Chapter

# Dynamic Stability of Open Two-Link Mechanical Structures

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## Abstract

The chapter deals with the assessment of the dynamic stability of elements selected from the truss or frame construction, which contains input and output parts (links) connected by a force line. From the aggregate of all factors, the resulting force factors and reactions are considered. Instead of the commonly used study of the moving of parts, a new method has been applied, consisting in the study of fluctuations in the speeds of movement and stresses. For this purpose, two partial differential equations are derived that relate the acceleration and the rate of voltage change to the gradients of these variables along the line of force. Using the Laplace transform obtained, the general equations of motion of the slave link. A technique for assessing the degree of distribution of force line parameters is derived, and the conditions for the loss of dynamic stability are identified. It is shown that in this mode, the destruction element of the truss or the frame is possible.


**Keywords:** frame, truss, two-link element, force line, speed, stress, partial derivative, differential equation, dynamic stability

## 1. Introduction

In various designs, parts that transmit any motion are often used. Any such design often consists of an input and an output link connected by a force line. With the perception of the load, either compression (stretching) or twisting takes place here. Usually, such elements are checked for longitudinal stability, according to Euler's criterion [1], or for the ultimate twisting. However, such devices often perceive variable loads, for example, wind, shock, etc., at which various vibrations occur. In this regard, it is advisable to evaluate the dynamic stability, which can manifest itself in the form of self-oscillatory regimes, both for the whole truss structure and for its elements, or in the form of sudden destruction. The proposed work is devoted to the study of the loss of dynamic stability of the elements of a truss or frame.

The stability problem of the movement of mathematics and mechanics has been studied since the nineteenth century. To solve such problems, the criteria and theories of Routh E., Gurwitz A., Lyapunov A., Chetayev N., Mikhailov A., Nyquist H., Bolotin V., Popov E. [2–8], etc. are used.

In the last years, many developments have been made, both in the theory and applications of the subject. However, accurate analytical solutions in the calculations of vibrations of a structural element were obtained in rare cases. Typically, calculations are performed approximately. Simplifications are made when choosing a design scheme for the mechanism. In such cases, negligible features of the system are neglected, and the main parameters that determine the nature of the phenomenon are distinguished.

In most cases, a method is specified in which parts of complex geometric shape (springs, crankshafts, etc.) are considered as equivalent straight bar or nonlinear elastic elements are replaced by linear elements. This approach allows replacing a mechanical system with concentrated masses with a system with distributed parameters [9]. Thus, simplifications are allowed that lead to the loss of objective data.

Some publications [10, 11] provide solutions to such problems by an approximate method with the replacement of the corresponding functional equations by suitable finite-dimensional difference schemes. As a result, the authors come to the problem of optimal control of the approximating system, which is described by equations in finite differences or the system of ordinary differential equations [9]. Then there is a need to consider the maximum principle and evaluate approximation methods. Such questions have not enough yet been investigated.

Some specialists of mechanical, for example, the authors of the Encyclopedia of Engineering Industry, Fedosov E., Krasovsky A., Popov E., propose to evaluate the stability of mechanical systems with distributed parameters by dispersion relations, i.e., according to the internal properties of the physical process. Here we use differential equations with variable coefficients that characterize the process under consideration. In this case, the solution of differential equations should be sought by numerical methods [12].

The condition for the stable operation of a system with distributed parameters was formulated in [13, 14]. The mathematical essence of the stability condition is formulated as follows:

If in the subspace $W_\varphi = 0$ the process $\varphi \equiv 0$ is stable under integrally small perturbations with respect to the measure $||\rho||$, and in the subspace $W_\varphi < 0$ – asymptotically stable under integrally small perturbations with measure $||\rho||$, then in a neighborhood $Z_R$ for any $\delta\ (\varepsilon, t_0) > 0$, there exists a number such that for $t \geq T$ it is true $\rho\ [\varphi\ (\cdot, t)] < 2\delta$, if $\rho\ [\varphi\ (\cdot, t_0)] < \delta$ and $\rho\ [h\ (x)] < \delta$. Here, $\varphi$ are the parameters of the process; $h\ (x)$ is a vector function of admissible solutions.

At present, it has not been possible to find scientific publications in which stability criteria are sufficiently clearly formulated in the study of open two-link mechanical systems with distributed parameters in the presence of significant nonlinearities.

These mechanisms are widely used by technicians. Therefore, it is very important to develop such methods that would make it possible to more accurately mathematically formalize the functioning processes and determine the zones of stable and unstable operation of these mechanisms.

In this regard, the proposed work attempts to consider in more detail the stability issues of these mechanical systems.

## 2. Statement of the problem

The reliability of the functioning of the noted mechanisms under external variable loads is largely determined by the speeds of the links and the stresses in the force lines. Therefore, there is a need to study the Equations [15]

$$\frac{d\Omega_j}{dt} = \sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial \xi_i} f_i, \tag{1}$$

where $\xi_i$ are the coordinates of the system, $f_i = d\,\xi_i/dt$, $t$ is the time, and $\Omega_j$ is the speed of the technological object.

Then the investigation reduces to solving equations

$$\frac{d\xi_i}{dt} = f_i(t, \xi_1, \xi_2, \ldots, \xi_n). \tag{2}$$

If such a path seems natural for specialists in control systems, then for the specialists-mechanics, it may seem unusual, since they often solve the problem of determining the change of coordinates and the shape of oscillations [16–21]. Such processes are usually investigated by methods of the theory of elasticity, for example, using the equation [22, 23]

$$v \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(Ef \frac{\partial u}{\partial x}\right) = Q(x, t), \tag{3}$$

whose solution is sought in the form

$$u(x, t) = \sum_{i=1}^{\infty} H_i \theta(x) \sin\left(p_i t + \alpha_i\right). \tag{4}$$

where $\nu$ and $E$ are mass and elastic characteristics of the mechanical highway, $f$ is cross-sectional area, $Q$ is intensity of external load, and $H_i$, $\theta$, $p_i$, and $\alpha_i$ are constants determined from the initial conditions.

In the case of using the Lagrange equation of the second kind, the oscillations of the kinetic ($T$) and potential energy ($U$) are considered. The Lagrange equation of the second kind has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q. \tag{5}$$

Here

$$Q = -\frac{\partial U}{\partial q}. \tag{6}$$

The parameters of the movement of the mechanism are determined from Eq. (5) after some transformations.

These Eqs. (5, 6) which are the basis of many papers on the dynamics of machines, for example, [21, 24, 25], etc., allow, under given boundary conditions, to estimate the change in the displacements of rod section, pipe string, etc. in time and space.

