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Chapter

Students' Productive Struggles in Mathematics Learning

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Abstract

Using a predetermined framework on students' productive struggles, the purpose of this study is to explore high school students' productive struggles during the simplification of rational algebraic expressions in a high school mathematics classroom. This study is foregrounded in the anthropological theory of the didactic, and its central notion of a "praxeology" – a praxeology refers to the study of human action, based on the notion that humans engage in purposeful behavior of which the simplification of rational algebraic expressions is an example. The research methodology comprised a lesson study involving a sample of 28 students, and the productive struggle framework was used for data analysis. Findings show that the productive struggle framework is a useful tool that can be used to analyze students' thinking processes during the simplification of rational algebraic expressions. Further research is required on the roles that noticing and questioning can play for mathematics teachers to respond to and effectively support the students' struggles during teaching and learning.

Keywords: anthropological theory of the didactic, productive struggles, lesson study, mathematical teacher noticing

1. Introduction

The purpose of this study is to explore high school mathematics students' productive struggles during the simplification of rational algebraic expressions. In recent research in mathematics learning and teaching [1–3], struggle is often associated with negative meanings of how mathematics is practiced in classrooms. Teachers of mathematics often view students' struggles in mathematics as something that should be avoided and/or as a learning problem that needs to be diagnosed and remediated or simply eradicated [4, 5]. Struggle in mathematics learning and teaching is an essential component of students' intellectual growth, and of deep learning of mathematical concepts with understanding [6]. Research suggests that the apparent confusion and/or doubt displayed by students during problem- solving provide students with opportunities for deepening their conceptual understanding of mathematical concepts during teaching [7, 8]. However, exposing students to complex problem-solving tasks which are beyond their cognitive levels, skills and abilities can result in productive failures on the part of students [7]. When students engage in complex problem-solving tasks, they are likely to experience productive failure unless support structures are put in place. Broadly, support structure refers to "[re-] structuring the problem itself, scaffolding, instructional facilitation,

provision of tools, expert help, and so on" ([7], p. 524). Research has shown that exposing students to complex problem-solving without putting in place efficient support structures can result in an unproductive cognitive process [9, 10]. The notion of productive failure is centered on view that students are not in position to find the solution to a mathematical problem on their own in the short term. With assistance from teachers and capable peers, and taping from their prior knowledge, students can overcome their productive failures. Students can also experience unproductive success when they experience immediate learning gains through drill-and-practice, and memorization approaches. Unproductive failure learning situations arise when the conditions in a learning environment do not favor neither learning in short term nor long term. While there is no "recipe" in avoiding and/or addressing unproductive failure situations when students are engaging in complex tasks, for example – simplifying rational algebraic expressions, teachers can adopt approaches that ameliorate unproductive situations. According to [11] and others [12, 13], learners who engage in unguided problem solving are likely to experience productive failure. ([13], p. 128) posits that "What can be conceivably be gained by leaving the learner [student] to search for a solution when the search is usually very time consuming, may result in ... no solution at all." Hence, to avoid unproductive failure learning situations, students must be provided with guidance during problem-solving. By guidance, we are referring to: scaffolding of problems; feedback through questioning, among others.

In other words, the struggle becomes a process in which students restructure their existing knowledge while moving towards a new understanding of what is being taught [14–16]. Students' struggles become productive in classrooms where they are afforded opportunities to solve complex problems, while being encouraged to try various approaches; even though in these classrooms, students can still fail and struggle, they will feel motivated and good about solving complex problems [17]. Equally, productive struggles ensue when students are given the support structure during problem-solving [7]. In classrooms, at the center of teaching and learning, teachers are expected to create a learning environment that values and promotes productive struggles among students by using challenging learning tasks that are nonetheless accessible to all students [18–21]. Productive struggle, which is stimulated by using challenging tasks during learning and teaching, supports students' cognitive growth and is essential for their learning of mathematics with understanding. While facilitating students' productive struggles teachers should avoid "reducing the cognitive load of the task such as [by] providing routine instructions tasks and over-modelling how to approach the task" ([17], p. 20). ([18], p. 178), similarly, encourages teachers to avoid "effortless achievement" by students; instead, teachers should value persistence and hard thinking.

