be a random variable having a $p.d.f.$ defined on $[a,b]$, such that $\mu = E(X), \sigma^2 = Var(X)$ and $\Omega \in C([a,b])$; $\int_a^x (b-t)^{\alpha-1}(\mu-t)f(t)dt = (b-x)^{\alpha-1}\sigma^2\Omega(x)f(x)$. Then, we have

$$Var_\alpha(g(X)) \geq \frac{\sigma^4(X)}{Var_\alpha(X)} E_\alpha^2(g'(X)\Omega(X)). \tag{29}$$

Proof: We have

$$Cov_\alpha^2(X,g(X)) = \left[\frac{1}{\Gamma(\alpha)}\int_a^b g'(x)dx(b-x)^{\alpha-1}\sigma^2\Omega(x)f(x)dx\right]^2. \tag{30}$$

On the other hand, we can see that

$$\left[\frac{1}{\Gamma(\alpha)}\int_a^b g'(x)dx(b-x)^{\alpha-1}\sigma^2\Omega(x)f(x)dx\right]^2 = \sigma^4 E_\alpha^2(g'(X)\Omega(X)) \tag{31}$$

Thanks to the fractional version of Cauchy Schwarz inequality [29], and using the fact that

$$Cov_\alpha^2(X,g(X)) \leq Var_\alpha(X)Var_\alpha(g(X)), \tag{32}$$

we obtain

$$\sigma^4 E_\alpha^2(g'(X)\Omega(X)) \leq Var_\alpha(X)Var_\alpha(g(X)). \tag{33}$$

This ends the proof.

Remark 1.29. Thanks to (30) and (31), we obtain the following fractional covariance identity

$$\sigma^2 E_\alpha(g'(X)\Omega(X)) = Cov_\alpha(X,g(X)).$$

It generalizes the good standard identity obtained in [28] that corresponds to $\alpha = 1$ and it is given by

$$\sigma^2 E(g'(X)\Omega(X)) = Cov(X,g(X)).$$

We end this section by proving the following fractional integral identity between covariance and expectation in the fractional case.

Theorem 1.30. Let X be a continuous random variable with a $p.d.f.$ having a support an interval $[a,b]$, $E(X) = \mu$. Then, for any $\alpha \geq 1$, the following general covariance identity holds

$$Cov_\alpha(h(X),g(X)) = E_\alpha(g'(X)Z(X)), \tag{34}$$

where $g \in C^1([a,b])$, with $E|Z(X)g'(X)| < \infty$, $h(x)$ is a given function and $Z(x)f(x)\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} = \int_a^x (E(h(X)) - h(t))\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt$.

Proof: We have

$$Cov_\alpha(h(X),g(X)) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}(h(x)-h(\mu))(g(x)-g(\mu))f(x)dx \tag{35}$$

and

$$E_\alpha(g'(X)Z(X)) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}g'(x)Z(x)f(x)dx. \tag{36}$$

The definition of $Z(X)$ implies that

$$\begin{aligned} E_\alpha(g'(X)Z(X)) = &\frac{1}{\Gamma(\alpha)}\int_a^\mu (g(\mu)-g(t))(b-t)^{\alpha-1}(h(\mu)-h(t))f(t)dt \\ &+\frac{1}{\Gamma(\alpha)}\int_\mu^b (g(t)-g(\mu))(b-t)^{\alpha-1}(h(t)-h(\mu))f(t)dt. \end{aligned} \tag{37}$$

Hence, we obtain

$$E_\alpha(Z(X)g'(X)) = Cov_\alpha(g(X),h(X)). \tag{38}$$

Remark 1.31. Taking $\alpha = 1$, in the above theorem, we obtain Theorem 2.2 of [10].

