Chapter

Optimal Control of Evolution Differential Inclusions with Polynomial Linear Differential Operators

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Abstract

In this chapter, we studied a new class of problems in the theory of optimal control defined by polynomial linear differential operators. As a result, an interesting Mayer problem arises with higher order differential inclusions. Thus, in terms of the Euler-Lagrange and Hamiltonian type inclusions, sufficient optimality conditions are formulated. In addition, the construction of transversality conditions at the endpoints of the considered time interval plays an important role in future studies. To this end, the apparatus of locally adjoint mappings is used, which plays a key role in the main results of this chapter. The presented method is demonstrated by the example of the linear optimal control problem, for which the Weierstrass-Pontryagin maximum principle is derived.

Keywords: Euler-Lagrange, differential inclusion, set-valued mapping, polynomial differential operators, linear problem, transversality, Weierstrass-Pontryagin maximum principle

1. Introduction

This chapter concerns with the special kind of optimal control problem with differential inclusions, where the left-hand side of the evolution inclusion is polynomial linear differential operators with variable coefficients; in fact, the main difficulty in the considered problems is to construct the Euler-Lagrange type higher order adjoint inclusions and the transversality conditions. That is why in the whole literature, only the qualitative properties of second-order differential inclusions are investigated (see [1–3] and references therein).

The paper [1] gives necessary and sufficient conditions ensuring the existence of a solution to the second-order differential inclusion with Cauchy initial value problem. Furthermore, second-order interior tangent sets are introduced and studied to obtain such conditions. The paper [2] studies, in the context of Banach spaces, the problem of three boundary conditions for both second-order differential inclusions and second-order ordinary differential equations. The results are obtained in several new settings of Sobolev type spaces involving Bochner and Pettis integrals. In the paper [3], the existence of viable solutions to the Cauchy problem $x'' \in F(x, x'), x(0) = x_0, x'(0) = y_0$ is proved, where F is a set-valued map defined on a locally compact set $M \subset \mathbb{R}^{2n}$, contained in the Frechet subdifferential of a φ-convex function of order two.

Some qualitative properties and optimization of first-order discrete and continuous time processes with lumped and distributed parameters have been expanding in all directions at an astonishing rate during the last few decades (see [4–13] and their references).

The optimization of higher order differential inclusions was first developed by Mahmudov in [14–21]. Since then this problem has attracted many author's attentions (see [22] and their references). The paper [14] studies a new class of problems of optimal control theory with Sturm-Liouville type differential inclusions involving second-order linear self-adjoint differential operators. By using the discretization method guaranteeing transition to continuous problem, the discrete and discrete-approximate inclusions are investigated. Necessary and sufficient conditions, containing both the Euler-Lagrange and Hamiltonian type inclusions, and "transversality" conditions are derived. The paper [15] deals with the optimization of the Bolza problem with third-order differential inclusions and arbitrary higher order discrete inclusions. The work [16] is devoted to the Bolza problem of optimal control theory given by second-order convex differential inclusions with second-order state variable inequality constraints. According to the proposed discretization method, problems with discrete-approximate inclusions and inequalities are investigated. Necessary and sufficient conditions of optimality including distinctive "transversality" condition are proved in the form of Euler-Lagrange inclusions. The paper [17] is concerned with the necessary and sufficient conditions of optimality for second-order polyhedral optimization described by polyhedral discrete and differential inclusions. The paper [18] is devoted to the study of optimal control theory with higher order differential inclusions and a varying time interval. Essentially, under a more general setting of problems and endpoint constraints, the main goal is to establish sufficient conditions of optimality for higher order differential inclusions. Thus with the use of Euler-Lagrange and Hamiltonian type of inclusions and transversal conditions on the "initial" sets, the sufficient conditions are formulated. The paper [21] studies a new class of problems of optimal control theory with state constraints and second-order delay discrete and delay differential inclusions. Under the "regularity" condition by using discrete approximations as a vehicle, in the forms of Euler-Lagrange and Hamiltonian type inclusions, the sufficient conditions of optimality for delay DFIs, including the peculiar transversality ones, are proved.

The present chapter is ordered in the following manner.

In Section 2 the necessary facts and supplementary results from the book of Mahmudov are given [23]; Hamiltonian function and locally adjoint mapping are introduced, and the problems with initial point constraints for polynomial linear differential operators governed by time-dependent set-valued mapping are formulated. In Section 3, we present the main results; on the basis of "transversality" conditions at the endpoints of the considered time interval, the sufficient conditions of optimality for differential inclusions with polynomial linear differential operators and with initial point constraints are proved. In particular, it is shown that our problems involve optimization of the so-called Sturm-Liouville type differential inclusions. To the best of our knowledge, there is no paper which considers optimality conditions for these problems in the literature, and we aim to fill this gap. Therefore, the novelty of our formulation of the problem is justified. To establish the Euler-Lagrange and Hamiltonian inclusions and the transversality conditions, we use the construction of a suitable rewriting of the primal polynomial linear differential operator and the rearrangement of its integration. The case of variable coefficients of polynomial linear differential operators turns out to be more complicated, unless transversality assumptions at the endpoints of the considered time interval are applied. It should be noted that the main proof can be easily generalized to the nonconvex case. Then, using the new approach given in Section 4 of this chapter, we construct the Weierstrass-Pontryagin maximum condition [24] for the linear optimal control problem. Consequently, in the particular case, the maximum principle follows from the Euler-Lagrange inclusion.

In Section 5 the optimality conditions are given for convex problem with second-order differential inclusions and endpoint constraints. By using second-order suitable Euler-Lagrange type adjoint inclusions and transversality conditions, Theorem 5.1 is proved.

The main results in this section can be extended to the case of Hilbert spaces ℓ_2, L_2^n. We remind that a Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product [2]. By definition, every Hilbert space is also a Banach space. Furthermore, in every Hilbert space, the following parallelogram identity $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ holds. Conversely, every Banach space in which the parallelogram identity holds is a Hilbert space. Remember that ℓ_2 is a space of numerical sequences, such that if $x = \{x_i\}$, then $\sum_{i=1}^{\infty} x_i^2 < \infty$. In fact ℓ_2 is an infinity dimensional coordinate-wise Hilbert space with the corresponding inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$. Endowing a relevant norm, we have a Banach space. Obviously, optimization of problem with PLDOs can be reduced to problem with geometric constraints in such finite-dimensional Hilbert space. As is known with all the pairs of elements of this space, a certain finite number is associated, i.e., inner product, existence of which is guaranteed by applying the familiar Cauchy Schwarz-Bunyakovskii [25] inequality. We remark that in our case for $x = \{x_0, x_1, x_2, \dots\} \in \ell_2$ and $x^* = \{x_0^*, x_1^*, x_2^*, \dots\} \in \ell_2^*$, the inner product $\langle x, x^*\rangle = \sum_{i=1}^{\infty} x_i x_i^*$ is finite numbers since this series is convergent. Besides it is known [25] that ℓ_2 is a self-adjoint space, i.e., $\ell_p = \ell_q^*$, and $1/p + 1/q = 1$, and so $\ell_2 = \ell_2^*$ for $p = 2$. Thus a dual cone constructed can be defined. The set of square integrable functions $L_2^n([0,1])$ is a Hilbert space with inner product $\langle x(t), y(t)\rangle = \int_0^1 x(t)y(t)dt$.

2. Preliminaries and problem statements

The basic concepts given in this section can be found in the book [23]; let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, v\rangle$ be an inner product of elements $x, v \in \mathbb{R}^n$, and (x, v) be a pair of x, v. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping from $\mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Therefore F is a convex set-valued mapping, if its graph $gph\ \ F = \{(x, v) : v \in F(x)\}$ is a convex subset of $\mathbb{R}^{2n}$. A set-valued mapping F is called closed if its $gph\ \ F$ is a closed subset in $\mathbb{R}^{2n}$. The domain of a set-valued mapping F is denoted by $dom\, F$ and is defined as $dom\, F = \{x : F(x) \neq \varnothing\}$. A set-valued mapping F is convex-valued if $F(x)$ is a convex set for each $x \in dom\, F$.

