Chapter

Electrostatically Driven MEMS Resonator: Pull-in Behavior and Non-linear Phenomena

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Abstract

This chapter deals with the investigation on stability and bifurcation analysis of a highly non-linear electrically driven micro-electro-mechanical resonator has been established. A non-linear model of this system will briefly be described considering both transverse and longitudinal displacement of the resonator. A short description to explore the need of incorporating higher-order correction of electrostatic pressure has been highlighted. The pull-in results and consequences of higher-order correction on the pull-in stability will be reported. In addition, consequences of air-gap, electrostatic forcing parameter, and effective damping on non-linear phenomena have been studied to highlight the possible undesirable catastrophic failure at the unstable critical points. Basins of attractions that postulate a unique response in multi-region state for a specific initial condition will also be studied. This chapter can enable a significant adaptation to identify the locus of instability in micro-cantilever-based resonator when subjected to AC voltage polarization with the understanding of theoretical ideas for controlling the systems and optimizing their operation.

Keywords: micro-beam, electrostatic actuation, pull-in analysis, higher-order-electrostatic distribution, non-linear phenomena, stability

1. Introduction and state-of-art research

The development of electrostatically actuated micro-system has been extensively carried out by the research community in order to develop low cost and high durability, and further improve the performance of sensors and actuators for wide applications. The use of electrostatic actuation offers a simplicity in design with low-cost fabrication, fast response, the ability to achieve rotary motion, and low power consumption. However, this actuation often leads into a complex non-linear phenomenon. As a result, structural movability becomes suspicious due to pull-in occurrence for any finite air-gap thickness. Furthermore, stable deflection range due to active electrostatic actuation is always being restricted since the movable substrate or electrode gets collapsed onto the stationary plate. Thus, computing pull-in voltage is inevitable and plays a decisive factor indicating a critical voltage under which stable operations and structural reliability may be asserted.

Generally, micro-electro mechanical system mathematically model by considering either a thin beam or a thin plate having cross-section in the order of microns and length in the order of hundreds of microns with an efficient electrostatic actuation. When the range of air-gap between stationary electrode and movable electrode is relatively large, typically of the order of 10^{-2}–10^{-1} or even higher for designing small-size of electro-statically actuated devices, parallel approximation theory of capacitor becomes ill-suited and in-valid. For a high air gap, it is unavoidable to develop a large deflection model for an electrostatically actuated micro-beam considering both higher order distribution of electrostatic pressure and mid-plane stretching exist. In the following section, a brief current research on modeling and dynamics of micro-electro-mechanical systems (MEMS) structures has been cited.