On the one hand, such information is redundant if it is necessary to take into account the interconnection of a large number of factors. For example, to assess the performance of the system, it is enough to know under what conditions self-oscillations occur (i.e., stability is lost), and at what not.

On the other hand, due to the lack of explicit information about the stresses developed in the dynamic process, it is difficult to estimate the probability of part failure.

In the above approaches, such methods of solving problems are specified in which a linear relationship between stresses and displacements of points of a solid body is adopted. According to the accepted linear dependence, these quantities are recalculated. These approaches may not always be applicable, since it is known from rheology that the elastic modulus can depend on the vibration frequency [26, 27].

In addition, depending on the stresses, the rod can be bent and thereby change the peculiarities of the formation of force factors at the links of the mechanism. At the same time, various nonlinear effects, including the essential ones, such as backlash, have a significant impact on the functioning. In this regard, there is a need to develop a method where the oscillations are clearly taken into account speeds and voltages, as well of various nonlinearities.

## 3. Basic equations

To solve this problem, it is assumed that in dynamics the elements of a truss or frame can be represented as models in **Figure 1**.

In accordance with the theory of strength of materials [1], part of the links of a mechanical system can be represented as a separate element, on which, in addition to external forces, bond reactions act. Therefore, during vibrations, the ends of such an element move with certain speeds, and force factors ($F_c$, $M_r$) are the corresponding resulting factors. To this we add viscous resistance ($h$), which we will consider as resistance to the movement of a particular unit from the side of the entire or adjacent part of the structure to this element (**Figure 1**).

When considering longitudinal vibrations in a straight solid rod, we use the equation quantity of motion in differential form for the case of the absence of mass forces [28]

$$\rho \frac{\partial \upsilon}{\partial t} = \frac{\partial \sigma}{\partial x} \tag{7}$$

and the equation of longitudinal oscillations [27]

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}. \tag{8}$$

where $\upsilon$ is a speed of longitudinal displacement ($\upsilon = \partial u/\partial t$), $u$ is the displacement along the $x$-axis, $\sigma$ are longitudinal (normal) stresses, $\rho$ is density of the material, and $E$ is modulus of elasticity.

Let us assume at this stage that $E$ = const and $\rho$ = const. We determine the derivative $\partial\upsilon/\partial t$ from Eq. (7) and substitute it in the left side of Eq. (8). We therefore have

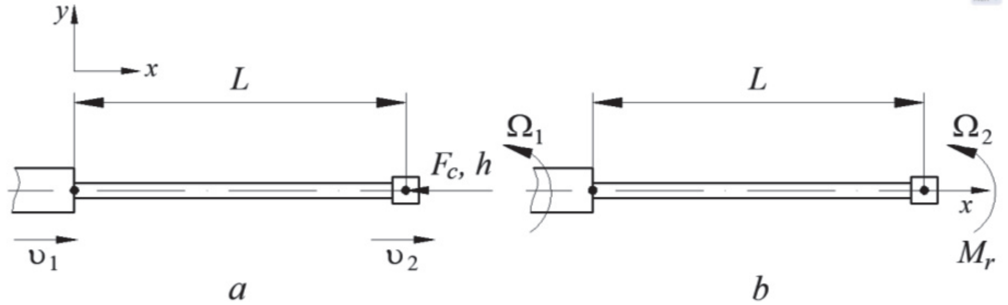


**Figure 1.**
*Models of a rod with a mass: (a) with longitudinal vibrations; (b) with torsional vibrations.*

$$\frac{1}{\rho}\frac{\partial\sigma}{\partial x} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}. \tag{9}$$

We integrate the Eq. (9) by x, assuming that for $x = 0$, $\sigma = \text{const}$:

$$\frac{1}{E}\int_0^x \frac{\partial\sigma}{\partial x}dx = \int_0^x \frac{\partial^2 u}{\partial x^2}dx. \tag{10}$$

We finally obtain

$$\frac{1}{E}(\sigma_x - \sigma_0) = \left.\frac{\partial u}{\partial x}\right|_x - \left.\frac{\partial u}{\partial x}\right|_{x=0}. \tag{11}$$

Denote the current value of the stress $\sigma_x$ by $\sigma$. Given that the surface forces acting on each point of the cross section of the elementary volume are directed in the opposite direction from the direction of the speed of movement, we rewrite the resulting equation in the form

$$\rho\frac{\partial \upsilon}{\partial t} = -\frac{\partial\sigma}{\partial x}. \tag{12}$$

We therefore have

$$\frac{1}{E}\frac{\partial\sigma}{\partial t} = -\frac{\partial \upsilon}{\partial x}. \tag{13}$$

When considering torsional vibrations, we assume that the movement of the sections is absent and the elastic vibrations are described by the equation

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2\varphi}{\partial x^2}. \tag{14}$$

where $\varphi$ and $x$ are the angle of rotation of the section of the rod and the coordinate (**Figure 1b**) and $G$ is the shear modulus of the material.

In addition, we use the equation quantity of motion in differential form for an elementary section of a rod with an outer radius $r$ at $\rho = \text{const}$ [28, 29]

$$\rho r\frac{\partial\Omega}{\partial t} = -\frac{\partial\tau}{\partial x}, \tag{15}$$

where $\Omega$ is the speed of rod cross section ($\Omega = \partial\varphi/\partial t$) and $\tau$ is maximum shear stresses of rod cross section.

From comparison Eqs. (14) and (15), we arrive at the equation

$$G\frac{\partial^2\varphi}{\partial x^2} = -\frac{1}{r}\frac{\partial\tau}{\partial x}. \tag{16}$$

We integrate this equation over coordinate $x$. We finally obtain

$$rG\frac{\partial\varphi}{\partial x} = rG\left.\frac{\partial\varphi}{\partial x}\right|_0 - (\tau_x - \tau_0) = -\tau + B_\tau. \tag{17}$$

Here $B_\tau$ is constant characterizing the stress at the initial conditions $x = x_0$ and $t = t_0$.

We differentiate the derived Eq. (17) over $t$. We have

$$rG\frac{\partial\Omega}{\partial x} = -\frac{\partial\tau}{\partial t}. \tag{18}$$

The system of Eqs. (12), (13), (15), and (18), first published in manuscript [29], makes it possible to describe changes in stresses in the elementary volume and velocity of movement of the elementary sections of solid-state lines. These are also applicable of the elementary sections of the solid (of the frames and of the trusses).