The Hamiltonian function and argmaximum set corresponding to a set-valued mapping F are defined by the relations correspondingly:

$$H_F(x, v^*) = \sup_v \{\langle v, v^*\rangle : v \in F(x)\}, \quad v^* \in \mathbb{R}^n,$$

$$F_A(x, v^*) = \{v \in F(x) : \langle v, v^*\rangle = H_F(x, v^*)\}$$

We set $H_F(x, v^*) = -\infty$ if $F(x) = \varnothing$. The interior and relative interior of a set $M \subset \mathbb{R}^{2n}$ are denoted by int$M$ and ri$M$, respectively.

A convex cone $K_M(z_0), z_0 \in M$ is a cone of tangent directions if from $\bar{z} = (\bar{x}, \bar{v}) \in K_M(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set M at a point $z_0 \in M$, i.e., there exists such function $q : \mathbb{R}^1 \to \mathbb{R}^{2n}$ that $z_0 + \alpha\bar{z} + q(\alpha) \in M$ for sufficiently small $\alpha > 0$ and $\alpha^{-1}q(\alpha) \to 0$, as $\alpha \downarrow 0$.

For a set-valued mapping F, the set-valued mapping $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$F^*(v^*; (x, v)) := \{x^* : (x^*, -v^*) \in K^*_{gph\,F}(x, v)\}$$
$$K_{gphF}(x, v) = \text{cone}\,[\,gph\,F - (x, v)], \forall (x^1, v^1) \in gph\,F.$$

It is called the LAM to F at a point $(x, v) \in gph\,F$, where $K^* = \{z^* : \langle \bar{z}, z^* \rangle \geq 0, \forall \bar{z} \in K\}$ denotes the dual cone to the cone K, as usual. Below by using the Hamiltonian function, associated to a set-valued mapping F, we will define another LAM. Thus, the LAM to "nonconvex" mapping F is defined as follows:

$$F^*(v^*; (x, v)) := \{x^* : H_F(x^1, v^*) - H_F(x, v^*) \leq \langle x^*, x^1 - x \rangle, \forall x^1 \in \mathbb{R}^n\},$$
$$(x, v) \in gph\,F, v \in F_A(x, v^*).$$

Clearly, for the convex mapping F, the Hamiltonian function $H_F(\cdot, \cdot, v^*)$ is concave, and the latter definition of LAM coincides with the previous definition of LAM ([23], p. 62). Note that prior to the LAM, the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [26] and for the smooth convex maps, the two notions are equivalent.

The aim of Section 3 is to obtain the Euler-Lagrange type adjoint inclusion and sufficient optimality conditions for a problem with polynomial linear differential operators:

$$\text{Minimize } \varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right), \tag{1}$$

$$(P_V)\ Lx(t) \in F(x(t), t), \ \text{a.e. } t \in [0, 1], \tag{2}$$

$$x(0) \in Q_0,\ x'(0) \in Q_1 x''(0) \in Q_2, \dots, x^{(s-1)}(0) \in Q_{s-1} \tag{3}$$

where $Lx = \sum_{k=1}^{s} p_k(t) D^k x$ is a PLDO of degree s with variable coefficients $p_k : [0, 1] \to \mathbb{R}^1$ and $D^k, k = 1, \dots, s$ is the operator of kth-order derivatives. In what follows for each k, a scalar function p_k is kth-order continuously differentiable function, $p_s(t) \neq 0$ on $[0, 1]$ identically, $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is time-dependent set-valued mapping, $\varphi : (\mathbb{R}^n)^s \to \mathbb{R}^1$ is continuous function, $Q_j \subseteq \mathbb{R}^n, j = 0, 1, \dots, s-1$ are nonempty subsets of $\mathbb{R}^n$, and s $(s \geq 2)$ is an arbitrary fixed natural number. It is required to find an arc $\tilde{x}(t)$ of the problem Eqs. (1)–(3) for the sth-order differential inclusions satisfying Eq. (2) almost everywhere (a.e.) on a considered time interval and minimizing the functional $\varphi\left(x(1), \dots, x^{(s-1)}(1)\right)$. An arc $x(\cdot)$ is absolutely continuous and $s-1$ order differentiable function, where $x^{(s)}(\cdot) \equiv \frac{d^s x(\cdot)}{dt^s} \in L_1^n([0, 1])$. Obviously, such class of functions is a Banach space, endowed with the different equivalent norms.

Remark 2.1. Notice that to get sufficient condition of optimality for the Mayer problem (P_V) described by ordinary evolution differential inclusions with PLDOs and with initial point constraints, using the discretized method, we consider the following sth-order discrete-approximate problem instead of the problem (P_V):

$$\text{minimize} \quad \varphi\left(x(1-(s-1)h), \Delta x(1-(s-1)h), \dots, \Delta^{s-1}x(1-(s-1)h)\right),$$

$$\sum_{k=1}^{s} p_k(t)\Delta^k x(t) \in F(x(t),t), t = 0, h, 2h, \dots, 1-sh;$$

$$\Delta^k x(0) \in Q_k, k = 0, \dots, s-1.$$

Here kth-order difference operator is defined as follows:

$$\Delta^k x(t) = \frac{1}{h^k}\sum_{s=0}^{k}(-1)^s C_k^s x(t+(k-s)h), \; C_k^s = \frac{k!}{s!(k-s)!}, \; t = 0, h, \dots, 1-h.$$

Thus by using the method of approximation [23, 26, 27], we can establish necessary and sufficient conditions for the rather complicated sth-order discrete-approximate problem. Then by passing to the limit in necessary and sufficient conditions of this problem as $h \to 0$, we can construct the optimality condition for the Mayer problem (P_V) described by higher order differential inclusions with PLDOs and with initial point constraints. But in this chapter to avoid long calculations, derivations of these conditions are omitted.

3. Optimization of evolution differential inclusions with PLDOs

In the present section, we study sufficient optimality conditions for the problem (P_V). Before all, we formulate the so-called sth-order Euler-Lagrange type differential inclusion and the transversality conditions:

i. $L^* x^*(t) \in F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t)$, a.e. $t \in [0,1]$,

where $L^* x^*(t) = \sum_{k=1}^{s}(-1)^k D^k\left[p_k(t)x^*(t)\right]$ is the adjoint PLDO of the primal operator L.

ii. $\sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}\left[p_{s-k}(0)x^*(0)\right] \in K_{Q_0}^*(\tilde{x}(0))$;

$\sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}\left[p_{s-k}(0)x^*(0)\right] \in K_{Q_1}^*(\tilde{x}'(0))$;
$D\left[p_s(0)x^*(0)\right] - p_{s-1}(0)x^*(0) \in K_{Q_{s-2}}^*\left(\tilde{x}^{(s-2)}(0)\right)$;
$-p_s(0)x^*(0) \in K_{Q_{s-1}}^*\left(\tilde{x}^{(s-1)}(0)\right)$.

iii. $\left(\sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}\left[p_{s-k}(1)x^*(1)\right], \sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}\left[p_{s-k}(1)x^*(1)\right], \dots, D\left[p_s(1)x^*(1)\right] - p_{s-1}(1)x^*(1), p_s(1)x^*(1)\right) \in \partial\varphi\left(\tilde{x}(1), \tilde{x}'(1), \dots, \tilde{x}^{(s-1)}(1)\right)$.

Later on we assume that $x^*(t), t \in [0,1]$ is absolutely a continuous function with the higher order derivatives until $s-1$ and $x^{*(s)}(\cdot) \in L_1^n([0,1])$. The following condition ensures that the LAM F^* is nonempty:

iv. $L\tilde{x}(t) \in F_A(\tilde{x}(t), x^*(t), t)$, a.e. $t \in [0,1]$ or, equivalently,
$\langle L\tilde{x}(t), x^*(t)\rangle = H_F(\tilde{x}(t), x^*(t)), L\tilde{x}(t) \in F(\tilde{x}(t), t)$.