4. A class of differential equations of fractional order

Inspired by the work in [4, 20], in what follows we will be concerned with a more general class of Langevin equations of fractional order. The considered class will contain a nonlinearity that depends on a fractional derivative of order δ. So, let us consider the following problem:

$$\begin{aligned} &{}^cD^\alpha\left(D^2+\lambda^2\right)u(t) = f\left(t,u(t),{}^cD^\delta u(t)\right), \\ &t\in[0,1], \quad \lambda\in\mathbb{R}_+^* \\ &0<\alpha\le 1, \quad 0\le\delta<\alpha, \end{aligned} \tag{39}$$

associated with the conditions

$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = \beta u(\eta), \eta\in(0,1), \tag{40}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of fractional order α, D^2 is the two-order classical derivative, $f : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, and $\beta\in\mathbb{R}$, such that $\beta sin(\lambda\eta) \neq sin(\lambda)$.

4.1 Integral representation

We recall the following result [20]:

Lemma 1.32. Let θ be a continuous function on $[0,1]$. The unique solution of the problem

$$\begin{aligned} &{}^{c}D^{\alpha}\left(D^{2}+\lambda^{2}\right)u(t)=\theta(t),\\ &t\in[0,1],\quad \lambda\in\mathbb{R}_{+}^{*}\\ &n-1<\alpha\leq n,\quad n\in\mathbb{N}^{*}, \end{aligned} \tag{41}$$

is given by

$$u(t)=\frac{1}{\lambda}\int_{0}^{t}sin\lambda(t-s)\left(\int_{0}^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\theta(\tau)d\tau+\sum_{i=1}^{n-1}c_{i}s^{i}\right)ds+c_{n}cos(\lambda t)+c_{n+1}sin(\lambda t), \tag{42}$$

where $c_i\in\mathbb{R}, i=1\ldots n+1$.

Thanks to the above lemma, we can state that

The class of Langevin equations (39) and (40) has the following integral representation:

$$\begin{aligned} u(t)=&\frac{1}{\lambda}\int_{0}^{t}sin\lambda(t-s)\left(\int_{0}^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),D^{\delta}(\tau))d\tau\right)ds\\ &+\frac{\sin(\lambda t)}{\Delta}\left[\beta\int_{0}^{\eta}sin\lambda(\eta-s)\left(\int_{0}^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),D^{\delta}(\tau))d\tau\right)ds\right.\\ &\left.-\int_{0}^{1}sin\lambda(1-s)\left(\int_{0}^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),D^{\delta}(\tau))d\tau\right)ds\right], \end{aligned} \tag{43}$$

where

$$\Delta:=\lambda(sin\lambda-\beta sin\lambda\eta). \tag{44}$$

4.2 Existence and uniqueness of solutions

Using the above integral representation (43), we can prove the following existence and uniqueness theorem.

Theorem 1.33. Assume that the following hypotheses are valid:

(H1): The function $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, and there exist two constants $\Lambda_1,\Lambda_2>0$, such that for all $t\in[0,1]$ and $u_i,v_i\in\mathbb{R}, i=1,2$,

$$|f(t,u_1,u_2)-f(t,v_1,v_1)|\leq\Lambda_1|u_1-v_1|+\Lambda_2|u_2-v_2|. \tag{45}$$

(H2): Suppose that $\Lambda\leq\frac{1}{(\Phi+\Upsilon)}$,

where

$$\Phi:=\frac{\Delta_1+\lambda+\beta_1\lambda\eta^{\alpha+1}}{\Gamma(\alpha+2)\lambda\Delta_1},\ \Psi:=\frac{\Delta_1 s^{\alpha}(\alpha+1)+\lambda^{2}+\beta_1\lambda^{2}\eta^{\alpha+1}}{\Gamma(\alpha+2)\lambda\Delta_1},\ \Upsilon:=\frac{\Psi}{\Gamma(2-\delta)},$$

$$\Lambda:=max\,(\Lambda_1,\Lambda_2),\quad \Delta_1=|\Delta|,\quad \beta_1=|\beta|.$$