It should be noted that the process of motion transmission in systems with hydraulic lines is characterized by the equations [30].

$$\frac{\partial \upsilon}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} - 2\tau_0\rho_0 r_0; \;\; \frac{\partial P}{\partial t} = -\kappa\frac{\partial \upsilon}{\partial x}, \tag{19}$$

where $\rho_0$ is the initial density of the medium, $P$ is line pressure, $\kappa$ is reduced modulus of elasticity of the line, $\tau_0$ is shear stress on the pipe wall, and $r_0$ is the radius of the pipe section.

Thus, the transfer of motion in solid and liquid media can be described by similar equations. This is shown in the analysis of the hydraulic drive operation [29].

## 4. Analysis of the Zener model

The equations of motion for the elementary volume of a substance with relatively low speeds of displacement are obtained above. However, in the event of any abrupt changes caused by either external influences or rapidly occurring vibration phenomena, there is a need for a deeper study of the process of motion transmission in mechanical systems [31].

A number of Maxwell, Voigt, and Zener phenomenological models have been developed for this problem. We consider the more general Zener model [26, 27] (**Figure 2**).

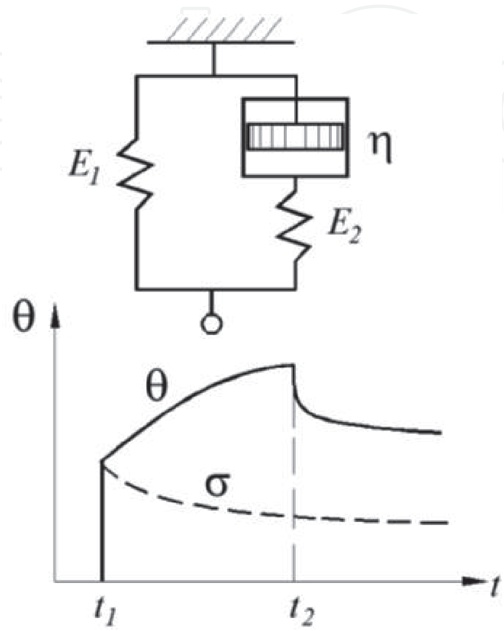


**Figure 2.**
*Zener rheological model: θ is deformation; η is viscosity; σ is normal stress;* t *is time.*

Here it is believed that there is a body that, under the action of stress, is elastically deformed and at the same time can flow. When stress is applied when $t = t_1$, the springs are instantly deformed by magnitudes $\sigma/E_1$ and $\sigma/E_2$, and the piston starts to move evenly with speed $(d\sigma/dt)/\eta$.

The differential equation is written in the form

$$\sigma + \frac{\eta}{E_2}\frac{d\sigma}{dt} = E_1\theta + \eta\frac{d\theta}{dt}. \tag{20}$$

Here, $E_1$ and $E_2$ are the isothermal and adiabatic modulus of elasticity, respectively.

We transform Eq. (20) into an operator form

$$D\sigma = E_2 D\theta - \frac{\sigma - E_1\theta}{\tau_\varepsilon}. \tag{21}$$

where

$$D \equiv \frac{d}{dt};\ \tau_\varepsilon = \frac{\eta}{E_2}. \tag{22}$$

Here $\tau_\varepsilon$ is the relaxation time under the condition of constant deformation.

We perform another transformation

$$\sigma\left(D + \frac{1}{\tau_\varepsilon}\right) = \theta E_2\left(D + \frac{1}{k_e\tau_\varepsilon}\right). \tag{23}$$

Here $k_e = E_2/E_1$.

Passing under zero initial conditions to Laplace transformations [29], we rewrite the Eq. (23) in the form

$$\sigma(s)\left(s + \frac{1}{\tau_\varepsilon}\right) = \theta(s)E_2\left(s + \frac{1}{k_e\tau_\varepsilon}\right). \tag{24}$$

Here, we replaced the operator $D$ with a complex variable with $(D = s)$, and $s = u + jv; j = (-1)^{1/2}$.

The Laplace image of the stress change from (18), taking into account the jump-like deformation $\theta\ (t) = \theta_0\ 1(t)$, is written in the form [32]

$$\sigma(s) = \theta_0 E_2 \frac{s + 1/k_e\tau_\varepsilon}{s(s + 1/\tau_\varepsilon)}. \tag{25}$$

We define the original by means of residues relative to the poles. We then have

$$\sigma(t) = \theta_0 E_2\left[\frac{1}{k_e} + \left(1 - \frac{1}{k_e}\right)\exp\left(-\frac{t}{\tau_\varepsilon}\right)\right] = \theta_0 E_1\left[1 + \left(\frac{E_2}{E_1} - 1\right)\exp\left(-\frac{t}{\tau_\varepsilon}\right)\right]. \tag{26}$$

From this expression, it follows that when $t_1 = 0$, i.e., at the time of a jump-like change in the relative deformation of the rod, the stress is $\sigma(0) = \sigma_0\ E_2$, but then with time the stress decreases, relaxes, at $t_2 > \tau_\varepsilon$ to the value $\sigma(t_2) = \theta_0\ E_1$. This conclusion is mathematically obtained in [29], and the process is illustrated by the graph in **Figure 2**.

Obviously, the elastic modulus $E_2$ corresponds to the adiabatic deformation process, and $E_1$ corresponds to the isothermal process.

Physically, this can be represented as follows. Initially, an adiabatic, without heat transfer, convergence of atoms in metal crystals takes place, but at the same time the entire atomic system becomes unbalanced—non-equilibrium. In order for the system to reach an equilibrium state, a relaxation time $\tau_\varepsilon = \eta/E_2$ is necessary, when the atoms, having received their share of thermal energy, occupy a new position.

*Note*. In physics [33], elastic oscillations in some cases are interpreted as the motion of a phonon gas. In this case, the relaxation of the internal energy in the crystal lattice is described by the kinematic equation for phonons. Acoustic relaxation is always accompanied by sound absorption, its dispersion, and the dependence of the speed of sound on frequency. The physical encyclopedia for solid dielectrics suggests estimating the relaxation constant from the phonon lifetime

$$\tau_\varepsilon \cong \tau_f = 3\lambda/\left(Cc_{av}^2\right), \tag{27}$$

where $C$ is the lattice heat capacity, $\lambda$ is thermal conductivity coefficient, and $c_{av}$ is the average value of the speed of sound.

If the deformation will change according to the harmonic law, then the stress will also change according to the harmonic law, but with a slightly different amplitude and phase advance, depending on frequency.