The following are sufficient optimality conditions for evolution differential inclusions with PLDOs.

Theorem 3.1. Let φ be a continuous and convex function and $F(\cdot, t)$ a convex set-valued mapping. Moreover, let

$$Q_j, j = 0, \dots, s-1$$

be convex sets. Then for optimality of the trajectory $\tilde{x}(\cdot)$ in the problem (P_V) with evolution differential inclusions and PLDOs, it is sufficient that there exists an absolutely continuous function $x^*(\cdot)$ with the higher order derivatives until $s-1$, satisfying a.e. the Euler-Lagrange type differential inclusion with PLDOs Eqs. (i) and (iv) and transversality conditions Eqs. (ii) and (iii) at the endpoints $t=0$ and $t=1$.

Proof. Using Theorem 2.1 ([23], p. 62), the definition of the Hamiltonian function and condition Eq. (i), we obtain

$$H_F(x(t), x^*(t), t) - H_F(\tilde{x}(t), x^*(t), t) \le \langle L^* x^*(t), x(t) - \tilde{x}(t) \rangle$$

which can be rewritten as follows

$$H_F(x(t), x^*(t), t) - H_F(\tilde{x}(t), x^*(t), t)$$
$$\le \left\langle \sum_{k=1}^{s} (-1)^k \left(p_k(t) x^*(t) \right)^{(k)}, x(t) - \tilde{x}(t) \right\rangle. \quad (4)$$

Further using the definition of the Hamiltonian function, Eq. (4) can be converted to the inequality

$$0 \ge \langle Lx(t) - L\tilde{x}(t), x^*(t) \rangle - \langle L^* x^*(t), x(t) - \tilde{x}(t) \rangle$$

or

$$0 \ge \left\langle \sum_{k=1}^{s} p_k(t) (x(t) - \tilde{x}(t))^{(k)}, x^*(t) \right\rangle - \left\langle \sum_{k=1}^{s} (-1)^k \left(p_k(t) x^*(t) \right)^{(k)}, x(t) - \tilde{x}(t) \right\rangle. \quad (5)$$

Integrating Eq. (5) over the interval $[0, 1]$, we have

$$0 \ge \int_0^1 \left[\left\langle \sum_{k=1}^{s} p_k(t) (x(t) - \tilde{x}(t))^{(k)}, x^*(t) \right\rangle \right.$$
$$\left. - \left\langle \sum_{k=1}^{s} (-1)^k \left(p_k(t) x^*(t) \right)^{(k)}, x(t) - \tilde{x}(t) \right\rangle \right] dt \quad (6)$$

Let us denote

$$B = \sum_{k=1}^{s} \left\langle (x(t) - \tilde{x}(t))^{(k)}, p_k(t) x^*(t) \right\rangle - \sum_{k=1}^{s} \left\langle (-1)^k \left(p_k(t) x^*(t) \right)^{(k)}, x(t) - \tilde{x}(t) \right\rangle$$

In what follows our approach lies in reducing B in a relationship consisting of s sums from k $(k = 1, \dots, s)$ to s of suitable derivatives of scalar products; thus, after some transformations we can deduce an important representation for a first term of B as follows:

$$
\begin{aligned}
\sum_{k=1}^{s}\left\langle (x(t)-\tilde{x}(t))^{(k)}, p_k(t)x^*(t)\right\rangle &= \sum_{k=1}^{s}\left[\frac{d}{dt}\left\langle x^{(k-1)}(t)-\tilde{x}^{(k-1)}(t), p_k(t)x^*(t)\right\rangle\right] \\
&-\sum_{k=2}^{s}\left[\frac{d}{dt}\left\langle x^{(k-2)}(t)-\tilde{x}^{(k-2)}(t), \left(p_k(t)x^*(t)\right)'\right\rangle\right] \\
&+\sum_{k=3}^{s}\left[\frac{d}{dt}\left\langle x^{(k-3)}(t)-\tilde{x}^{(k-3)}(t), \left(p_k(t)x^*(t)\right)''\right\rangle\right] \\
-\cdot\cdots+&\sum_{k=s-2}^{s}\left[\frac{d}{dt}\left\langle x^{(k-s+2)}(t)-\tilde{x}^{(k-s+2)}(t), (-1)^{m-3}\left(p_k(t)x^*(t)\right)^{(s-3)}\right\rangle\right] \\
&+\sum_{k=s-1}^{s}\left[\frac{d}{dt}\left\langle x^{(k-s+1)}(t)-\tilde{x}^{(k-s+1)}(t), (-1)^{s-2}\left(p_k(t)x^*(t)\right)^{(s-2)}\right\rangle\right] \\
&+\frac{d}{dt}\left\langle x(t)-\tilde{x}(t), (-1)^{m-1}\left(p_s(t)x^*(t)\right)^{(s-1)}\right\rangle \\
&+\sum_{k=1}^{s}\left[\left\langle x(t)-\tilde{x}(t), (-1)^{k}\left(p_k(t)x^*(t)\right)^{(k)}\right\rangle\right].
\end{aligned}
\tag{7}
$$

Then in view of Eq. (7) in the definition of B, we have an efficient formula:

$$
\begin{aligned}
B = &\sum_{k=1}^{s}\left[\frac{d}{dt}\left\langle x^{(k-1)}(t)-\tilde{x}^{(k-1)}(t), p_k(t)x^*(t)\right\rangle\right] \\
&-\sum_{k=2}^{s}\left[\frac{d}{dt}\left\langle x^{(k-2)}(t)-\tilde{x}^{(k-2)}(t), \left(p_k(t)x^*(t)\right)'\right\rangle\right] \\
&+\sum_{k=3}^{s}\left[\frac{d}{dt}\left\langle x^{(k-3)}(t)-\tilde{x}^{(k-3)}(t), \left(p_k(t)x^*(t)\right)''\right\rangle\right] \\
-\cdot\cdots+&\sum_{k=s-2}^{s}\left[\frac{d}{dt}\left\langle x^{(k-s+2)}(t)-\tilde{x}^{(k-s+2)}(t), (-1)^{s-3}\left(p_k(t)x^*(t)\right)^{(s-3)}\right\rangle\right] \\
&+\sum_{k=s-1}^{s}\left[\frac{d}{dt}\left\langle x^{(k-s+1)}(t)-\tilde{x}^{(k-s+1)}(t), (-1)^{s-2}\left(p_k(t)x^*(t)\right)^{(s-2)}\right\rangle\right] \\
&+\frac{d}{dt}\left\langle x(t)-\tilde{x}(t), (-1)^{s-1}\left(p_s(t)x^*(t)\right)^{(s-1)}\right\rangle.
\end{aligned}
\tag{8}
$$

Then taking into account the structure of B in Eq. (8), we can compute the integral on the right-hand side of Eq. (6) as follows:

$$
\int_0^1 Bdt = \sum_{k=1}^{s}\left[\int_0^1 d\left\langle x^{(k-1)}(t)-\tilde{x}^{(k-1)}(t), p_k(t)x^*(t)\right\rangle\right]
$$

$$-\sum_{k=2}^{s}\left[\int_0^1 d\left\langle x^{(k-2)}(t)-\tilde{x}^{(k-2)}(t),\left(p_k(t)x^*(t)\right)'\right\rangle\right]$$

$$+\sum_{k=3}^{s}\left[\int_0^1 d\left\langle x^{(k-3)}(t)-\tilde{x}^{(k-3)}(t),\left(p_k(t)x^*(t)\right)''\right\rangle\right]$$

$$-\cdot\cdots+\sum_{k=s-2}^{s}\left[\int_0^1 d\left\langle x^{(k-s+2)}(t)-\tilde{x}^{(k-s+2)}(t),(-1)^{s-3}\left(p_k(t)x^*(t)\right)^{(s-3)}\right\rangle\right]$$

$$+\sum_{k=s-1}^{s}\left[\int_0^1 d\left\langle x^{(k-s+1)}(t)-\tilde{x}^{(k-s+1)}(t),(-1)^{s-2}\left(p_k(t)x^*(t)\right)^{(s-2)}\right\rangle\right]$$

$$+\int_0^1 d\left\langle x(t)-\tilde{x}(t),(-1)^{s-1}\left(p_s(t)x^*(t)\right)^{(s-1)}\right\rangle.$$