Then problem (39) and (40) has a unique solution on $[0,1]$.
Proof: We introduce the space

$$E = \{u; u \in \mathcal{C}([0,1]), D^{\delta}u \in \mathcal{C}([0,1])\},$$

endowed with the norm $\|u\|_E := \|u\|_\infty + \|D^{\delta}u\|_\infty$.
Then, $(E, \|.\|_E)$ is a Banach space.
Also, we consider the operator $T : E \to E$ defined by

$$\begin{aligned}(Tu)(t) := \frac{1}{\lambda}\int_0^t sin\lambda(t-s)J_0^{\alpha} f\left(s, u(s), D^{\delta}u(s)\right)ds \\ + \frac{\sin(\lambda t)}{\Delta}\left[\beta\int_0^{\eta} sin\lambda(\eta - s)J_0^{\alpha} f(s, u(s), D^{\delta}(s))ds\right. \\ \left. - \int_0^1 sin\lambda(1-s)J_0^{\alpha} f(s, u(s), D^{\delta}u(s))ds\right]\end{aligned} \tag{46}$$

We shall prove that the above operator is contractive over the space E.
Let $u_1, u_2 \in E$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}|Tu_1(t) - Tu_2(t)| \le \frac{1}{\lambda}\int_0^t |sin\lambda(t-s)| \quad J_0^{\alpha}|f\left(s, u_1(s), D^{\delta}u_1(s)\right) - f\left(s, u_2(s), D^{\delta}u_2(s)\right)|ds \\ + \frac{|\sin(\lambda t)|}{|\Delta|}\left[|\beta|\int_0^{\eta} |sin\lambda(\eta - s)| \quad J_0^{\alpha}|f(s, u_1(s), D^{\delta}u_1(s)) - f(s, u_2(s), D^{\delta}u_2(s))|ds\right. \\ \left. + \int_0^1 |sin\lambda(1-s) \quad |J_0^{\alpha}|f(s, u_1(s), D^{\delta}u_1(s)) - f(s, u_2(s), D^{\delta}u_2(s))|ds\right] := \mathcal{A}\end{aligned}$$

By (H1), we have

$$\mathcal{A} \le \frac{\Lambda}{\Gamma(\alpha+2)}\left(\frac{1}{\lambda} + \frac{1}{|\Delta|} + \frac{|\beta|}{|\Delta|}\eta^{\alpha+1}\right)\left(|u_1 - u_2| + |D^{\delta}u_1 - D^{\delta}u_2|\right).$$

Hence, it yields that

$$\|Tu_1 - Tu_2\|_\infty \le \Lambda\Phi\|u_1 - u_2\|_E. \tag{47}$$

With the same arguments as before, we can write.

$$\begin{aligned}|T'u_1(t) - T'u_2(t)| \le \frac{1}{\lambda}|sin\lambda(t-s)|J_0^{\alpha}|f\left(s, u_1(s), D^{\delta}u_1(s)\right) - f\left(s, u_2(s), D^{\delta}u_2(s)\right)|ds \\ + \frac{|\lambda cos(\lambda t)|}{|\Delta|}\left[|\beta|\int_0^{\eta} |sin\lambda(\eta - s)|J_0^{\alpha}|f\left(s, u_1(s), D^{\delta}u_1(s)\right) - f\left(s, u_2(s), D^{\delta}u_2(s)\right)|ds\right. \\ \left. + \int_0^1 |sin\lambda(1-s)|J_0^{\alpha}|f(s, u_1(s), D^{\delta}u_1(s)) - f(s, u_2(s), D^{\delta}u_2(s))|ds\right] := \mathcal{B}.\end{aligned}$$

Again, by (H1), we obtain

$$\mathcal{B} \leq \Lambda\left(\frac{\Delta_1 s^{\alpha}(\alpha+1)+\lambda^2+\beta_1\lambda^2\eta^{\alpha+1}}{\Gamma(\alpha+2)\lambda\Delta_1}\right)\left(|u_1-u_2|+|D^{\delta}u_1-D^{\delta}u_2|\right).$$

Consequently, we get

$$\|T'u_1-T'u_2\|_{\infty} \leq \Lambda\Psi\|u_1-u_2\|_E.$$

This implies that

$$\|D^{\delta}Tu_1-D^{\delta}Tu_2\|_{\infty} \leq \Lambda\Upsilon\|u_1-u_2\|_E. \tag{48}$$

Using (47) and (48), we can state that

$$\|Tu_1-Tu_2\|_E \leq \Lambda(\Phi+\Upsilon)\|u_1-u_2\|_E.$$

Thanks to (H2), we can say that the operator T is contractive.