The ratio modulus $\sigma(\omega)/\theta(\omega)$ and phase shift $\varphi$ are calculated from (23) using expressions [29].

$$\left|\frac{\sigma(\omega)}{\theta(\omega)}\right| = E_1 \frac{\sqrt{1+(k_e\tau_\varepsilon\omega)^2}}{\sqrt{1+(\tau_\varepsilon\omega)^2}}; \tag{28}$$

$$\phi = arctg(k_e\tau_\varepsilon\omega) - arctg(\tau_\varepsilon\omega). \tag{29}$$

where $\omega$ is the circular oscillation frequency.

The peculiarity of the passage of a harmonic signal through a metal is illustrated in **Figure 3**.

Here are graphs of the functions $E = E(\omega)$ and $\varphi = \varphi(\omega)$. And here is $\tau'_\varepsilon = k_e\, \tau_\varepsilon$.

The physical meaning of the function $E(\omega)$ is illustrated by two cases.

The first case: when $\omega < 1/\tau'_\varepsilon$, we have

$$|E(\omega)| = E_1. \tag{30}$$

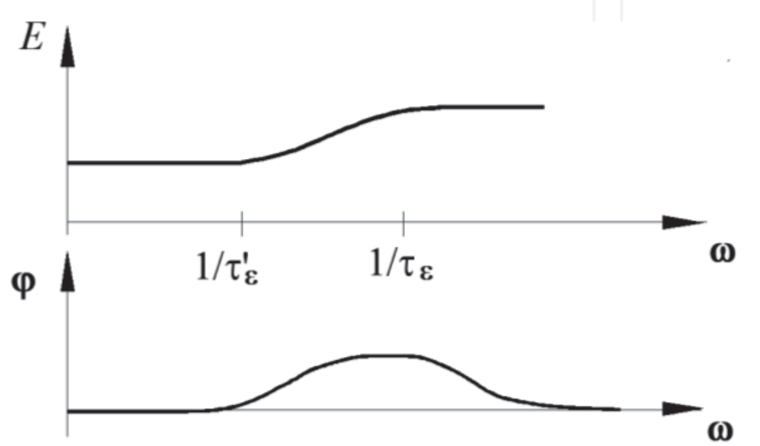


**Figure 3.**
*Passage of a harmonic signal through a metal.*

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The modulus function $E(\omega)$ is equal to the isothermal modulus of elasticity.

The second case: when $\omega > 1/\tau_\varepsilon$, we have

$$|E(\omega)| = E_2. \tag{31}$$

The modulus function $E(\omega)$ is equal to the adiabatic modulus of elasticity.

From Eq. (23), it follows that the ratio $\sigma(\omega)/\theta(\omega)$ describes some complex function, which can be called the function of the generalized elastic modulus $E_\omega(s)$, i.e., we have

$$E_\omega(s) = E_2 \frac{s + (k_e \tau_\varepsilon)^{-1}}{s + (\tau_\varepsilon)^{-1}} = E_u(\omega) + jE_\upsilon(\omega), \tag{32}$$

where $E_u(\omega)$ and $E_\upsilon(\omega)$ are, respectively, the values of the function $E_\omega(\omega)$ along the real and imaginary axes of the complex plane $(U, jV)$.

It follows from the above that the value of the elastic modulus strongly depends on the experimental conditions, in particular, on the oscillation frequency, and on the relaxation spectrum. So, for casting steel 1X15H15M2K3BT, the static modulus of elasticity is $E_{st}$ = 1.617 * $10^5$ MPa, and dynamic is $E_\omega$ = 2.058 * $10^5$ MPa.

If the frequency value is equal to $\omega = (\tau_\varepsilon)^{-1}$, then this may have a certain impact on the properties of the mechanical system.

Since alloys, such as steel, contain different phases, it is likely that each of them will have its own combination of $E_2$, $E_1$, and $\tau_\varepsilon$. In this case, you may have to take into account the average relaxation time, the width relaxation of spectrum, etc.

## 5. Derivation of the initial equations

Let us turn further to the longitudinal oscillations.

The interaction of the elastic wave is largely with the interface of the media due to the wave impedance, which is determined by the relation [34]. It follows that

$$-\frac{\sigma}{\upsilon} = \rho a_1, \tag{33}$$

where $a_1$ is the velocity of propagation of longitudinal oscillations in the medium.

When considering harmonic oscillations propagating along a line, we usually study the wave resistance in operator form or mechanical impedance

$$Z_b(j\omega) = \frac{\sigma(j\omega)}{\upsilon(j\omega)}. \tag{34}$$

Referring to Formula (34) and taking into account that

$$a_1 = \Theta\sqrt{\frac{E}{\rho}}, \tag{35}$$

where [34]

$$\Theta = \sqrt{\frac{1-\mu}{(1+\mu)(1-2\mu)}}, \tag{36}$$

we write

$$Z_b(j\omega) = \rho(\omega)\Theta\sqrt{\frac{E_\omega(j\omega)}{\rho(j\omega)}} = \Theta\sqrt{E_\omega(j\omega)\rho(j\omega)}. \tag{37}$$

Note that the ratio $E_\upsilon/E_u$ = $tg\ \xi$ characterizes the magnitude of internal friction and $\xi$ determines the phase on which the change of stress is ahead of the change in deformation.

Then Eq. (37) becomes

$$Z_b(j\omega) = \frac{\sigma(j\omega)}{\upsilon(j\omega)} = \Theta\sqrt{E_u(\omega)\rho(\omega)}\sqrt{1 + j\frac{E_\upsilon(\omega)}{E_u(\omega)}}. \tag{38}$$

If $\rho$ = const and $\Theta$ = 1, then the last equation may be written in the form

$$Z_b(j\omega) = \frac{j\omega E_u(\omega)\sqrt{\rho/E_u(\omega)}\sqrt{1 + jE_\upsilon(\omega)/E_u(\omega)}}{j\omega} = \frac{E_u(\omega)\theta(j\omega)}{j\omega}. \tag{39}$$

Here

$$\theta(j\omega) = \pm\sqrt{j\frac{\omega}{E_u(\omega)}[\rho j\omega + \psi(\omega)]};\ \psi(\omega) = -\frac{\rho\omega E_\upsilon(\omega)}{E_u(\omega)}. \tag{40}$$

Internal friction in solids $\psi$ can play a significant role. For example, it is known that magnesium alloys and a number of other materials have very good vibration-insulating properties, largely due to internal friction. At the same time, for steels this value is small and it is often neglected.