Thus, integrating B, we can obtain

$$\begin{aligned}
\int_0^1 Bdt = {} & \sum_{k=1}^{s}\Big[\left\langle x^{(k-1)}(1)-\tilde{x}^{(k-1)}(1),p_k(1)x^*(1)\right\rangle \\
& -\left\langle x^{(k-1)}(0)-\tilde{x}^{(k-1)}(0),p_k(0)x^*(0)\right\rangle\Big] \\
& -\sum_{k=2}^{s}\Big[\left\langle x^{(k-2)}(1)-\tilde{x}^{(k-2)}(1),\left(p_k(1)x^*(1)\right)'\right\rangle-\left\langle x^{(k-2)}(0)-\tilde{x}^{(k-2)}(0),\left(p_k(0)x^*(0)\right)'\right\rangle\Big] \\
& +\sum_{k=3}^{s}\Big[\left\langle x^{(k-3)}(1)-\tilde{x}^{(k-3)}(1),\left(p_k(1)x^*(1)\right)''\right\rangle \\
& -\left\langle x^{(k-3)}(0)-\tilde{x}^{(k-3)}(0),\left(p_k(0)x^*(0)\right)''\right\rangle\Big] \\
& -\cdots+\sum_{k=s-2}^{s}\Big[\left\langle x^{(k-s+2)}(1)-\tilde{x}^{(k-s+2)}(1),(-1)^{s-3}\left(p_k(1)x^*(1)\right)^{(s-3)}\right\rangle \\
& -\left\langle x^{(k-s+2)}(0)-\tilde{x}^{(k-s+2)}(0),(-1)^{s-3}\left(p_k(0)x^*(0)\right)^{(s-3)}\right\rangle\Big] \\
& +\sum_{k=s-1}^{s}\Big[\left\langle x^{(k-s+1)}(1)-\tilde{x}^{(k-s+1)}(1),(-1)^{s-2}\left(p_k(1)x^*(1)\right)^{(s-2)}\right\rangle \\
& -\left\langle x^{(k-s+1)}(0)-\tilde{x}^{(k-s+1)}(0),(-1)^{s-2}\left(p_k(0)x^*(0)\right)^{(s-2)}\right\rangle\Big] \\
& +\left\langle x(1)-\tilde{x}(1),(-1)^{s-1}\left(p_s(1)x^*(1)\right)^{(s-1)}\right\rangle \\
& -\left\langle x(0)-\tilde{x}(0),(-1)^{s-1}\left(p_s(0)x^*(0)\right)^{(s-1)}\right\rangle.
\end{aligned}$$

Here by suitable rearrangement and necessary simplification, we have

$$\begin{aligned}\int_0^1 B dt = &\sum_{k=1}^{s}\left[\left\langle x^{(k-1)}(1)-\tilde{x}^{(k-1)}(1), p_k(1)x^*(1)\right\rangle\right]\\ &-\sum_{k=2}^{s}\left[\left\langle x^{(k-2)}(1)-\tilde{x}^{(k-2)}(1), \left(p_k(1)x^*(1)\right)'\right\rangle\right]\\ &+\sum_{k=3}^{s}\left[\left\langle x^{(k-3)}(1)-\tilde{x}^{(k-3)}(1), \left(p_k(1)x^*(1)\right)''\right\rangle\right]\\ &-\dots+\sum_{k=s-2}^{s}\left[\left\langle x^{(k-s+2)}(1)-\tilde{x}^{(k-s+2)}(1), (-1)^{s-3}\left(p_k(1)x^*(1)\right)^{(s-3)}\right\rangle\right]\\ &+\sum_{k=s-1}^{s}\left[\left\langle x^{(k-s+1)}(1)-\tilde{x}^{(k-s+1)}(1), (-1)^{s-2}\left(p_k(1)x^*(1)\right)^{(s-2)}\right\rangle\right]\\ &+\left\langle x(1)-\tilde{x}(1), (-1)^{s-1}\left(p_s(1)x^*(1)\right)^{(s-1)}\right\rangle\\ &+\left\langle x(0)-\tilde{x}(0), \sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\left\langle x'(0)-\tilde{x}'(0), \sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\left\langle x''(0)-\tilde{x}''(0), \sum_{k=0}^{s-3}(-1)^{s-k-2}D^{s-k-3}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\cdot\dots-\left\langle x^{(s-2)}(0)-\tilde{x}^{(s-2)}(0), -D\left[p_s(0)x^*(0)\right]+p_{s-1}(0)x^*(0)\right\rangle\\ &-\left\langle x^{(s-1)}(0)-\tilde{x}^{(s-1)}(0), p_s(0)x^*(0)\right\rangle.\end{aligned} \tag{9}$$

In order to make use of the transversality condition Eq. (ii), we rewrite it in a more relevant form:

$$\begin{aligned}&\left\langle x(0)-\tilde{x}(0), \sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\left\langle x'(0)-\tilde{x}'(0), \sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\left\langle x''(0)-\tilde{x}''(0), \sum_{k=0}^{s-3}(-1)^{s-k-2}D^{s-k-3}\left[p_{s-k}(0)x^*(0)\right]\right\rangle\\ &+\cdot\dots-\left\langle x^{(s-2)}(0)-\tilde{x}^{(s-2)}(0), -D\left[p_s(0)x^*(0)\right]+p_{s-1}(0)x^*(0)\right\rangle\\ &-\left\langle x^{(s-1)}(0)-\tilde{x}^{(s-1)}(0), p_s(0)x^*(0)\right\rangle\geq 0; \forall x^{(k)}(0)\in K_{Q_k}\left(\tilde{x}^{(k)}(0)\right),\end{aligned}$$

$k = 0, \dots, s-1$. Thus, from Eqs. (6) and (9), we have

$$
\begin{aligned}
0 \geq & \sum_{k=1}^{s}\left[\left\langle x^{(k-1)}(1)-\tilde{x}^{(k-1)}(1), p_k(1)x^*(1)\right\rangle\right] \\
& -\sum_{k=2}^{s}\left\langle x^{(k-2)}(1)-\tilde{x}^{(k-2)}(1), \frac{d\left(p_k(1)x^*(1)\right)}{dt}\right\rangle \\
& +\sum_{k=3}^{s}\left[\left\langle x^{(k-3)}(1)-\tilde{x}^{(k-3)}(1), \frac{d^2\left(p_k(1)x^*(1)\right)}{dt^2}\right\rangle\right]-\cdots \\
& +\sum_{k=s-2}^{s}\left[\left\langle x^{(k-s+2)}(1)-\tilde{x}^{(k-s+2)}(1), (-1)^{s-3}\frac{d^{s-3}\left(p_k(1)x^*(1)\right)}{dt^{s-3}}\right\rangle\right] \\
& +\sum_{k=s-1}^{s}\left[\left\langle x^{(k-s+1)}(1)-\tilde{x}^{(k-s+1)}(1), (-1)^{s-2}\frac{d^{s-2}\left(p_k(1)x^*(1)\right)}{dt^{s-2}}\right\rangle\right] \\
& +\left\langle x(1)-\tilde{x}(1), (-1)^{s-1}\frac{d^{s-1}\left(p_s(1)x^*(1)\right)}{dt^{s-1}}\right\rangle.
\end{aligned}
$$