Hence, by Banach fixed point theorem, the operator has a unique fixed point which corresponds to the unique solution of our Langevin problem.

4.3 Existence of solutions

We prove the following theorem.

Theorem 1.34. Assume that the following conditions are satisfied:

(H3): The function $f : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is jointly continuous.

(H4): There exists a positive constant $\mathcal{M}$; $|f(t,u,v)| \leq \mathcal{M}$ for any $t\in[0,1], u,v\in\mathbb{R}$.

Then the problem (39), (40) has at least one solution on $[0,1]$.

Proof: We use Schaefer fixed point theorem to prove this result. So we proceed into three steps.

Step 1: We prove that T is continuous and bounded.

Since the function f is continuous by (H3), then the operator is also continuous; this proof is trivial and hence it is omitted.

Let $\Omega\subset E$ be a bounded set. We need to prove that $T(\Omega)$ is a bounded set.

Let $u\in\Omega$. Then, for any $t\in[0,1]$, we have

$$|(Tu)(t)| \leq \left(\frac{1}{\lambda}+\frac{1}{|\Delta|}\right)\int_0^1 J_0^{\alpha}|f\left(s,u(s),D^{\delta}u(s)\right)|ds+\frac{|\beta|}{|\Delta|}\int_0^{\eta} J_0^{\alpha}|f\left(s,u(s),D^{\delta}(s)\right)|ds := \mathcal{C}$$

Using (H4), we get

$$\|Tu\|_{\infty} \leq \Phi\mathcal{M}. \tag{49}$$

In the same manner, we find that

$$\|D^{\delta}Tu\|_{\infty} \leq \Upsilon\mathcal{M}. \tag{50}$$

From (49) and (50), we have

$$\|Tu\|_E \leq (\Phi+\Upsilon)\mathcal{M}.$$

The operator is thus bounded.

Step 2: Equicontinuity.

Let $u \in E$. Then, for each $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}|Tu(t_2) - Tu(t_1)| \leq |\frac{1}{\lambda}\left[\int_0^{t_2} sin\lambda(t_2 - s)J_0^\alpha f(s,u(s),D^\delta u(s))ds - \int_0^{t_1} sin\lambda(t_1 - s)J_0^\alpha f(s,u(s),D^\delta u(s))ds\right]\\ + \frac{\sin(\lambda t_2) - \sin(\lambda t_1)}{\Delta}\left[\beta\int_0^{\eta} sin\lambda(\eta - s)J_0^\alpha f(s,u(s),D^\delta u(s))ds\right.\\ \left. + \int_0^{1} sin\lambda(1 - s)J_0^\alpha f(s,u(s),D^\delta u(s))ds\right]|\\ \leq \frac{|\Theta|}{|\Delta|}|\sin(\lambda t_2) - \sin(\lambda t_1)| + \frac{1}{\lambda}\int_{t_1}^{t_2} sin\lambda(t_2 - s)J_0^\alpha|f\left(s,u(s),D^\delta u(s)\right)|ds\\ + \frac{1}{\lambda}\int_0^{t_2}|(sin\lambda(t_2 - s) - sin\lambda(t_1 - s)|J_0^\alpha|f\left(s,u(s),D^\delta u(s)\right)|ds,\end{aligned} \tag{51}$$

where

$$\Theta := \beta\int_0^{\eta}|sin\lambda(\eta - s)J_0^\alpha|f\left(s,u(s),D^\delta u(s)\right)|ds + \int_0^{1}|sin\lambda(1 - s)|J_0^\alpha|f\left(s,u(s),D^\delta u(s)\right)|ds.$$