The dynamic features of lines with parameters distributed over length (in principle, parameters are distributed in any line) are characterized by the operator coefficient of wave propagation, which, in Laplace images, can be written in the form [29, 31]

$$\theta(s) = \pm\sqrt{\frac{s}{E_u(\omega)}[\rho s + \psi(\omega)]}. \tag{41}$$

Then, when $E_\upsilon$ = 0, $\rho$ = const, and $E_u$ = $E$, the wave resistance will be

$$Z_b(s) = \frac{\sigma(s)}{\upsilon(s)} = \frac{\theta(s)E}{s}. \tag{42}$$

From where it also follows

$$\theta(s) = \frac{sZ_b(s)}{E}. \tag{43}$$

Conducting a one-dimensional Laplace transform [30] of Eqs. (13) and (14) for longitudinal oscillations with zero initial conditions and taking into account that with the accepted assumptions

$$\theta(s) = \pm s\sqrt{\frac{\rho}{E}}, \tag{44}$$

we obtain

$$\rho s \upsilon(s) = -\frac{d\sigma(s)}{dx}; \tag{45}$$

$$E\frac{d\upsilon(s)}{dx} = -s\sigma(s). \tag{46}$$

The solution of the system of Eqs. (45) and (46) allows to find for the selected section the instantaneous deviations from the steady-state values of stress and speed of movement sections of rod. Each of these quantities will be the sum of the quantities of the same name, determined in the front of the perturbation propagating in the forward and reverse directions. The instantaneous deviations of the marked variables, as well as the peculiarities of the disturbance propagation along the line, depend on the physical and geometric properties of the line.

Differentiating Eq. (23) with respect to x, then eliminating the derivative $d\upsilon(s)/dx$ using Eq. (24), and applying relation (20), we obtain

$$\frac{\partial^2\sigma(s)}{\partial x^2} - \theta^2(s)\sigma(s) = 0. \tag{47}$$

This equation is a second-order differential equation with constant coefficients. The solution is

$$\sigma(s) = C_1 \exp\left[\theta(s)x\right] + C_2 \exp\left[-\theta(s)\right]. \tag{48}$$

The integration constants $C_1$ and $C_2$ are determined by the boundary conditions. Let at $x = 0$

$$\sigma(s,x) = \sigma_1(s,0); \frac{\partial\sigma(s,x)}{\partial x} = -\frac{E}{s}\theta^2(s)\upsilon_1(s,0). \tag{49}$$

The last condition from (49) is obtained from (45) by replacing $\rho s = \theta^2(s)E/s$. Then, taking into account (49), we get

$$C_1 = \frac{\sigma_1(s,0) - s^{-1}\theta(s)E\upsilon_1(s,0)}{2}; C_2 = \frac{\sigma_1(s,0) + s^{-1}\theta(s)E\upsilon_1(s,0)}{2}. \tag{50}$$

After substituting these dependencies, the solution will be

$$\sigma(s,x) = \frac{\sigma_1(s,0)\{\exp\left[\theta(s)x\right] + \exp\left[-\theta(s)x\right]\}}{2} - \frac{s^{-1}\theta(s)E\sigma_1(s,0)\{\exp\left[\theta(s)x\right] - \exp\left[-\theta(s)x\right]\}}{2}. \tag{51}$$

When we introduce hyperbolic functions, then we get

$$\sigma(s,x) = \sigma_1(s,0)ch[\theta(s)x] - \theta(s)s^{-1}E\upsilon_1(s,0)sh[\theta(s)x]. \tag{52}$$

Having solved the system of Eqs. (47) and (48) with respect to $\upsilon(s,x)$ in the manner described, we obtain

$$\frac{\partial^2\upsilon(s)}{\partial x^2} - \theta^2(s)\upsilon(s) = 0. \tag{53}$$

For boundary conditions with $x = 0$

$$v(s,x) = v_1(s,0);\ \frac{\partial v(s,x)}{\partial x} = -\frac{s\sigma(s,0)}{E}, \tag{54}$$

We finally obtain

$$v(s,x) = v_1(s,0)ch[\theta(s)x] - \frac{s\sigma_1(s,0)sh[\theta(s)x]}{\theta(s)E}. \tag{55}$$

Movement in two-link elements that occurs within elastic limits can be viewed as the movement of the driven point (link) from the movement of the leading point (link), which is affected by the previous links of the truss, for example, which perceive wind load. If, in the process of oscillation, the output link does not allow the input impulse to pass, then waves of disturbance are reflected from the end of the lines.

Consider the case of a matched load when there are no reflected waves in the system and oscillations in the system do not affect the movement of the driven link due to the attached large mass. In this case, the boundary conditions are the following relations:

$$\begin{aligned} &v(s,l) = v_2(s);\ v(s,0) = v_1(s);\ \sigma(s,l) = \sigma_2(s);\ \sigma(s,0) = \sigma_1(s);\\ &\sigma_2(s) = \frac{F_c(s) + h_n v_2(s) + msv_2(s)}{f_2}. \end{aligned} \tag{56}$$

Here $f_2$ is the sectional area of the line in front of the slave link of mass $m$; $h_n$ and $F_c$ are coefficient of friction loss, proportional to the speed of movement, as well as the resistance force acting on the slave link; $l$ is the length of the line.

Together we solve (52)–(56) by performing the following transformations

$$\begin{aligned} &\sigma_1(s) = \frac{1}{chA}\left[\sigma_2(s) + \frac{1}{s}Ev_1(s)\theta(s)shA\right];\\ &v_2(s) = v_1(s)chA - \frac{s}{E\theta(s)}shA\left\{\frac{1}{chA}\left[\sigma_2(s) + \frac{1}{s}Ev_1(s)\theta(s)shA\right]\right\};\\ &v_2(s) = \frac{v_1(s)}{chA} - \frac{s}{E\theta(s)}\sigma_2(s)thA;\\ &\left[\frac{v_1(s)}{chA} - v_2(s)\right]\frac{E}{s}\frac{\theta(s)}{thA} = \frac{1}{f_2}[F(s) + h_n v_2(s) + msv_2(s)]. \end{aligned} \tag{57}$$

After bringing similar members, we obtain the equation of motion of the driven link of the mechanical system in the form

$$v_2(s)\left[1 + h_n\vartheta_n(s)s + m\vartheta_n(s)s^2\right] = \frac{v_1(s)}{ch[\theta(s)l]} - F_c(s)\vartheta_n(s)s. \tag{58}$$

Here

$$\vartheta_n(s) = \vartheta_{n0}Z_n(s); \vartheta_{n0} = \frac{l}{Ef_2}; Z_n(s) = \frac{thA}{A}; A = \theta(s)l. \tag{59}$$