Using the derivative operator D, it is not hard to see that the relation described above can be expressed in a more compact form:

$$
\begin{aligned}
0 \geq & \left\langle \sum_{k=0}^{s-1}(-1)^{s-k-1}D^{s-k-1}\left[p_{s-k}(1)x^*(1)\right], x(1)-\tilde{x}(1)\right\rangle \\
& +\left\langle \sum_{k=0}^{s-2}(-1)^{s-k-2}D^{s-k-2}\left[p_{s-k}(1)x^*(1)\right], x'(1)-\tilde{x}'(1)\right\rangle+\cdots \\
& -\left\langle D\left[p_s(1)x^*(1)\right]-p_{s-1}(1)x^*(1), x^{(s-2)}(1)-\tilde{x}^{(s-2)}(1)\right\rangle \\
& +\left\langle p_s(1)x^*(1), x^{(s-1)}(1)-\tilde{x}^{(s-1)}(1)\right\rangle.
\end{aligned}
\tag{10}
$$

Furthermore, applying the definition of the transversality condition Eq. (iii) for all feasible arc $x(\cdot)$, we have

$$
\begin{aligned}
& \varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right)-\varphi\left(\tilde{x}(1), \tilde{x}'(1), \dots, \tilde{x}^{(s-1)}(1)\right) \\
& \geq \left\langle \sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}\left[p_{s-k}(1)x^*(1)\right], x(1)-\tilde{x}(1)\right\rangle \\
& +\left\langle \sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}\left[p_{s-k}(1)x^*(1)\right], x'(1)-\tilde{x}'(1)\right\rangle+\cdots \\
& +\left\langle D\left[p_s(1)x^*(1)\right]-p_{s-1}(1)x^*(1), x^{(s-2)}(1)-\tilde{x}^{(s-2)}(1)\right\rangle \\
& -\left\langle p_s(1)x^*(1), x^{(s-1)}(1)-\tilde{x}^{(s-1)}(1)\right\rangle
\end{aligned}
\tag{11}
$$

Then from the last two inequalities Eqs. (10) and (11) for all feasible arc $x(\cdot)$, we have immediately $\varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right) \geq \varphi\left(\tilde{x}(1), \tilde{x}'(1), \dots, \tilde{x}^{(s-1)}(1)\right)$, that is, $\tilde{x}(\cdot)$ is an optimal trajectory. □

Remark 3.1. It can be noted that in the particular case, if $p_2(t) = p(t), p_1(t) = p'(t)$, where $p(\cdot) : [0,1] \to (0,\infty)$, the second-order linear differential operator $Lx = p_2(t)x'' + p_1(t)x'$ is a well-known self-adjoint Sturm-Liouville operator $Lx \equiv (px')'$.

Corollary 3.1. Let $F(\cdot,t)$ be a closed set-valued mapping. Then under the assumptions of Theorem 3.1, the conditions Eqs. (i) and (iii) can be rewritten in terms of Hamiltonian function as follows:

$$L^*x^*(t) \in \partial_x H_F(\tilde{x}(t), x^*(t), t); .L\tilde{x}(t) \in \partial_{v^*} H(\tilde{x}(t), x^*(t), t), \text{a.e.} t \in [0,1]$$

Proof. Indeed, by Theorem 2.1 ([23], p. 62) and Lemma 5.1 [14], we can write.

$$F^*(v^*; (x,v), t) = \partial_x H_F(x, v^*, t), \text{and } F_A(x, v^*, t) = \partial_{v^*} H_F(x, v^*, t)$$

respectively. Then it is easy to see that the result of corollary are equivalent with the conditions Eqs. (i) and (iv) of Theorem 3.1. □

Below nonconvexity of a set-valued mapping $F(\cdot,t)$ means that its Hamilton function in general is a nonconcave function satisfying the condition Eq. (a).

Theorem 3.2. Suppose that we have the "nonconvex" problem $(\mathrm{P_V})$, that is, $\varphi : (\mathbb{R}^n)^s \to \mathbb{R}^1$ and $F(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in general are nonconvex function and set-valued mapping, respectively. Moreover, suppose that $K_{Q_j}\left(\tilde{x}^{(j)}(0)\right), \tilde{x}^{(j)}(0) \in Q_j$ is the cones of tangent directions to $Q_j, j = 0, \ldots, s-1$.

Then for optimality of the trajectory $\tilde{x}(\cdot)$, it is sufficient that there exists an absolutely continuous function $x^*(\cdot)$, satisfying the following conditions:

a. $L^*x^*(t) \in F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t)$, a.e. $t \in [0,1]$,

b. $\sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j}[p_{s-k}(0)x^*(0)] \in K^*_{Q_j}\left(\tilde{x}^{(j)}(0)\right), j = 0, \ldots, s-1,$

c. $\varphi(\nu_0, v_1, \ldots, v_{s-1}) - \varphi\left(\tilde{x}(1), \tilde{x}'(1), \ldots, \tilde{x}^{(s-1)}(1)\right)$

$$\geq \sum_{j=0}^{s-1} \left\langle \sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j}[p_{s-k}(1)x^*(1)], \nu_j - \tilde{x}^{(j)}(1) \right\rangle,$$
$\forall v_j \in \mathbb{R}^n, j = 0, \ldots, s-1,$

d. $\langle L\tilde{x}(t), x^*(t) \rangle = H_F(\tilde{x}(t), x^*(t), t)$, a.e. $t \in [0,1]$.

Proof. In the proof of Theorem 3.1, we have used the following inequality:

$$\begin{aligned} &H_F(x(t), x^*(t), t) - H_F(\tilde{x}(t), x^*(t), t) \\ &\leq \left\langle \sum_{k=1}^{s} (-1)^k \left(p_k(t)x^*(t)\right)^{(k)}, x(t) - \tilde{x}(t) \right\rangle. \end{aligned} \tag{12}$$

Hence, from the inequality Eq. (12), immediately we have the inequality Eq. (10). Moreover, setting $\nu_j = \tilde{x}^{(j)}(1)\,(j = 0, \ldots, s-1)$ for all feasible trajectories $x(\cdot)$, it is not hard to see that for nonconvex φ the following inequality holds:

$$\varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right) - \varphi\left(\tilde{x}(1), \tilde{x}'(1), \dots, \tilde{x}^{(s-1)}(1)\right)$$
$$\geq \sum_{j=0}^{s-1} \left\langle \sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j}\left[p_{s-k}(1)x^*(1)\right], x^{(j)}(1) - \tilde{x}^{(j)}(1) \right\rangle, j = 0, 1, \dots, s-1,$$

Then for the furthest proof, we proceed by analogy with the preceding derivation of Theorem 3.1. □

4. Some applications to optimal control problems with PLDOs

In this section we give two applications of our results. The first one is the particular Mayer problem for differential inclusions involving PLDOs with constant coefficients, and the second one concerns optimization of "linear" differential inclusions with PLDOs and constant coefficients. Thus, suppose now we have the following optimization problem (for simplicity we consider a convex problem) with sth-order PLDO with constant coefficients:

$$\text{Minimize } \varphi_0(x(1)),$$
$$(\mathrm{P_C})\ Lx(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1], Lx = D^s x + p_1 D^{s-1} x + \dots + p_{s-1} Dx$$
$$x(0) = \alpha_0, x'(0) = \alpha_1, x''(0) = \alpha_2, \dots, x^{(s-1)}(0) = \alpha_{s-1}, . \tag{13}$$

where L is the sth-order polynomial operator, $p_k, k = 1, \dots, s-1$ are some real constants, $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $\varphi_0 : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous convex function, and $\alpha_j \in \mathbb{R}^n,\ j = 0, \dots, s-1$ are fixed n-dimensional vectors. It is known that the multiplication operation is commutative for polynomial linear differential operators with constant coefficients. On the other hand, the sth-order adjoint operator is defined as follows:

$$L^* x^* = (-1)^s D^s x^* + (-1)^{s-1} p_1 D^{s-1} x^* + \dots - p_{s-1} D x^*.$$

Corollary 4.1. Let φ_0 and $F(\cdot, t)$ be a convex function and a set-valued mapping, respectively. Then, for the trajectory $\tilde{x}(\cdot)$ to be optimal in the problem $(\mathrm{P_C})$, it is sufficient that there exists an absolutely continuous function $x^*(\cdot)$ satisfying the Euler-Lagrange type differential inclusion.