A number of researchers have attempted to develop numerous models over the times to improve the design characteristics and investigating related dynamics. Luo and Wang [1] investigated analytically and numerically the chaotic motion in the certain frequency band of a MEMS with capacitor non-linearity. Pamidighantam et al. [2] derived a closed-form expression for the pull-in voltage of fixed-fixed micro-beams and fixed-free micro-beams by considering axial stress, non-linear stiffening, charge re-distribution, and fringing fields. They carried out an extensive analysis of the non-linearities in a micro-mechanical clamped-lamped beam resonator. Abdel-Rahman et al. [3] presented a non-linear model of electrically actuated micro-beams with consideration of electrostatic forcing of the air-gap capacitor, restoring force of the micro-beam, and axial load applied to the micro-beam. The response of a resonant micro-beam subjected to an electrical actuation has been investigated by Younis and Nayfeh [4]. Xie et al. [5] performed the dynamic analysis of a micro-switch using invariant manifold method. They considered micro-switch as a clamped-clamped micro-beam subjected to a transverse electrostatic force. An analytical approach and resultant reduced-order model to investigate the dynamic behavior of electrically actuated micro-beam-based MEMS devices have been demonstrated by Younis et al. [6]. The natural frequency and responses of electrostatically actuated MEMS with time-varying capacitors have been investigated by Luo and Wang [7]. Authors have demonstrated that the numerically and analytically obtained predictions were in good agreement with the findings obtained experimentally. A simplified discrete spring-mass mechanical model has been considered for the dynamic analysis of MEMS device. In Teva et al. [8], a mathematical model for an electrically excited electromechanical system based on lateral resonating cantilever has been developed. The authors obtained static deflection and the frequency response of the oscillation amplitude for different voltage-polarization conditions. Kuang and Chen [9] and Najar et al. [10] studied the dynamic characteristics of non-linear electrostatic pull-in behavior for shaped actuators in micro-electro-mechanical systems (MEMS) using the differential quadrature method (DQM). Zhang and Meng [11] analyzed the resonant responses and non-linear dynamics of idealized electrostatically actuated micro-cantilever-based devices in micro-electromechanical systems (MEMS) by using the harmonic balance (HB) method. Rhoads et al. [12] proposed a micro-beam device, which couples the inherent benefits of a resonator with purely parametric excitation with the simple geometry of a micro-beam. Krylov and Seretensky [13] developed higher-order correction to the parallel capacitor approximation of the electrostatic pressure acting on micro-structures taking into account the influence of the curvature and slope of the beam on the electrostatic pressure. The higher-order approximation has validated through a comparison with analytical solutions for simple geometries as well as numerical results. Decuzzi et al. [14] investigated the dynamic response of a micro-cantilever beam used as a transducer in a biomechanical sensor. Here, Euler-Bernoulli beam theory was introduced to model the cantilever motion of the transducer. They also considered Reynolds equation of lubrication for the analysis of hydrodynamic interactions. A number of review papers [15–18] provided an overview of the fundamental research on modeling and dynamics of electrostatically actuated MEMS devices under working different conditions. Nayfeh et al. [19] studied that the characteristics of the pull-in phenomenon in the presence of AC loads differ from those under purely DC loads. Zhang et al. [20] furnished a survey and analysis of the electrostatic force of importance in MEMS, its physical model, scaling effect, stability, non-linearity, and reliability in details. Chao et al. [21] predicted the DC dynamics pull-in voltages of a clamped-clamped micro-beam based on a continuous model. They derived the equation of motion of the dynamics model by considering beam flexibility, inertia, residual stress, squeeze film, distributed electrostatic forces, and its electrical field fringing effects. Shao et al. [22] demonstrated the non-linear vibration behavior of a micro-mechanical clamped-lamped beam resonator under different driving conditions. They developed a non-linear model for the resonator by considering both mechanical and electrostatic non-linear effects, and the numerical simulation was verified by experimental findings. Moghimi et al. [23] investigated the non-linear oscillations of micro-beams actuated by suddenly applied electrostatic force, including the effects of electrostatic actuation, residual stress, mid-plane stretching, and fringing fields in modelling. Chatterjee and Pohit [24] introduced a non-linear model of an electrostatically actuated micro-cantilever beam considering the non-linearities of the system arising out of electric forces, geometry of the deflected beam and the inertial terms. Furthermore, one may use the review articles [15–18] as a source of information to the overall images about the electromechanical model of MEMS devices actuated by electrostatically and related dynamics. A detailed review of perturbation techniques to obtain the non-linear solution of such systems/structures can be found in [25]. A detailed description of the forced and parametrically excited systems has been highlighted in [26–28].

Several researchers have studied the pull-in behavior of micro-mechanical system under various driving conditions about its static beam positions. In addition, it has been learnt that researchers are still considering simple geometry ignoring the non-linear effect or components in their mathematical model to investigate the theoretical and experimental aspects of dynamic performance of MEMS devices. Moreover, in order to highlight a proper insight and a better understanding into the MEMS devices, the accurate simulations of mechanical behaviors with a faithful mathematical model is fairly inevitable that can exhibit a more realistic shape of the bending deflection of the micro-beam and the development of resulting electrostatic pressure distribution. Here, author has been attempted to investigate the dynamic stability and bifurcation analysis of electrostatically actuated MEMS cantilever along with pull-in behavior, both statically and dynamically accounting for the effect of mid-plane stretching and non-linear distribution of electrostatic pressure. The main focus here is to investigate the assessment of the system stability and subsequent bifurcations, which usually demonstrate the locus of instability. Pull-in voltage and its response under the non-linear effects have been computed. The method of multiple scales has been used to analyze the stability and bifurcation of the steady state solutions via frequency-response characteristics, time responses, and basin of attractions.