Substituting (58) into the last equation of system (57), we obtain an equation describing the stresses fluctuations in the force line in the vicinity of the slave link in form

$$\sigma_2(s) = \frac{F_c(s) + \upsilon_1(s)(h_n + ms)/chA}{f_2[1 + h_n\vartheta_n(s)s + m\vartheta_n(s)s^2]}. \tag{60}$$

Performing such transformations in relation to torsional vibrations, we obtain

$$\Omega_2(s)\left[1 + h_k\vartheta_k(s)s + J\vartheta_k(s)s^2\right] = \frac{\Omega_1(s)}{chA_k} - M_r(s)\vartheta_n(s)s. \tag{61}$$

Here

$$\vartheta_k(s) = \vartheta_{k0}Z_k(s); \vartheta_{k0} = \frac{l}{GrW_{p2}}; Z_k(s) = \frac{thA_k}{A_k}; A_k = \theta_k(s)l; \theta_k(s) = \pm\sqrt{\frac{\rho}{G}}. \tag{62}$$

Comparing (15), (18), and (19), we can see that the processes of motion transfer in solid and liquid media can be described by similar equations. This is shown when analyzing the operation of the hydraulic drive [9, 35–37].

## 6. Lemma on the degree of distribution of force line parameters

Eqs. (47) and (48) make it possible to calculate the frequency characteristics of the system, i.e., determine the response of the model to the harmonic change in the speed of the lead link or the resistance force acting on the driven link.

For example, for $\upsilon_1 \equiv 0$ and $s = j\omega$, we obtain the characteristic $W_F\ (j\omega)$, illustrating the influence of $F$ on $\upsilon_2$, and the characteristic $W_{F\sigma}\ (j\omega)$, illustrating the influence of $F$ on $\sigma$:

$$W_F(j\omega) = \frac{\upsilon_2(j\omega)}{F_c(j\omega)} = -\frac{\vartheta_n(j\omega)j\omega}{1 + h_n\vartheta_n(j\omega)j\omega + m\vartheta_n(j\omega)(j\omega)^2}; \tag{63}$$

$$W_{F\sigma}(j\omega) = \frac{\sigma_2(j\omega)}{F_c(j\omega)} = \frac{1}{f_2\left[1 + h_n\vartheta_n(j\omega)j\omega + m\vartheta_n(j\omega)(j\omega)^2\right]}. \tag{64}$$

For F $\equiv$ 0, we obtain the frequency characteristics $W_\upsilon\ (j\omega)$, illustrating the effect of $\upsilon_1$ on $\upsilon_2$, and $W_{F\sigma}\ (j\omega)$, illustrating the influence of $\upsilon_1$ on $\sigma_2$. So we have

$$W_\upsilon(j\omega) = \frac{\upsilon_2(j\omega)}{\upsilon_1(j\omega)} = -\frac{ch^{-1}[\theta(j\omega)l]}{1 + h_n\vartheta_n(j\omega)j\omega + m\vartheta_n(j\omega)(j\omega)^2}, \tag{65}$$

$$W_{F\sigma}(j\omega) = \frac{\sigma_2(j\omega)}{\upsilon_1(j\omega)} = \frac{h_n + mj\omega}{f_2 ch[\theta(j\omega)l]\left[1 + h_n\vartheta_n(j\omega)j\omega + m\vartheta_n(j\omega)(j\omega)^2\right]}. \tag{66}$$

From (63) to (66), it can be seen that changes in the voltage and speed of movement of the output link are lagging behind changes in input impacts.

We introduce the notation $\alpha = l\ \omega\ (\rho/E)^{1/2}$.

Insofar as

$$Z_n(j\omega) = \frac{th[\theta(j\omega)l]}{\theta(j\omega)l} = \frac{jtg\alpha}{j\alpha} = \frac{tg\alpha}{\alpha};\ ch(j\alpha) = \cos\alpha, \tag{67}$$

then

$$\vartheta_n(j\omega) = \vartheta_n(\alpha) = \vartheta_{n0} Z_n(\alpha) \tag{68}$$

and cosα are not complex functions.

The graph of the function $Z_n(\alpha)$ is shown in **Figure 4**.

On the whole we conclude that at $k = 1$ $\alpha \to 0$, $Z_n \to 1$; at $\pi/2 + k\pi > \alpha > \pi + k\pi$, $Z_n < 0$. Here $k = 0, 1, 2....n$.

Eq. (58), given this, can be rewritten in the form

$$\upsilon_2(s)\left[1 + h_n\vartheta_n(\alpha)s + m\vartheta_n(\alpha)s^2\right] = \upsilon_1(s)\cos^{-1}(\alpha) - F_c(s)s\vartheta_n(\alpha). \tag{69}$$

If $\alpha \to 0$, then Eq. (69) is reduced to the well-known equation describing dynamic processes in the mechanism with short lines of force:

$$\upsilon_2(s)\left[1 + h_n\vartheta_{n0}s + m\vartheta_{n0}s^2\right] = \upsilon_1(s) - F_c(s)s\vartheta_{n0}. \tag{70}$$

The breaks shown in **Figure 4** are mathematically related to the function of tangent. For a real mechanism, this means that motion parameters are rebuilt. Apparently, in this instant, the form of oscillations changes abruptly. Below this feature is discussed in more detail.

The appearance of resonance is described by another expression that defines the conditions for the formation of the maximum amplitude of oscillations.

$$A_F(\omega) = \frac{\vartheta_n(\alpha)\omega}{\sqrt{\left[1 - m\vartheta_n(\alpha)\omega^2\right]^2 + \left[h_n\vartheta_n(\alpha)\omega\right]^2}} = \left(\sqrt{\frac{\left[1 - m\vartheta_n(\alpha)\omega^2\right]^2}{\left[\vartheta_n(\alpha)\omega^2\right]} + h_n^2}\right)^{-1}. \tag{71}$$

Thus $A_F$ achieves its maximum when the condition

$$\frac{1 - m\vartheta_n(\alpha)\omega^2}{\vartheta_n(\alpha)\omega} = 0. \tag{72}$$

We show that this condition is similar to the rule for determining natural frequencies. This rule was formulated by Babakov [22].

Consider the following problem. Let the rod hang vertically. At the end of the rod, a load is fixed. The load is assumed to be point (**Figure 5**).