While substantial research has been carried out on the types of errors that are committed by students when simplifying rational algebraic expressions in high school mathematics [22–24], this study explores the students' productive struggles during the simplification of rational algebraic expressions in real time, unlike the previous studies that focused only on students' errors. It is apropos to mention that students' productive struggles also include an understanding of how students deal with conceptual errors and misconceptions. As such, this study uses a predetermined framework [6] to analyze students' productive struggles as well as for analyzing the teachers' responses to the students' productive struggles. Existing research has focused on the difficulties encountered by students in understanding the equivalence of rational algebraic expressions through simplification and by valuing the importance of working and/or manipulating these expressions accurately with great flexibility [25–28]. The challenge here lies in the ability of students to work with more than one rational algebraic expressions and to find their

equivalences. Thus, to explore students' productive struggles during the simplification of rational algebraic expressions in high school, this study is guided by the following research questions: What are the types of productive struggles experienced by the students while simplifying rational algebraic expressions in a high school lesson? How do teachers notice, and respond to the students' productive struggles during classroom activities? What questioning techniques are used by the teachers to support the students' productive struggles?

2. Theoretical framework

In this section, we define the anthropological theory of the didactic – in which the study is based, and the students' productive struggle framework that is used for analyzing students' learning activities.

2.1 Anthropological theory of didactics

This study is founded in the anthropological theory of didactics and its central notion of a "praxeology" – a praxeology refers the study of human action, based on the notion that humans engage in purposeful behavior of which learning mathematics is an example [29, 30]. Nicaud et al. [28] argues that anthropological theory of the didactic as a general epistemological model for mathematical knowledge can be used to understand human mathematical activities, such as, in the context of this chapter, the simplification of rational algebraic expressions. Like any praxeology, the mathematical knowledge emerging from human activities is constituted by an amalgamation of four critical components, namely: type of task; technique; technology; and theory [28]. In human activities related to the learning of mathematics, Nicaud et al. [28] further re-classified the four critical components into two main praxeological models – the practical block and the knowledge block. The practical block is made up of the type of task and the technique. In the context of this study, the specific task is the simplification of rational algebraic expressions, whereas the technique refers to the tools that students need to carry out these simplifications. Examples of tools include: factorizations; finding common denominators, expanding expressions, and cancelation procedures among others. The knowledge block consists of a technology – which is used to explain the technique, and a theory – which is used to justify the technology. A point to be stressed here is that the word "technology" is used here to refer to a discourse on a given technique. In other words, "this discourse is supposed, at least in the best-case scenario, both to justify the technique as a valid way of performing tasks and throw light on the logic and workings of that technique" ([31], p. 2616). For instance, in this study, the technique is the "know how" to simplify the rational algebraic expressions, while the technology consists of what mathematical knowledge or logic justifies the way these techniques are operationalized.

At the core of ATD [29] is the notion of an epistemological model aimed at understanding the "ecology of mathematical knowledge that emerges from human practices" ([30], p. 1). Research shows that there many traditions of didactics at the core of teaching and learning in schools – the German Didaktik, whose origins hail from the seventeenth century, is one them [32, 33]. In general, the word "didaktik" refers to both the art of teaching, and to a theory of teaching. It is worthwhile noting that the German Didaktik does not cover subject areas issues but covers general issues of theory and practice of teaching [34]. However, the German Didaktik is guided by three core tenets: bildung; theory of educational content; and the notion of teaching as a meaningful endeavor which is encountered between students and

content [35–37]. The bildung – encapsulates the aims and values of the education system centered on "formation of the mind, the unfolding of capability, and the development of the sensitivity of the learner [student]" ([35], p. 544). In the German Didaktik, theory of educational content is construed as: the nature of content; educational value of the content; and the general organization of the content for educational purposes [38]. Also, at the core of German Didaktik, is the notion of "productive encounter" between content and students, which is analyzed and facilitated by teachers during teaching and learning [39, 40]. To provide context to this discussion between ATD and the German Didaktik, our position is that the German Didaktik is a general theory on the art of teaching and learning, while ATD seeks to address teaching and learning issues within a subject area – for example mathematics. In this study, the focus is exploring students' productive struggles when simplifying rational algebraic expressions. As such, the ATD with its praxeologies is used as theory for understanding how students conceptualize the simplification of rational algebraic expressions in mathematics.

2.2 Productive struggles

In the previous section, this study has alluded to the importance of students' struggle during learning activities on simplification of rational algebraic expressions and explained how this leads to overcoming conceptual difficulties and achieving deeper and more long-lasting learning [41]. Kapur [11] posits that, during productive struggles, a failed initial attempt on a certain task can lead to improved learning. This learning process envisioned by [11] occurs in two stages. Firstly, students are given a learning activity or problem they cannot solve immediately, and thus the teacher encourages them to conjecture on the possible solutions to the problem. Secondly, once the initial attempts have failed, students receive instruction on possible ways to solve the problem and are given another opportunity to try to solve the problem themselves. In other words, productive struggle "can prime students for subsequent instruction by making them more aware of their own knowledge gaps and more interested in filling those gaps" ([41], p. 85). Depending on individual students' levels of conceptual understanding, it is apropos to say that they experience different types of struggles. After observing these different types of productive struggles in a classroom situation when working on challenging problems, Warshauer [42] developed a productive struggle framework that consists of four types.