$$L^* x^*(t) \in F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t); \langle L\tilde{x}(t), x^*(t)\rangle = H_F(\tilde{x}(t), x^*(t)),$$
$$\text{a.e. } t \in [0, 1], L^* x^* = (-1)^s x^{*(s)} + (-1)^{s-1} p_1 x^{*(s-1)} + \dots - p_{s-1} x^{*\prime}$$

and transversality condition at the endpoint $t = 1$.

$$(-1)^s x^{*(s-1)}(1) \in \partial\varphi_0(\tilde{x}(1)), x^{*(j)}(1) = 0, j = 0, \dots, m-2.$$

Proof . We conclude this proof by returning to the conditions Eqs. (i)–(iii) of Theorem 3.1. Clearly, a problem $(\mathrm{P_C})$ can be reduced to the problem of form $(\mathrm{P_V})$, where

$$\varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right) \equiv \varphi_0(x(1)).$$

It follows that $\partial\varphi\left(x(1), x'(1), \dots, x^{(s-1)}(1)\right) = \partial_x \varphi_0(x(1)) \times (\underbrace{0, \dots, 0}_{s-1})$. On the other hand, since $p_s(t) \equiv 1, p_j(t) = p_{s-j}$, and $j = 1, \dots, s-1$ are constants, by sequentially substitution in the transversality condition Eq. (iii), we derive that

$$\sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j}\left[p_{s-k}(1)x^*(1)\right] = x^{*(s-1-j)}(1) = 0, j = 1, \dots, s-1,$$

and therefore for $j = 0$.

$$\sum_{k=0}^{s-1} (-1)^{s-k} D^{s-k-1}\left[p_{s-k}(1)x^*(1)\right] = (-1)^s x^{*(s-1)}(1) \in \partial\varphi_0(\tilde{x}(1)).\square$$

Suppose now that we have the so-called linear Mayer problem with PLDOs:

$$\text{Minimize } \varphi_0(x(1)), \tag{14}$$

$$Lx(t) \in F(x(t), t), \ \text{ a.e. } t \in [0,1],$$

$$x^{(j)}(0) = \alpha_j, j = 0, \dots, s-1, F(x,t) = A(t)x + B(t)U \tag{15}$$

where φ_0 differentiable convex function; $A(t)$ and $B(t)$ are $n \times n$ and $n \times r$ continuous matrices, respectively; U is a convex compact of $\mathbb{R}^r$; $\alpha_j, j = 0, \dots, s-1$ are constant vectors. It is required to find a control function $\tilde{u}(\cdot)$ such that the corresponding trajectory $\tilde{x}(\cdot)$ minimizes the Mayer functional $\varphi_0(x(1))$.

In fact, this is optimization of Cauchy problem for "linear" differential inclusions with PLDO. The controlling parameter $u(\cdot)$ is called admissible if it only takes values in the given control set U which is a nonempty, convex compact.

Theorem 4.1. The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ is a solution to Eqs. (14) and (15) if there exists an absolutely continuous function $x^*(\cdot)$, satisfying the Euler-Lagrange type differential equation, the transversality condition, and Weierstrass-Pontryagin maximum principle:

$$L^*x^*(t) = A^*(t)x^*(t), \text{a.e. } t \in [0,1],$$

$$(-1)^s x^{*(s-1)}(1) = \varphi_0'(\tilde{x}(1)), .x^{*(j)}(1) = 0, j = 0, \dots, s-2$$

$$\langle B(t)\tilde{u}(t), x^*(t)\rangle = \max_{u \in U} \langle B(t)u, x^*(t)\rangle.$$

Proof. Obviously, the Hamiltonian is

$$H_F(x, v^*, t) = \langle A(t)x, v^*\rangle + \max_{u \in U} \langle B(t)u, v^*\rangle.$$

Hence,

$$F^*(v^*; (x, \tilde{v}), t) = \partial_x H_F(x, v^*, t) = A^*(t)v^*, \tilde{v} \in F_A(x, v^*, t)\tilde{v} = A(t)x + B(t)\tilde{u}$$

where the argmaximum inclusion $\tilde{v} \in F_A(x, v^*, t)$ implies that $\langle B(t)\tilde{u}, v^*\rangle = \max_{u \in U}\langle B(t)u, v^*\rangle$ and $F^*(v^*; (x, \tilde{v}), t) \neq \varnothing$. Thus, applying Theorem 3.1, we obtain

$$L^*x^*(t) = A^*(t)x^*(t), \ L\tilde{x}(t) \in F_A(\tilde{x}(t), x^*(t), t) \ ,$$

$$\langle B(t)\tilde{u}(t), x^*(t)\rangle = \max_{u \in U} \langle B(t)u, x^*(t)\rangle.$$

Consequently, the transversality condition Eq. (ii) of Theorem 3.1 is unnecessary and by Corollary 4.1 $(-1)^s D^{s-1}x^*(1) = \varphi_0'(\tilde{x}(1))$, $D^j x^*(1) = 0, j = 0, \ldots, s-2$. □

Remark 4.1. Suppose that in the definition of sth-order PLDO (see Eq. (13)) $s = 1, p_i = 0, i = 1, .., s-1$ and U is a convex closed polyhedron. Then we have linear equations with variable coefficients $x' = A(t)x + B(t)u, u \in U$ in the finite time interval $t \in [0, 1]$. Obviously, for such problems an adjoint Euler-Lagrange type differential equation and transversality condition at a point $t = 1$ consist of the following: $x^{*\prime}(t) = -A^*(t)x^*(t), x^*(1) = -\varphi_0'(\tilde{x}(1))$. We remind that along with Pontryagin's maximum principle (see, e.g., [24]) under the condition for generality of position for time-optimal problem, the existence results of optimal control are proved.

Example 4.1. Let us consider the following Mayer problem with second-order PLDO $Lx = D^2x = x''$:

$$\text{Infimum } \varphi(x(1), x'(1)) \text{ is subject to } x'' = u, u \in [-1, 1], x(0) \in Q_0, x'(0) \in Q_1. \tag{16}$$

Here $\varphi(x(1), x'(1)) = x'^2(1) - x(1)$ and $Q_0 = \{0\}, Q_1 = \{1\}$.

It should be noted that substituting $F(t) = u(t), x''(t) = a(t), m = 1$ into Newton's second law $F(t) = ma(t)$, we have $x'' = u$.

Obviously, in this problem $F(x, t) \equiv F(x) = \{u : |u| \leq 1\}, s = 2$.

Then Eq. (16) has the form:

$$\text{Infimum } \varphi(x(1), x'(1)) \text{ is subject to } Lx \in F(x), t \in [0, 1], x(0) = 0, x'(0) = 1. \tag{17}$$

It can be easily seen that in the adjoint inclusion Eq. (i).
$-D\big(p_1(t)x^*(t)\big) + D^2\big(p_2(t)x^*(t)\big) \in F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)))$
of Corollary 3.1 $p_2(t) \equiv 0$ and $p_2(t) \equiv 1$, and so we have

$$x^{*\prime\prime}(t) \in F^*(x^*(t); (\tilde{x}(t), \tilde{x}''(t))).$$

Now, it is not hard to see that

$$H_F(x, v^*) = \max_u \{uv^* : |u| \leq 1\} = |v^*| \tag{18}$$

and

$$F^*(v^*; (x, v)) = \partial_x H_F(x, v^*) \equiv 0, \ v \in F_A(x, v^*) = \{-1, +1\}. \tag{19}$$

Then taking into account $Lx^* = d^2x^*/dt^2$, as a result of Theorem 3.1 (see also Corollary 3.1) from Eq. (19), we deduce that

$$x^{*\prime\prime} = 0, \ t \in [0, 1],$$

for which the solution is a linear function of the form $x^*(t) = C_1 t + C_2$, where C_1, C_2 are arbitrary constants. Then Eq. (18) implies that $\tilde{u}(t)x^*(t) = |x^*(t)|$ or

$$\tilde{u}(t) = \begin{cases} \operatorname{sgn} x^*(t), & \text{if } x^*(t) \neq 0, \\ \forall u_0 \in [-1,1], & \text{if } x^*(t) = 0. \end{cases} \tag{20}$$

Further, from the linearity of $x^*(\cdot)$ and from Eq. (20), we insure that each optimal control function is a piecewise constant function.