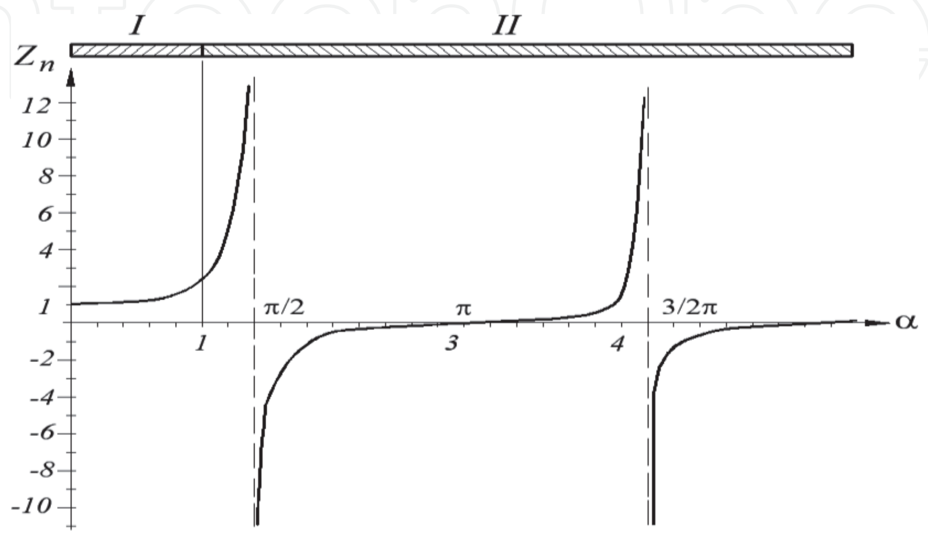


**Figure 4.**
*Change of function $Z_n$ of dimensionless parameter α: I is the range of admissible function values for systems with lumped parameters; II is the range of admissible values of the function for systems with distributed parameters.*

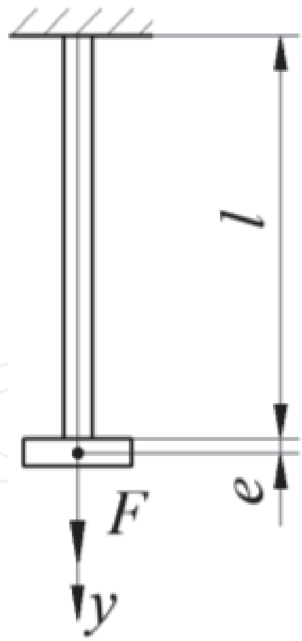


**Figure 5.**
*Cargo suspended on a rod.*

Let the mass of the load be $m$ and the ratio of the mass of the rod with the cross-sectional area $f$ to the mass of the load be $a = l\mu/m$. Here μ is the linear mass of the rod.

It is believed that the longitudinal tension of the rod during oscillations is balanced by the force of inertia of the load. This leads to the following condition at the lower end of the rod:

$$Ef\left(\frac{\partial y}{\partial x}\right)_{x=l} = -m\left(\frac{\partial^2 y}{\partial t^2}\right)_{x=l}. \tag{73}$$

At the top end, which is fixed we have $y(0, t) = 0$. At the initial time, the rod is stretched by the force F applied to the lower end and then without the initial speed is left to itself, so that

$$y(x.0) = \frac{Fx}{Ef}; \frac{dy(x, 0)}{dt} = 0. \tag{74}$$

The solution of the problem is reduced to the calculation of the constant B, D, and parameter b values in the equation of the vibration modes

$$\varphi(x) = B\cos(bx) + D\sin(bx), \tag{75}$$

and, moreover, to the calculation of the constants $A_i$ and $B_i$ of the general solution

$$u(x,t) = \sum_{i-1}^{\infty}\left[A_i \cos(p_i t) + B_i \sin(p_i t)\right]\varphi(x) \tag{76}$$

in accordance with the initial conditions (74).

From the first boundary condition (41), it follows that $B = 0$.

After that, from the second condition is the equation of frequencies

$$\beta tg\beta = a, \tag{77}$$

where $\beta = al$.

Thus, the equation of the own modes of oscillations of the rod has the form

$$\varphi_k(x) = D_k \sin\left(\frac{\beta_k x}{l}\right)_{, \text{(k=1,2,3, ... )}.} \tag{78}$$

Here $\beta_k$ is the roots of Eq. (77).

The solution of Eq. (78) can be carried out graphically [21].

The lowest natural frequencies corresponding to these values are calculated by the formula

$$\omega = p_1 = \frac{\beta_1}{l}\sqrt{\frac{Ef}{\mu}}. \tag{79}$$

Note that the linear mass of the rod $\mu = \rho f$. Expanding the coefficients, we obtain the original equation

$$\beta tg\beta = \frac{lf\rho}{m}. \tag{80}$$

We now turn to Eq. (79) and divide the numerator of the right side by the denominator

$$Ef\frac{\sqrt{\rho/E}}{tg\alpha} = m\omega. \tag{81}$$

We will still carry out a number of transformations. We have

$$Efl\frac{\sqrt{\rho/E}}{m} = l\omega tg\alpha; \frac{fl}{m} = \frac{l\omega tg\alpha}{\sqrt{E\rho}}. \tag{82}$$

After multiplying both sides of the last equation by $\rho$ and after performing the corresponding transformations, we get

$$\frac{\rho fl}{m} = \alpha tg\alpha. \tag{83}$$

Since here the parameters $\alpha$ and $\beta$ are equivalent, it can be argued that the roots of Eqs. (72) and (80) are the same. This is confirmed by the results of calculations.

The results obtained make it possible to formulate a lemma on the degree of distribution of the parameters of power lines. But for this, we introduce the notation

$$\alpha = l\omega\sqrt{\rho/\chi}, \tag{84}$$

where $\chi$ is the elastic modulus of the material of the force line.

We state the lemma as follows. *If the parameters of the mechanical two-link system, characterized by the oscillation frequency* $\omega$, *length* $l$, *density* $\rho$, *and the elastic modulus of the material of the force line* $\chi$ *connecting these links, are such that magnitude of the dimensionless coefficient* $\alpha$ *lies in the interval* $0 \le \alpha \le 1$, *then the wave processes in the lines of force can be neglected.*

This lemma is also valid for systems where torsional vibrations of rods and vibrations in hydraulic systems take place. The criterion on the degree of distribution of the system parameters was first described in the book [29].

## 7. Dynamic stability of rod systems

Wave phenomena take place in hydraulic, mechanical, electrical environments [4]. As shown above, wave processes take place in systems with lines of any length, but depending on the conditions, they can either be neglected or not ignored. If at the same time there is a transfer of energy to perform any effective work, then the problem of ensuring sustainable functioning inevitably arises.