The main four types of productive struggles identified by [6, 16, 42] relate to the following aspects: getting started; carrying out a process; experiencing uncertainty in explaining and sense making; and expressing misconceptions and errors. **Table 1** shows the types students' productive struggles and their respective general descriptions [6, 16, 42].

The study uses the above four pre-determined types of students' productive struggles as a framework for analyzing students' ways of simplifying rational algebraic expressions.

2.3 Responses to productive struggles

The construct of "noticing" in mathematics teaching is a widely researched phenomenon in mathematics education, particularly in the high school context [43–45]. Mathematical noticing or simply noticing during teaching consists of three interrelated skills: "*attending* to children's [students'] strategies, *interpreting* children [students'] understandings, and *deciding* how to respond based on children's [students'] understandings" ([45], p. 117). Huang and Li [46] further elaborates on this, positing

Type of productive struggle	Description of the productive struggle
Getting started	Students feel cognitively overloaded and confused about the task – this is evidenced by the fact that there are no written answers or attempts on paper. Students also claim that they do not remember the work and/or the type of problems, and there could be gestures of uncertainty, and resignation. As a sign of frustration, students' utterances could be: "I do not know what to do," "Oh dear! I am very confused," "I wish I knew where to start," etc. In term of simplifying rational algebraic expressions, students might not fully understand the illustration from the question.
Carrying out a process	This relates to students encountering an impasse while attempting to solve a given task. For example, students may find it difficult to demonstrate or follow a known procedure or algorithm. Also, students may fail to recall the facts or formulae required to successfully implement a process, such as factorizations, multiplication of factors, or division of factors required to obtain an equivalent fraction.
Experiencing uncertainty in explaining and sense-making	Students may find it difficult to explain their work to other members of the group, when working in small groups, or to the whole class, when asked to do so by the teacher. In many cases, students fail to verbalize their thinking processes or to justify their correct answers. For example, the student may say "I know this is the correct simplification, but I cannot explain how I got it."
Expressing misconception and errors	Errors can be classified as: careless and conceptual. On one hand, conceptual errors occur when students fail to observe the correct relational ideas when solving problems. On the other hand, careless errors relate to unintentional and yet avoidable procedures that students commit during problem solving. A misconception is usually not wrong thinking; however, it can be interpreted as an indication of deep-seated misplaced ideas that are used to justify the process of finding a solution to a problem – these can manifest as local generalizations made by students.

Table 1.
Types of students' productive struggles and their descriptions.

that *attending* is also about identifying what is noteworthy, that *interpreting* is about making general connections between specific classroom interactions and broader theories of teaching and learning, and that *deciding* is also about how teachers use what they know and understand about their learning contexts to decide how to respond or reason about classroom activities. Teachers use the construct of noticing to identify students' productive struggles. Once the students' struggles have been identified, the teachers will make intentional efforts to support these struggles – in this context, the simplification of rational algebraic expressions. In other words, supporting students' productive struggles requires the teacher to find ways of addressing or responding to the struggles by converting them into positive learning endeavors that create further opportunities for deep learning, rather than episodes in which learners experience difficulties and frustration [4, 6, 16]. Recent studies have illustrated many possible ways teachers can use to respond to the students' productive struggles in mathematics [4–6, 47] - these ways are not mutually exclusive.

Teachers mainly respond to the students' productive struggles in the following four ways: Firstly, they can use *telling* – in other words, after evaluating the nature of the students' productive struggles, a teacher can help a student by: suggesting new approaches to solve the problem; directly correcting the student's errors and/or misconceptions; and giving the student a simpler problem to work on first. [48, 49] stress the notion of "judicious telling," which requires teachers to support students'