2. Pull-in

Though the major focus of this study is to explore the non-linear behavior of an electrostatically actuated MEMS device, it is of vital importance to study the pull-in behavior of electrostatically driven MEMS device as well. Both static and dynamic pull-in have been albeit briefly discussed in the coming sub-section followed by the system's non-linear behaviors. Before proceeding with an understanding of the MEMS dynamics, especially non-linear dynamics, it is prudent to briefly explore the range of operating applied voltage under which the system model and attendant analysis are considered to be sufficiently accurate for the predictive design.

2.1 Problem description

Differential equation of motion of a continuous micro-cantilever beam subjected to AC potential difference by stationary electrode has been shown in **Figures 1** and **2**, while the associated boundary conditions are being expressed in [25, 29]. However, the electrostatic force is considered to be uniform across the width, while transverse $v(x,t)$ and axial $u(x,t)$ displacement component holds a constraint equation known as in-extensibility condition

$$w'^2 + (1+u')^2 = 1.$$

$$\ddot{w} + 2\varsigma\dot{w} + w'''' + (d/l)^2\left[(w'')^3 + 4w'w''w''' + (w')^2 w''''\right] + (d/l)^2$$

$$\left[w'\int_0^{\bar{\xi}}\left\{\ddot{w}'w' + (\dot{w}')^2\right\}d\bar{\xi}\right] - (d/l)^2\left[w''\int_{\bar{\xi}}^{1}\int_0^{\bar{\eta}}\left\{\ddot{w}'w' + (\dot{w}')^2\right\}d\bar{\xi}\,d\bar{\eta}\right]$$

$$= \frac{6\varepsilon_0 l^4 V^2}{Eh^3 d^3}\left(1 + 2w + 3w^2 + HOD - \left\{\frac{(d/l)}{3}\left(2w'' + 2ww'' + w'^2 + HOD\right)\right\}\right). \quad (1)$$

2.2 Static analysis

It has been practically observed that the most common failure mode considered in the design of electrostatically driven MEMS devices is static pull-in condition beyond which it leads to desterilize the system for any further applied voltage. This failure is majorly occurred in the excess of electrostatic load in comparison to the static load-bearing capacity. As a result, system undergoes a negative stiffness in the

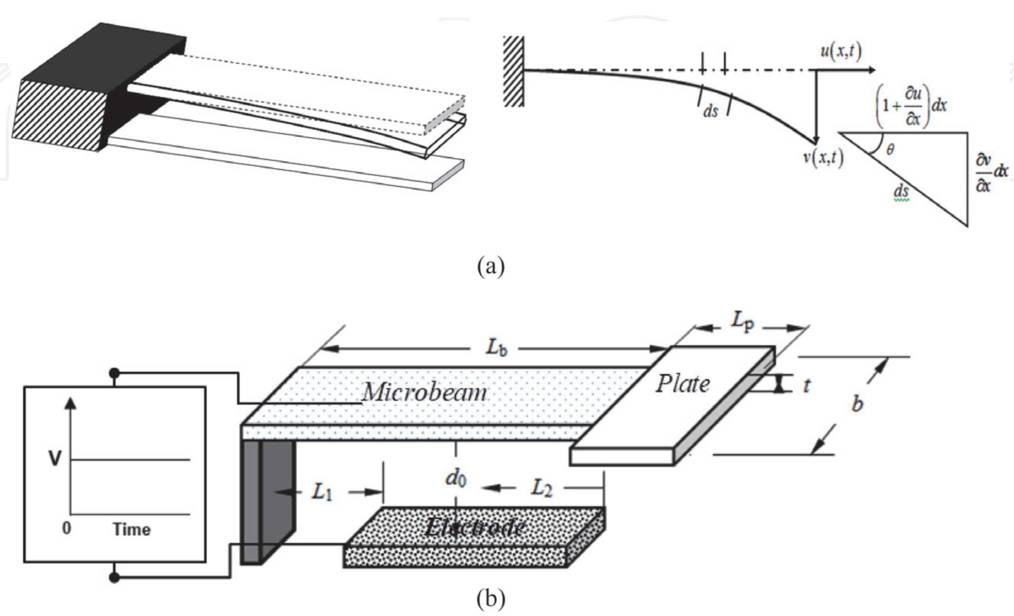

Figure 1.
(a) A pictorial diagram of micro-cantilever beam separated from a stationary electrode at a distance of d [25]. (b) Graphical representation of cantilever-based micro-structure coupled with rigid plate [29].