In addition, by the transversality condition Eq. (iii) of Corollary 3.1, we can write.

$(x^{*\prime}(1), -x^*(1)) \in \partial\varphi(\tilde{x}(1), \tilde{x}'(1))$. On the other hand, it is not hard to see that $\varphi(x,y) = y^2 - x$ is a convex function; in fact, the 2×2 Hessian matrix

$$\varphi''(x,y) = \begin{bmatrix} \varphi''_{xx}(x,y) & \varphi''_{xy}(x,y) \\ \varphi''_{yx}(x,y) & \varphi''_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

is a positive semidefinite, that is, all eigenvalues of $\varphi''(x,y)$ are nonnegative. Indeed, denoting this matrix by A, we see that the characteristic equation $|A - \lambda E| = \lambda^2 - 2\lambda = 0$ (E is a 2×2 unique square matrix) has two real nonnegative eigenvalues $\lambda_1 = 0, \lambda_2 = 2$. Consequently, $\varphi(x,y)$ is convex and $\partial\varphi(x,y) = (-1, 2y)$. It follows that $\partial\varphi(\tilde{x}(1), \tilde{x}'(1)) = (-1, 2\tilde{x}'(1))$. Comparing this relation with $(x^{*\prime}(1), -x^*(1)) \in \partial\varphi(\tilde{x}(1), \tilde{x}'(1))$, we immediately have $x^{*\prime}(1) = -1, x^*(1) = -2\tilde{x}'(1)$. Then from a general solution of the adjoint Euler-Lagrange type inclusion (equation) $x^*(t) = C_1 t + C_2$, we have $-2\tilde{x}'(1) = x^*(1) = C_1 + C_2, -1 = x^{*\prime}(1) = C_1$ (C_1, C_2 are arbitrary constants), and so $x^*(t) = 1 - t - 2\tilde{x}'(1)$, whence $x^*(t) \neq 0$, if $t \neq \tau = 1 - 2\tilde{x}'(1)$. Therefore, Eq. (20) implies that for optimal control $\tilde{u}(\cdot)$, there are four possibilities:

$$\tilde{u}(t) = 1, \;\; x^*(t) > 0, \;\; t \in [0,1]. \tag{21}$$

$$\tilde{u}(t) = -1, \;\; x^*(t) < 0, t \in [0,1]. \tag{22}$$

$$\tilde{u}(t) = \begin{cases} 1, & \text{if } \;\; 0 \le t < \tau, \\ -1, & \text{if } \;\; \tau < t \le 1. \end{cases} \tag{23}$$

$$\tilde{u}(t) = \begin{cases} -1, & \text{if } \;\; 0 \le t < \tau, \\ 1, & \text{if } \;\; \tau < t \le 1. \end{cases} \tag{24}$$

(observe that τ is a point of discontinuity of $\tilde{u}(\cdot)$ and the values of the control functions $\tilde{u}(\cdot)$ at a point of discontinuity τ are unessential). As a consequence, it follows that either the sign of the linear function $x^*(t)$ does not change for the whole interval $[0,1]$ or $x^*(t) > 0, 0 \le t < \tau; x^*(t) < 0, \tau < t \le 1$ for a some τ in the interval $0 < \tau < 1$ (the case Eq. (24) is excluded). Therefore, since $\tilde{u}(t)$ is a piecewise constant function, having not more than two intervals of constancy, we have either the cases Eqs. (21) and (22) or the case Eq. (23). In general, using Eqs. (21)–(23), by solving the Cauchy problem

$$x''(t) = u(t), \;\; x(0) = 0, x'(0) = 1 \tag{25}$$

we have a unique solution of the initial value problem Eq. (25). Thus for the time interval on which $u = 1$, we have $x'(t) = t + c_1$; $x(t) = t^2/2 + c_1 t + c_2$ (c_1, c_2 are constants). From Eq. (25) we obtain.

$$x'(t) = t + 1; x(t) = t^2/2 + t. \tag{26}$$

By analogy, for $u = -1$ we have.

$$x'(t) = 1 - t;\ x(t) = -t^2/2 + t. \tag{27}$$

Now, let $x_1(t)$ and $x_2(t)$ be parabolas of Eqs. (26) and (27), respectively. Here, in the case Eq. (21), $\tilde{u}(t) = 1, t \in [0,1]$, and so from Eq. (26), we have $\tilde{x}_1(1) = 0.5 + 1 = 1.5; \tilde{x}_1'(1) = 2$. Consequently, the value of problem Eq. (16) is $\varphi(\tilde{x}_1(1), \tilde{x}_1'(1)) = \tilde{x}_1'^2(1) - \tilde{x}_1(1) = 2^2 - (1.5) = 2.5$, if $\tilde{u}(t) = 1, t \in [0,1]$. By a similar way, for a control function $\tilde{u}(t) = -1, t \in [0,1]$ from Eq. (27), we obtain that $\tilde{x}_2(1) = -1^2/2 + 1 = 0.5, \tilde{x}_2'(1) = 0$ and $\varphi(\tilde{x}_2(1), \tilde{x}_2'(1)) = 0^2 - (0.5) = -0.5$.

On the other hand, in the case Eq. (23), the control function $\tilde{u}(t)$ first is equal to $+1$ and then equal to -1, and the trajectory $\tilde{x}(t)$ consists of two pieces of parabolas $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ ($\tilde{x}(t)$ is continuous and piecewise smooth on the interval $0 \le t \le 1$). Then the solution of the equation Eq. (25) on the interval $0 \le t \le \tau$ is given by Eq. (26); at a point τ are satisfied $x_1(\tau) = \tau^2/2 + \tau$, $x_1'(\tau) = 1 + \tau$. Consider now the initial value problem:

$$x_2''(t) = -1,\ \ x_2(\tau) = \tau^2/2 + \tau, x_2'(\tau) = 1 + \tau, t \in [\tau, 1]. \tag{28}$$

It is clear that $(\tau^2/2) + \tau = x_2(\tau) = -(\tau^2/2) + c_1\tau + c_2$ and $1 + \tau = x_2'(\tau) = -\tau + c_1$ from which we obtain that the solution of the initial value problem Eq. (28) is $\tilde{x}_2(t) = -(t^2/2) + (1 + 2\tau)t - \tau^2$. Substituting the value $\tau = 1 - 2\tilde{x}'(1)$ into equation $\tilde{x}_2'(t) = 1 - t + 2\tau$, we have $5\tilde{x}_2'(1) = 2$, $\tilde{x}_2'(1) = \tilde{x}'(1) = 0.4$ (it follows that $\tau = 0.2$). Moreover, $\tilde{x}_2(1) = 2\tau - \tau^2 + (0.5)$ and $\tilde{x}_2(1) = \tilde{x}(1) = (3/2) - 4\tilde{x}'^2(1) = 0.86$. Thus, the value of our Mayer problem is $\varphi(\tilde{x}(1),\ \tilde{x}'(1)) = \tilde{x}'^2(1) - \tilde{x}(1) = (2/5)^2 - (43/50) = -0.7$, where $\tilde{u}(t)$ is defined as in Eq. (23). Comparing the values $2.5,\ -0.5,\ -0.7$, we believe that the value of Mayer problem is -0.7.