productive struggles by repeating the students' own contributions with the aim of highlighting the mathematical ideas that students have already grasped and understood to enable students to better understand the contexts and terminology in the specific tasks. Secondly, teachers can utilize *directed guidance*, which involves the teacher breaking down the problem given to the student into manageable parts, which can assist him/her to anticipate the next step in solving the problem. Directed guidance can also be used, as in this study, for instance, to allow a student to do operations on numerical fractions before he/she attempts simplifications of rational algebraic expressions. Teachers can also use "advancing questions," which can "extend students' current mathematical thinking towards a mathematical goal (simplifications of rational algebraic expressions) of a lesson" ([47], p. 178). Thirdly, teachers could use *probing guidance*, in which the teacher assesses the student's thinking by asking him/her to justify and explain his/her proposed solution. This is done by asking assessing questions and advancing questions (as explained above). Asking assessing questions allows the teacher: to discover students' thinking processes, evaluate their cognitive capabilities, and encourage them to share their thinking on the simplification of rational algebraic expressions [47]. Lastly, teachers can use *affordance*, which involves the teacher's ability to engage students by emphasizing justifications and sense-making with the entire group or with individuals. The term also refers to affording the students time and space to think and solve the problem with encouragement from the teacher but with minimum help. By using these four ways to respond to students' productive struggles, and teachers are afforded the opportunity to deepen their own understanding and more appropriately access students' thinking processes, while positioning themselves to effectively support students' learning – in this case, their learning on the simplifications of rational algebraic expressions. The teachers' questioning techniques will allow the teachers to deepen his/her understanding of the nature of the struggles students harbor.

As already alluded to in this chapter, support structures need to be put in place to ameliorate situations where students' productive struggles can be obstacles for student learning or barriers to students' conceptual development in mathematics [7]. In addition, where students struggle as expected during learning, it is apropos for teachers not to rush to provide a support structure, but to wait until students reach an impasse – as evidenced by utterances such as "I am stuck,", and "I have no idea on how to proceed," among others. By extension, support structures can also refer to questioning techniques of teachers, teacher explanations, or feedback in real time on students' work. It is worthwhile noting that a delayed support structure, for instance – teacher's explanation, can lead to performance failure in the short term, but in the longer term benefits the student as it gives the student time to discern the concepts of the problems being solved.

3. Methodology

In this section, we describe the research sample within the context of the study, the lesson study as a research methodology, and the data sources and analysis techniques.

3.1 Participants

This study sought to explore Grade 11 mathematics students' productive struggles during simplification of rational algebraic expression, and the ways in which the teacher noticed, and responded in a high school located in South Carolina in the

United States of America. Twenty-eight students participated, constituting all Grade 11 students at the high school. Since the study involved minors, ethical clearance was sought from the South Carolina County School District, the school principal, and the legal guardians of the students. In addition, consent was also sought from the participating teacher who was responsible for teaching the concept of simplification of rational algebraic expressions.

3.2 Data sources

In this study, data was collected by using a pre-determined research instrument, in other words, a lesson on the simplification of rational algebraic expressions, which was co-planned and co-implemented by the teacher and the researchers. The lesson which is the subject of investigation is part of a series of lessons that were taught on the simplifications of rational algebraic expressions. To be more precise it is the third lesson of the series lessons. Lesson 1 dealt with simplification of rationale algebraic expressions of the form: $\frac{2}{ab} + \frac{5}{bd}$. Lesson 2 dealt with the simplification of single rational algebraic expressions where factorization was envisaged, for example: $\frac{2x-6}{x^2-9}$. Lesson 3, which is the focus of this study, deals with the simplification of two rationale algebraic expressions being added or subtracted, for example, $\frac{x}{x^2+x-2} - \frac{2}{x^2-5x+4}$, where factorization and finding common denominators are envisaged. All the problem solved by students during the three lessons outlined are foregrounded in the South African Grade 11 mathematics syllabus and come from a prescribed textbook that students used.

This study uses a lesson plan as research methodology with specific focus on exploring the types of productive struggles students experienced during the simplification of rational algebraic expressions. While the stages of the lesson study of setting goals and planning are given less prominence in the data analyses, they are nonetheless important because they foreground the activities of the implementation and debriefing stages. Since the students' productive struggles manifest during the implementation stage of the lesson study, the study has prioritized the implementation stage to explore the students' productive struggles. The debriefing stage affords the teacher the opportunity to discuss with the researcher the students' productive struggles as he observed and responded to them in class, and in real time.