Analogously, we can obtain

$$|T'u(t_2) - T'u(t_1)| \leq \lambda\frac{|\Theta|}{|\Delta|}|cos(\lambda t_2) - cos(\lambda t_1)| + \frac{1}{\lambda}|sin\lambda(t_2 - s) - sin\lambda(t_1 - s)|J_0^\alpha|f\left(s,u(s),D^\delta u(s)\right)|.$$

Consequently, we can write

$$|D^\delta Tu(t_2) - D^\delta Tu(t_1)| \leq J^{1-\delta}|T'u(t_2) - T'u(t_1)| \tag{52}$$

As $t_1 \to t_2$, the right-hand sides of (51) and (52) tend to zero.

Therefore,

$$\|Tu(t_2) - Tu(t_1)\|_E \to 0.$$

The operator T is thus equicontinuous.

As a consequence of Step 1 and Step 2 and thanks to Arzela-Ascoli theorem, we conclude that T is completely continuous.

Step 3: We prove that $\Sigma := \{u \in E; u = \lambda Tu, 0 < \lambda < 1\}$ is a bounded set.

Let $u \in \Sigma$. Then, for each $t \in [0,1]$, the following two inequalities are valid:

$$|u(t)| = |\lambda Tu(t)| \leq |Tu(t)| \leq \mathcal{M}\Phi$$

and

$$|D^\delta u(t)| = |\lambda D^\delta Tu(t)| \leq |D^\delta Tu(t)| \leq \mathcal{M}\Upsilon.$$

Therefore,

$$\|u\|_E \leq \mathcal{M}(\Upsilon + \Phi).$$

Thanks to steps 1, 2, and 3 and by Schaefer fixed point theorem, the operator T has at least one fixed point. This ends the proof of the above theorem.

5. Conclusions

In this chapter, the fractional calculus has been applied for some classes of integral inequalities. In fact, using Riemann-Liouville integral, some Minkowski and Hermite-Hadamard-type inequalities have been established. Several other fractional integral results involving a family of positive functions have been also generated. The obtained results generalizes some classical integral inequalities in the literature. In this chapter, we have also presented some applications on continuous random variables; new identities have been established, and some estimates have been discussed.

The existence and the uniqueness of solutions for nonlocal boundary value problem including the Langevin equations with two fractional parameters have been studied. We have used Caputo approach together with Banach contraction principle to prove the existence and uniqueness result. Then, by application of Schaefer fixed point theorem, another existence result has been also proved. Our approach is simple to apply for a variety of real-world problems.

Author details

Zoubir Dahmani[1*] and Meriem Mansouria Belhamiti[2]

1 Laboratory LMPA, Faculty of SEI, UMAB, University of Mostaganem, Algeria

2 Department of Math-Info, Faculty of SEI, University of Mostaganem, Algeria

*Address all correspondence to: zzdahmani@yahoo.fr

References

[1] Belarbi S, Dahmani Z. On some new fractional integral inequalities. Journal of Inequalities in Pure and Applied Mathematics. 2009;**10**(3):1-12

[2] Kilbas AA, Srivastava M, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam: Elsevier Science B.V; 2006

[3] Podlubny I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. San Diego: Academic Press; 1999

[4] Salem A, Alzahrani F, Almaghamsi L. Fractional Langevin equation with nonlocal integral boundary condition. Mathematics. 2019;**7**:110

[5] Barnett NS, Cerone P, Dragomir SS, Roumeliotis J. Some inequalities for the expectation and variance of a random variable whose PDF is n-time diferentiable. Journal of Inequalities in Pure and Applied Mathematics. 2000; **1**(21):1-29

[6] Barnett NS, Cerone P, Dragomir SS, Roumeliotis J. Some inequalities for the dispersion of a random variable whose PDF is defined on a finite interval. Journal of Inequalities in Pure and Applied Mathematics. 2001;**2**(1):1-18