Publications devoted to the study of the stability of open-loop systems with distributed parameters in the presence of significant nonlinearities, where sufficiently precise criteria are stated, could not be found. Therefore, here outlines the main points of this issue, which were developed in more detail in [9, 29, 35–37].

We algebraically decompose Eq. (58) into two. We have

$$\frac{\upsilon_1}{\cos\alpha} = \upsilon_2(s) + \vartheta_n(\alpha) f_2 s\sigma(s); \tag{85}$$

$$\sigma(s) f_2 = F_c(s) + h_n \upsilon_2(s) + m s \upsilon_2(s). \tag{86}$$

Here α corresponds to the expression (84).

The accuracy of the decomposition is easily verified by the inverse solution.

Moving on to the originals of Eq. (58), we do the inverse transform Laplace of functions

$$F_1(s) = \vartheta_n(\alpha) f_2 s\sigma(s) \text{ and } F_2(s) = \upsilon_1(s)/\cos\alpha. \tag{87}$$

We have then

$$L^{-1}|F_1(s)| = \frac{1}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty} \vartheta_n(\alpha) f_2 \sigma(s) e^{st} ds = \vartheta_n(\alpha) f_2 \frac{1}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty} s\sigma(s) e^{st} ds = \vartheta_n(\alpha) f_2 \frac{d\sigma}{dt}. \tag{88}$$

$$L^{-1}|F_2(s)| = \upsilon_1(t)/\cos\alpha. \tag{89}$$

Then the originals (31) can be written in the form

$$\upsilon_1(t)/\cos\alpha = \upsilon_2(t) + \vartheta_n(\alpha) f_2 \frac{d\sigma}{dt}; \tag{90}$$

$$\sigma(t) f_2 = F_c(t) + h_n \upsilon_2(t) + m\frac{d\upsilon_2}{dt}. \tag{91}$$

The system of Eqs. (90, 91) may contain various nonlinearities (yield zone in the stress diagram, nonlinear friction, etc.) and is solved by the updated Runge-Kutta method [9, 38]. The admissibility of such a technique was checked by comparing the frequency characteristics constructed by formulas (63) to (66) and using the above method when introducing harmonic oscillations with different frequencies [29, 35].

In addition, given that the fluctuations of the speeds of movement and stresses (pressure) in mechanical and hydraulic systems can be described by similar equations, we checked the adequacy of the proposed method by means of full-scale and numerical experiments in an electric drive. The results had good convergence [9, 29, 31, 35–37].

In the process of modeling, it turned out that when $Z < 0$ (**Figure 4**), the solution becomes unstable. **Figure 6** shows an example of the process of loss of

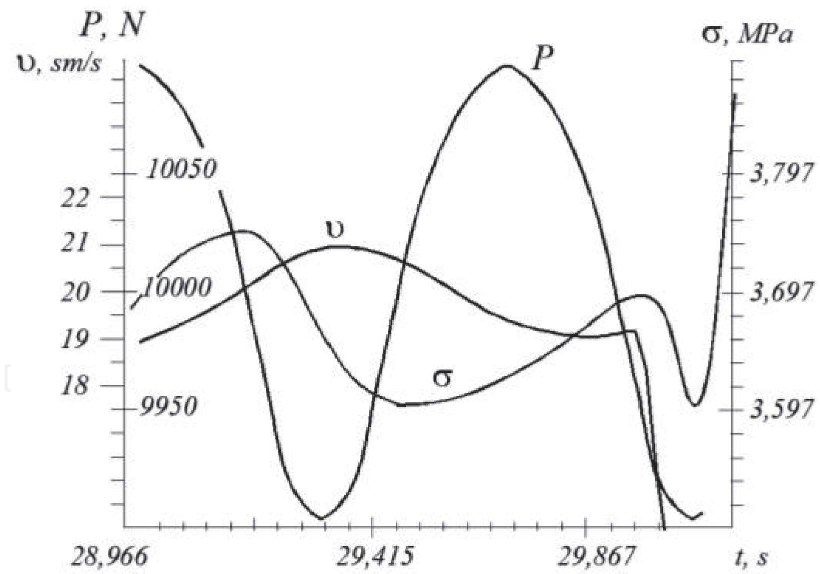


**Figure 6.**
*Loss of stable operation in the mechanical element during longitudinal movement.*

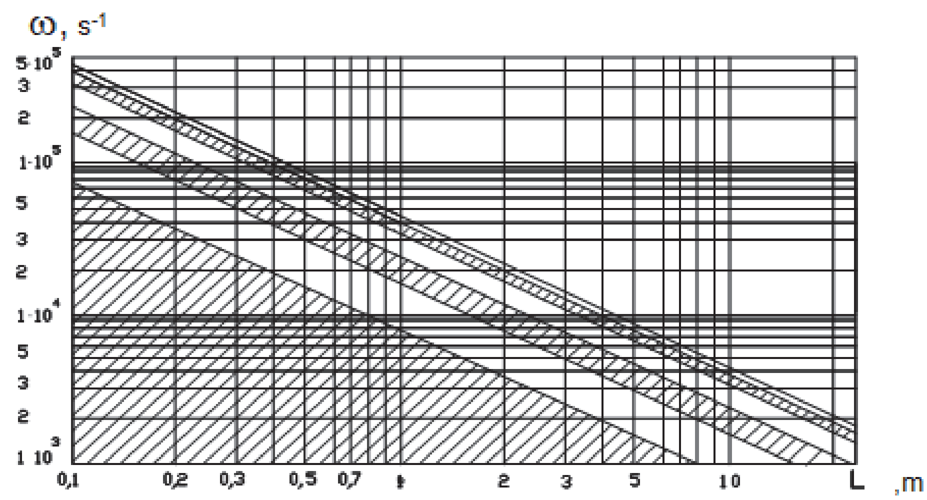


**Figure 7.**
*Zones of steady (shaded) and unstable operation of the mechanical system of longitudinal movement.*

dynamic stability in a mechanical element with longitudinal vibrations. For a truss, this means breaking one of the rods.

The areas of dynamic stability for mechanical elements with longitudinal vibrations are shown in **Figure** 7.

## 8. Conclusions

In trusses under the influence of variable loads, vibrations occur periodically in different elements, characterized by fluctuations in speed and stress. Under certain conditions, due to wave motions in the rods, the shape of the oscillations may change, i.e., lost stability of motion. At this instant, the destruction of the carrier element will occur. The proposed chapter considers the conditions for the occurrence of such an event.

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