Research studies position the lesson study, which originated from East Asia, as a form of practice-based continuous professional development of mathematics teachers, which has since been adopted by many other countries [50]. In each of these countries, the emphasis of the lesson study varies, however, its major role of school-based continuous professional development for mathematics teachers remain. For example, in China, the focus is on "developing best teaching strategies for specific subject content for student learning," and in Japan, the focus is on "general and long-term educational goals, such as developing students' mathematical thinking through observing student learning in order to collect evidence to improve it" ([51], p. 271). Regardless of the country, the lesson study has three salient features, it is: a deliberate practice – meaning that the task of a lesson study is goal-oriented aimed at improving teacher performance, and affords opportunities for repetition and refinement; a research methodology – aimed at improving both professional and academic knowledge; and an improvement science – through the use of "plan-do-study-act" ([52], p. 54) innovations. In this study, we chose the lesson study as a research methodology to explore the students' productive struggles when simplifying rational algebraic expressions over the design-based research methodology. The lesson study, as a deliberate practice and research methodology can be used in a similar way as the design-based research to narrow the gap between research and practice during

teaching [51–53]. Proponents of the lesson study argue that "not only is this (lesson study) real research, but the methodology of lesson study has huge benefits as means of developing knowledge that is useful for improving teaching (and learning)" ([54], p. 584). As a research methodology, the lesson study, seeks to address specific research questions using a research lesson – a research lesson is a lesson that is a subject of an investigation by researchers. A case in point here is the simplification of rational algebraic expressions. When using the lesson study as a research methodology, the research lesson is often proceeded by an evaluation of the students' conceptual understandings of concepts taught – this can be done through documentary analysis of students' written work in tests and/or carrying out focus group interviews with students who participated in the research lesson.

Data collection processes were informed by the stages of a lesson study approach [55]. Moreover, all the stages were video-recorded. The lesson study was thus used in this study as the research methodology [51], as this allowed both teacher and researchers to study students' thinking [47, 51]. In the context of this study, the lesson study consisted of four stages. The first involved *setting goals* by identifying specific students' learning and development goals and achievements, as agreed upon beforehand by the teacher and the researchers, pertaining to the simplification of rational algebraic expressions. The second stage was *planning*, which meant using the goals identified to plan a "research lesson" that would be used for data collection on the topic. During this stage, discussions took place on how to anticipate students' questions and the teacher's responses. During the third stage, *implementing*, the teacher taught the class, while the researchers observed and collected the data. Focus group interviews took place with the students, who were given opportunities to explain their understanding of the lesson topic. In the final stage, *debriefing*, the teacher and the researchers met to discuss and/or reflect on the data collected; samples of students' work that had been collected were also analyzed to validate some of their productive struggles during the lesson [51, 55–57].

3.3 Data analysis

Video-recordings of the classroom interactions during the research lesson and focus group interviews for students were transcribed verbatim. Thereafter, the transcriptions were analyzed using a pre-determined productive struggles framework (see **Table 1**) thus exploring the types of students' productive struggles encountered and the teachers' responses to these. In addition, documentary analysis was used to analyze students' written work to see how they were simplifying rational algebraic expressions.

4. Findings and discussion

In this section, a pre-determined framework (see **Table 1**) is used to explore the types of struggles experienced by the high school students, and the ways in which the teacher noticed and responded to these are discussed from examples given within a lesson situation. To be more specific during the implementation stage of the lesson study. For anonymity, the letters T and S represent the teacher and student respectively. The word episode is used to refer to a lesson excerpt.

4.1 Getting started

Below is an excerpt that describes the classroom interactions between the teacher and the students when the students were asked to simplify two rational

algebraic expressions: $\frac{2}{x^2-x-2} - \frac{6}{x^2+6x+5}$. Prior to this lesson, students had been simplifying single rational algebraic expressions that require factorization – students were expected to factorize the numerator and denominator, and then perform a cancelation. However, in this episode from the lesson, the simplification of rational algebraic expressions was extended sums and differences of two rational algebraic expressions which required factorization.

Learning episode 1.

S: I am stuck.

T: Sweet, I am on my way. Where are you stuck?

S: I factored it, and this is what I got $\frac{2}{(x-2)(x+1)} - \frac{6}{(x+5)(x+1)}$.

T: Alright. What are you going to do?

S: I've got to make them [*the denominators*] the same.

T: Ok what do you have to do to make them the same? [*Teacher notices that the students are struggling]*

T: What factors are common in denominators of both fractions? [*sensing that there might be an overall conceptual problem in the class by listening to students' chit chat*]. Hey guys, let us back up. Are sure you know what going on [*referring to the last student*]? Alright, we are going to revisit the problem we did yesterday:$\frac{2}{3x} + \frac{2}{7x^2} - \frac{3}{2xy^2}$, which we expanded to: $\frac{2}{3x} + \frac{2}{7xx} - \frac{3}{2xyy}$