5. Sufficient conditions of optimality for second-order evolution differential inclusions with endpoint constraints

Note that in this section the optimality conditions are given for second-order convex differential inclusions (P_M) with convex endpoint constraints. These conditions are more precise than any previously published ones since they involve useful forms of the Weierstrass-Pontryagin condition and second-order Euler-Lagrange type adjoint inclusions. In the reviewed results, this effort culminates in Theorem 5.1:

$$\text{Minimize}\, g(x(1), x'(1)),$$

$$(P_M)\ x''(t) \in F(x(t), x'(t), t), \text{a.e. } t \in [0,1],$$

$$x(0) = x_0, x'(0) = x_1; x(1) \in M_0,\ \ x'(1) \in M_1,$$

where g is a convex continuous function, $F(\cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is convex set-valued mapping, and $M_0, M_1 \subseteq \mathbb{R}^n$ are convex sets.

The following adjoint inclusion is the second-order Euler-Lagrange type inclusion for the problem (P_M):

a$_1$. $(x^{*''}(t) + v^{*'}(t), v^*(t)) \in F^*(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t)), t)$, a.e. $t \in [0,1]$, where

b$_1$. $\tilde{x}''(t) \in F_A(\tilde{x}(t), \tilde{x}'(t); x^*(t), t)$, a.e. $t \in [0,1]$.

In what follows we assume that $x^*(t)$, $t \in [0,1]$ is absolutely continuous function together with the first-order derivatives for which $x^{*\prime\prime}(\cdot) \in L_1^n([0,1])$. Besides the auxiliary function $v^*(t), t \in [0,1]$ is absolutely continuous and $v^{*\prime}(\cdot) \in L_1^n([0,1])$.

The transversality conditions at the endpoint $t=1$ consist of the following:

c$_1$. $(v^*(1) + x^{*\prime}(1), -x^*(1)) \in \partial_{(x,u)} g(\tilde{x}(1), \tilde{x}'(1)) - K^*_{M_0}(\tilde{x}(1)) \times K^*_{M_1}(\tilde{x}'(1))$.

Now we are ready to formulate the following theorem of optimality.

Theorem 5.1. Suppose that g is a continuous and convex function, $F(\cdot, t)$ is a convex set-valued mapping, and M_0, M_1 are convex sets. Then for optimality of the feasible trajectory $\tilde{x}(t)$ in the problem (P$_M$), it is sufficient that there exists a pair of absolutely continuous functions:

$$\{x^*(t), v^*(t)\}, t \in [0,1]$$

satisfying a.e. the second-order Euler-Lagrange type inclusions Eqs. (a_1) and (b_1) and the transversality condition Eq. (c_1) at the endpoint $t=1$.

Proof. By the proof idea of Theorem 3.1 from Eqs. (a_1) and (b_1), we obtain the adjoint differential inclusion of second order:

$$(x^{*\prime\prime}(t) + v^{*\prime}(t), v^*(t)) \in \partial_{(x,u)} H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t), t), t \in [0,1].$$

On the definition of subdifferential set of the Hamiltonian function $H_F(\cdot, t)$ for all feasible trajectory $x(t), t \in [0,1]$, we rewrite the last relation in the equivalent form:

$$\begin{aligned} &H_F(x(t), x'(t), x^*(t), t) - H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t), t) \\ &\le \langle x^{*\prime\prime}(t) + v^{*\prime}(t), x(t) - \tilde{x}(t)\rangle + \langle v^*(t), x'(t) - \tilde{x}'(t)\rangle. \end{aligned} \tag{29}$$

Now by using definition of the Hamiltonian function, the inequality Eq. (29) can be reduced to the inequality

$$0 \ge \langle (x(t) - \tilde{x}(t))'', x^*(t)\rangle - \langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t)\rangle - \frac{d}{dt}\langle v^*(t), x(t) - \tilde{x}(t)\rangle. \tag{30}$$

Integrating of the inequality Eq. (30) over the interval $[0,1]$, we derive that

$$\begin{aligned} 0 \ge &\int_0^1 [\langle (x(t) - \tilde{x}(t))'', x^*(t)\rangle - \langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t)\rangle] dt \\ &+ \langle v^*(0), x(0) - \tilde{x}(0)\rangle - \langle v^*(1), x(1) - \tilde{x}(1)\rangle. \end{aligned} \tag{31}$$

For convenience we transform the expression in the square parentheses on the right-hand side of Eq. (31) as follows

$$\langle (x(t) - \tilde{x}(t))'', x^*(t)\rangle - \langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t)\rangle$$

$$= \frac{d}{dt}\left\langle (x(t)-\tilde{x}(t))', x^*(t)\right\rangle - \frac{d}{dt}\langle x^{*\prime}(t), x(t)-\tilde{x}(t)\rangle.$$

Thus by elementary property of the definite integrals, we can compute the integral on the right-hand side of Eq. (31):

$$\begin{aligned}&\int_0^1 \left[\left\langle (x(t)-\tilde{x}(t))'', x^*(t)\right\rangle - \langle x^{*\prime\prime}(t), x(t)-\tilde{x}(t)\rangle\right]dt\\ &= \langle x'(1)-\tilde{x}'(1), x^*(1)\rangle - \langle x'(0)-\tilde{x}'(0), x^*(0)\rangle\\ &-\langle x^{*\prime}(1), x(1)-\tilde{x}(1)\rangle + \langle x^{*\prime}(0), x(0)-\tilde{x}(0)\rangle.\end{aligned} \tag{32}$$

Then substituting Eq. (32) into Eq. (31), we have

$$\begin{aligned}0 \ge \langle x'(1)-\tilde{x}'(1), x^*(1)\rangle - \langle x'(0)-\tilde{x}'(0), x^*(0)\rangle\\ -\langle v^*(1)+x^{*\prime}(1), x(1)-\tilde{x}(1)\rangle + \langle v^*(0)+x^{*\prime}(0), x(0)-\tilde{x}(0)\rangle.\end{aligned} \tag{33}$$

Now, remember that $x(\cdot), \tilde{x}(\cdot)$ are feasible trajectories and $x(0) = \tilde{x}(0) = x_0$ and $x'(0) = \tilde{x}'(0)$.
$= x_1$. Then it follows from Eq. (33) that

$$0 \ge \langle x'(1)-\tilde{x}'(1), x^*(1)\rangle - \langle v^*(1)+x^{*\prime}(1), x(1)-\tilde{x}(1)\rangle. \tag{34}$$

Now, thanking to the transversality conditions Eq. (c_1) at the endpoint $t = 1$, we can rewrite

$$\begin{aligned}g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1)) \ge \langle v^*(1)+x^{*\prime}(1)+x^*(1), x(1)-\tilde{x}(1)\rangle\\ +\langle x^{*\prime}(1)-x^*(1), x'(1)-\tilde{x}'(1)\rangle,\ \ x^*(1)\in K^*_{M_0}(\tilde{x}(1)),\ \ x^{*\prime}(1)\in K^*_{M_1}(\tilde{x}'(1))\end{aligned}$$

or, in other words

$$\begin{aligned}g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1))\\ \ge \langle v^*(1)+x^{*\prime}(1), x(1)-\tilde{x}(1)\rangle - \langle x^*(1), x'(1)-\tilde{x}'(1)\rangle\end{aligned} \tag{35}$$

Thus, summing the inequalities Eqs. (34) and (35) for all feasible trajectories $x(\cdot)$, satisfying the initial conditions $x(0) = x_0$ and $x'(0) = x_1$ and endpoint constraints $x(1)\in M_0, x'(1)\in M_1$, we have the needed inequality:
$g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1)) \ge 0$ or $g(x(1), x'(1)) \ge g(\tilde{x}(1), \tilde{x}'(1))$. □

6. Conclusion

According to proposed method, the problem with the differential inclusions described by polynomial linear differential operators is investigated. Obviously, this problem is an important generalization of problems with first-order differential inclusions. Thus, sufficient conditions of optimality for such problems are deduced. Here the existence of nonfunctional initial point or endpoint constraints generates different kinds of transversality conditions. Besides, there can be no doubt that investigations of optimality conditions of problems with second- and fourth-order Sturm-Liouville type differential inclusions can play an important role in the development of modern optimization and there is every reason to believe that this role will be even more significant in the future. Thus, the suggested problem with linear differential operators and variable coefficients can be used in various forms in applied problems.

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