[7] Kumar P, Singh SP, Dragomir SS. Some inequalities involving beta and gamma functions. Nonlinear Analysis Forum. 2001;**6**(1):143-150

[8] Kumar P. Moment inequalities of a random variable defined over a finite interval. Journal of Inequalities in Pure and Applied Mathematics. 2002;**3**(3): 1-24

[9] Kumar P. Inequalities involving moments of a continuous random variable defined over a finite interval. Computers and Mathematics with Applications. 2004;**48**:257-273

[10] Mohtashami Borzadaran GR, Shanbhag DN. Further results based on Chernoff-type inequalities. Statistics & Probability Letters. 1998;**39**:109-117

[11] Niezgoda M. New bounds for moments of continuous random variables. Computers & Mathematcs with Applications. 2010;**60**(12): 3130-3138

[12] Sharma R, Devi S, Kapoor G, Ram S, Barnett NS. A brief note on some bounds connecting lower order moments for random variables defined on a finite interval. International Journal of Applied Science and Research. 2009; **1**(2):83-85

[13] Dahmani Z. Fractional integral inequalities for continuous random variables. Malaya Journal of Matematik. 2014;**2**(2):172-179

[14] Dahmani Z, Tabharit L. On weighted Gruss type inequalities via fractional integrals. Journal of Advanced Research in Pure Mathematics. 2010; **2**(4):31-38

[15] Dahmani Z, Bouziane AE, Houas M, Sarikaya MZ. New w-weighted concepts for continuous random variables with applications. Note di Matematica. 2017; **37**(1):2340

[16] Sarikaya MZ, Yaldiz H. New generalization fractional inequalities of Ostrowski-Gruss type. Lobachevskii Journal of Mathematics. 2013;**34**(4): 326-331

[17] Sarikaya MZ, Yaldiz H, Basak N. New fractional inequalities of Ostrowski-Gruss type. Le Matematiche. 2014;**69**:227-235

[18] Dahmani M, Anber A, Dahmani Z. Speech movements on vocal tract:

Fractional nonlinear dynamics. Journal of Information and Optimization Sciences. 2019;**40**(6):1307-1315

[19] Mainradi F, Pironi P. The fractional Langevin equation: Brownian motion revisited. Extracta Mathematicae. 1996; **11**(1):140-154

[20] Salem A, Alzahrani F, Almaghamsi L. Langevin equation involving one fractional order with three point boundary conditions. Journal of Nonlinear Sciences and Applications. 2019;**12**:791-798

[21] Dahmani Z. On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. Annals of Functional Analysis. 2010;**1**(1):51-58

[22] Dahmani Z. New classes of integral inequalities of fractional order. Le Matematiche. 2014;**LXIX**(1):237-247

[23] Dahmani Z. New identities and lower bounds for CUD and beta distributions. ROMAI Journal. 2019

[24] Gorenflo R, Mainardi F. Fractional Calculus: Integral and Differential Equations of Fractional Order. Wien: Springer Verlag; 1997. pp. 223-276

[25] Bougoffa L. On Minkowski and Hardy integral inequality. Journal of Inequalities in Pure and Applied Mathematics. 2003;**7**(2):60

[26] Set E, Ozdemir ME, Dragomir SS. On the Hermite-Hadamard inequality and other integral inequalities involving two functions. Journal of Inequalities and Applications. 2010:148102. DOI: 10.1155/2010/148102

[27] Liu W, Ngo QA, Huy VN. Several interesting integral inequalities. Journal of Mathematical Inequalities. 2009; **10**(2):201-212

[28] Caccoulos T, Papathanasiou V. Characterizations of distribution by variance bounds. Statistics & Probability Letters. 1989;**7**:351-356

[29] Dahmani Z. About some integral inequalities using Riemann-Liouville integrals. General Mathematics. 2012; **20**(4):63-69