At the beginning of the episode, a student remarked that he/she was stuck – meaning that they could not initiate the simplification the rational algebraic expression. The teacher used probing guidance, by asking questions, such as *Where are you stuck? … What are you going to do?* This prompted the student to explain his/her thinking processes to the teacher. While listening to the students chatting, the teacher noticed that they were experiencing challenges of simplifying rational algebraic expressions' particularly the factorization. From an ATD perspective, some students lacked the technique or the tools such as factorizations to simplify the rational algebraic expressions [29, 31]. In responding to this productive struggle, the teacher decided, together with the students, to revisit a much simpler example they had done the previous day. The teacher's action constituted directed guidance by redirecting the students' attention to a much simpler example with the aim of trying to deepen their understanding of the related concepts. In the debriefing interview, the teacher referred to the *get started* stage as a "freak out" moment, positing, "I definitely think there was a get out there and a freak out moment and they don't understand anything." He continued to say that, whenever his students were stuck, he reminded them to calm down and think about the concepts they had already covered and to try to apply them to the novel problem. Intuitively, the teacher alludes to the notion of *delay of structure* [7] – this notion is about a teacher delaying, giving a student a support structure, for example, in the form questions, explanations or feedback, immediately when the student experiences an impasse [58, 59].

4.2 Carrying out a process

In another episode during a lesson, the students were tasked with simplifying the following two rational algebraic expressions: $\frac{x^2-7x+23}{x^2-36} + \frac{6x-19}{36-x^2}$. The student in question did nearly all the work correctly but failed to factorize the last step – this work was done on the board during the lesson. As the student was busy simplifying the rational algebraic expressions, he/she came to an impasse and failed to reduce the final rational algebraic expression. In this excerpt, S represents a student working the problem on the board, while C_1, C_2, and C_3 are other students in the class. While S was simplifying the rational algebraic fraction on the board, the other students (C_1, C_2, and C_3) were comparing their own solutions to that of S.

Learning episode 2.

S: $\frac{x^2-7x+23}{x^2-36}+\frac{6x-19}{36-x^2}=\frac{x^2-7x+23}{x^2-36}+-\left(\frac{6x-19}{36-x^2}\right)$

[The student tries to use the notion that $(x-y)=-(y-x)$ having noticed that (x^2-36) and $(36-x^2)$ exhibit a similar trait – the student's work was written on the board].

S: $\frac{x^2-7x+23}{x^2-36}+\frac{6x-19}{36-x^2}=\frac{x^2-7x+23}{x^2-36}+-\left(\frac{6x-19}{36-x^2}\right)=\frac{x^2-7x+23}{(x+6)(x-6)}+\frac{-6x+19}{(x+6)(x-6)}=\frac{x^2-13x+42}{(x+6)(x-6)}$

S: I am stuck [*the student fails to recognize that the numerator of the last fraction can be factored as* $(x-6)(x-7)$ *and the fraction would consequently reduce to* $\frac{x-7}{x+6}$]

C_1: That is not what I got, teacher [C_1 *seeks to help S*].

C_2: We did the other side, teacher, we got +13 and − 42. *The classmate alludes to the fact that he multiplied* $\frac{x^2-7x+23}{x^2-36}$ *by negative one to get* $\frac{-\left(x^2+7x-23\right)}{(36-x^2)}=\frac{-x^2+7x-23}{(6+x)(6-x)}$

$\frac{-x^2+7x-23}{(6+x)(6-x)}+\frac{6x-19}{(6+x)(6-x)}=\frac{-x^2+13x-42}{(6+x)(6-x)}$.

C_3: Factor out the top [*the numerator*] and then you can cross [*cancel with denominator*] out.

S: $\frac{x^2-13x+42}{(x+6)(x-6)}=\frac{(x-7)(x-6)}{(x+6)(x-6)}=\frac{(x-7)}{(x+6)}$.

The classroom interactions in this episode were student–student interactions, where the teacher did not participate in the simplifications of the algebraic rational expression. When the student working at the board encountered an impasse, he/she said, "*I am stuck*" – thus calling for help. In this episode, the teacher did not comment or respond; instead, one of the students did so, stating, "That is not what I got, teacher." By not responding immediately to the students' classroom interactions, the teacher was using the affordance technique – where students were afforded the space and time to think through and solve the problem with the teachers' encouragement but with minimum help [49]. In this kind of approach, students are encouraged to use other students' thinking processes as resources to simplify rational algebraic expressions; for instance, student (C_2) suggested an alternative step of writing the expression $-x^2+13x-42$ instead of $x^2-13x+42$. Having noticed that the student at the board (S) needed help in simplifying the rational algebraic fraction, a fellow student (C_3) told him/her how to proceed: "factor out the top [the numerator] and then you can cross [cancel with denominator] out." Finally, with this assistance, student (S) was able to simplify the algebraic rational expression successfully.

During the debriefing interview, the teacher alluded to the fact that some students failed to carry out a procedure: "the main thing with today's lesson was about finding the common denominator … but I think other than that they got it pretty good." Interestingly, when asked about why he/she did not respond or comment on the students' interactions, the teacher said, "it is one way that I use to create an interactive and engaging learning environment among students during the lesson." In addition, the teacher was concerned that, as the problems would become more complex in subsequent lessons, his students were likely to struggle with identifying common denominators.

4.3 Experiencing uncertainty in explaining and sense-making

In the next episode, the focus is on how a student simplified two rational algebraic expressions:

$\frac{x+4}{x^2+15x+56}+\frac{6}{x^2+16x+63}$ on the board.

Using this example, we illustrate how a student found it difficult to verbalize his/her thinking processes and failed to justify his/her answers even though they were correct.

Learning episode 3.

S: I am going to factorize the two denominators $(x+7)(x+8)$ and $(x+9)(x+7)$

T: Ok, what are you going to do now? Ok, I understand that you factored the bottom [*denominator*]. So, what do you think comes next?

S: I will multiply $\frac{(x+4)(x+9)}{(x+7)(x+8)(x+9)} + \frac{6(x+9)}{(x+8)(x+7)(x+9)}$

T: Why did you put $(x+9)$ there? Why did you write $(x+9)$ on the left? [*student shrugging his shoulders to indicate he does not know why he/she wrote what he wrote*]

T: [*sensing the uncertainty*] … What is our goal now? What are we trying to accomplish? Before we add those on top [*numerators*] what do we need to have?

S: Common factor?

T: Close. We must have a common what?

S: Denominator.

In this episode, while the student was using the correct method, there came a point where he/she could not explain and/or verbalize his/her strategy for simplifying the problem. For example, when asked by the teacher why he/she had multiplied both fractions by the factor $(x+9)$, the student could not answer, but instead shrugged his/her shoulder as a way of saying "I do not know." When the teacher sensed this uncertainty, he responded by asking probing questions to guide the student towards achieving the goal of the question – "why did you put $(x+9)$ there?" The teacher wants to get to a point where the student says that he/she wants to find a common denominator between the two rational algebraic expressions.

During the interview with the teacher, he remarked that uncertainty was also expressed through the student's unwillingness to go to the board to work out the problems given to the class. The teacher said, "I really like to see the people that are struggling more at the board," and that he would like to hear more students saying, "I don't know what I am doing, but I am going up there" – the teacher acknowledges that the latter is a challenge which he/she hopes could be resolved by exposing students to more practice questions.

4.4 Expressing misconceptions and errors

In this section, we discuss the types of errors that manifested in the students' written work in learning episodes 1, 2, and 3 above. **Figure 1** below shows a student's conceptual error that was committed when simplifying the rational algebraic expression in learning episode 1.

Figure 1 reveals that although the student completed the question, he/she committed a conceptual error by making both denominators the same by observing that the first denominator had a "-2" and the second denominator had a "5" – the numbers "2" and "5" are the independent variables of the two denominators of the fractions to be simplified – the student ignored the "minus" sign for "2," opting instead to use a positive "2." In other words, the student seems to have ignored the letters and reduced the rational algebraic expressions into simple numerical fractions [23, 24]. The student, however, succeeded in simplifying his/her own numerical fractions from $-\frac{4}{20}$ to $-\frac{1}{5}$ [58].

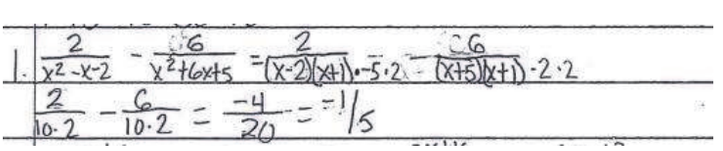


Figure 1.
An example of a student's conceptual error from learning episode 1.

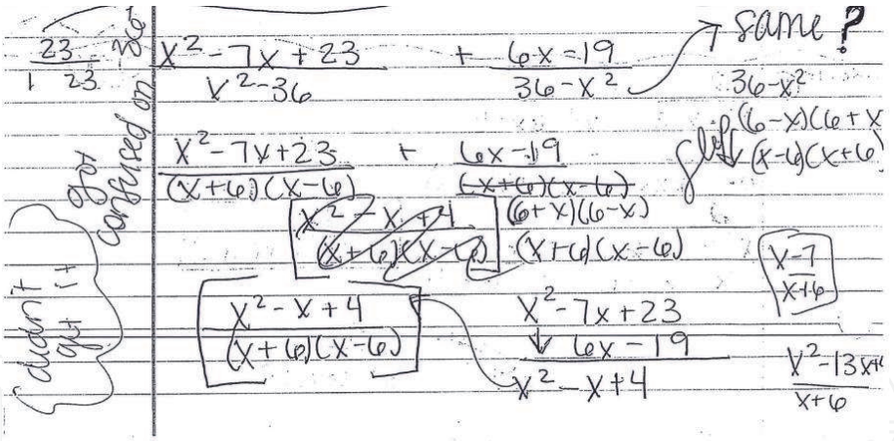


Figure 2.
An example of a student's misconception from learning episode 2.

In **Figure 2**, the student did not realize that $x^2 - 36 = -(36 - x^2)$ or more generally that $(a - b) = -(b - a)$, and thus had a misconception that the two denominators from both rational algebraic expressions were the same. This misconception resulted in the student not being able to simplify the resulting rational algebraic expression, because he/she could not factorize its numerator – in fact, the numerator cannot be factorized, hence the cancelation between the numerator and denominator cannot be done.

In **Figure 3**, the student committed an error by forgetting to follow through the multiplication of the numerator and denominator of the first and second fractions by $(x + 9)$ and $(x + 8)$ respectively – as a result the student had incorrect numerators and could not simplify the two rational algebraic expression.

4.5 Limitations

This study is based on a very small sample of 28 Grade 11 mathematics students in one school from a county. It is not the intention of the authors to draw upon any generalizations on the students' productive struggles on the simplifications of rational algebraic expressions from the small sample used in the study. Ii is our contention, that some of the observations made on the students' productive struggles are attributed to sample of 28 Grade 11 students who participated, their mathematical skills and abilities on the topic under discussion. As such, this study merely highlights some of the potential productive struggles that students are likely to encounter when solving problems on the simplifications of rational algebraic expressions. In a way the study can be used to give directions on the future research

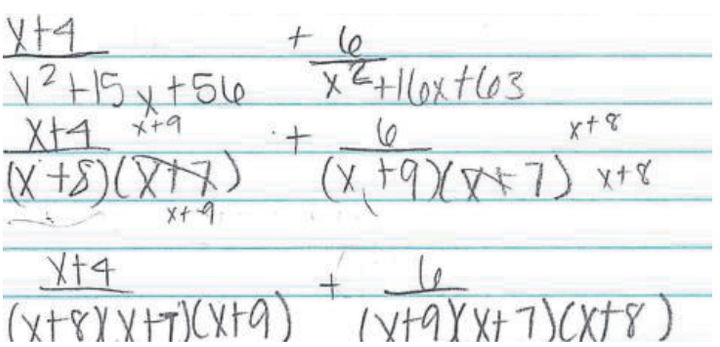


Figure 3.
An example of the student's error from learning episode 3.

work on students' productive struggles, mathematics teachers noticing and questioning techniques during lessons.

4.6 Instructional implications

Given the limitations of the study – the use of a single and small study, we are cautious about drawing generalized instructional implications that can be drawn from this study. Having said that, we believe that the study can highlight issues related to: struggle, support structures, and delay structure. Struggle – often struggle in mathematics is viewed as something negative, however, this study construes struggle as something essential for the student's intellectual growth, and a necessity which used during mathematics lessons. Support structures – during problem solving the role of support structures during learning in the form of feedback, questions, scaffolding questions, among others, is critical for students learning. Delay of structure - an important instructional implication here is that the role of delay of structure when students reach an impasse opens-up opportunity for learning – where there is no impasse, despite rigorous support structure provision, learning is not guaranteed [7].

5. Conclusion

In this chapter, our aim was to explore students' productive struggles on the simplifications of rational algebraic expressions, and of how teachers notice and respond to these productive struggles. Using the pre-determined productive struggle framework developed by Warshauer [42], we were able to identify and categorize the types of productive struggles that students experienced in the classroom and to look at the different ways in which the teacher addressed these struggles. Throughout the paper, it was not our intention to deal with constructs of noticing and questioning separately, but rather to discuss them within the types of productive struggles. In addition, the types of errors discussed in this paper are not exhaustive [23, 24], since they only pertain to the problems discussed in learning episodes 1, 2, and 3.

In conclusion, while this study contributes to the mathematics classroom discourses on the students' productive struggles on the simplification of rational algebraic expressions, using a bigger sample, further research is required on: the roles that mathematical teachers' noticing and questioning can play, and on how teachers respond to and effectively provide support structures to students' productive struggles during the teaching and learning of specific mathematics concepts.

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