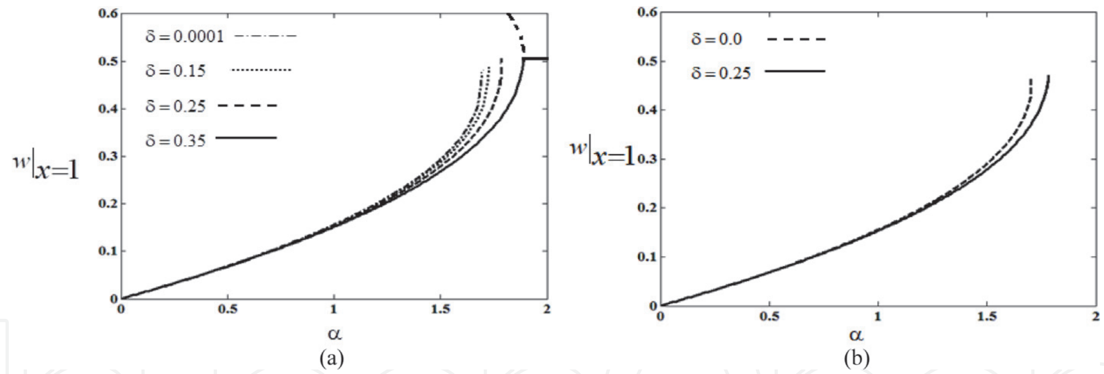

Figure 2.
(a) Variations of the non-dimensional tip deflection $w|_{x=1}$ *with variable* α *for various values of* δ*. (b): Effect of higher-order correction of electrostatic pressure on the non-dimensional tip deflection* $w|_{x=1}$ *with variable* α *[25].*

system's equation of motion. Pull-in instability as to calculate the pull-in voltage is inevitable to understand its limit to perform the desire task under a static voltage beyond which the movable electrode collapses onto the stationary electrode. As a result, the system is statically unstable as the electrostatic force overshadows the internal resistance as restoring force.

In most of the communication and power-circuits systems, electrostatically actuated micro-switch works only to alternate ON and OFF conditions by tuning a bias voltage across the pull-in back and forth. Therefore, it is advisable to have a micro-system, which can operate at low actuation voltage for performing the task mostly suited in power communications. The pull-in voltage can be obtained by demonstrating the static deflection of the tip of the micro-cantilever beam directly solving the boundary value problem by setting all time derivatives in Eq. (1) equal to zero. **Figure 1** shows the tip-deflection with the voltages ranging from zero to forcing level, where the pull-in instability takes place as explained details in [25, 29]. Recalling the fact that system leads to a pull-in condition when the system's net stiffness becomes negative. Here, the pull-in condition starts at α equal to 1.69 that indicates 66.83 compared to the pull-in voltage 66.78 obtained in Ref. [24] and 68.5 obtained in Ref. [23] considering the same design parameters.

However, the obtained static pull-in voltage may increase with increase in gap-length ratio ($\delta = {}^{d_0}\!/_{l}$). It can be noted that the pull-in voltage may occur at a lower when the effect of non-linear curvature is considered while calculating the electro-static pressures. Further, the higher-order correction factor may lead to lower value of pull-in voltage, which provides a most suitable for the design of a micro-system having significant gap-length.

2.3 Dynamic analysis

The loss of stability in dynamic responses occurs when the deformable electrode comes into contact with the fixed electrodes under an instantaneous electrostatic actuation that is lower than the static pull-in voltages known as dynamic pull-in phenomenon. Analysis of dynamic pull-in of an electrostatically actuated is complex due to its non-linear nature of electrostatic forces along with time integration of the momentum equations. Along with time-dependent terms, the transient part of the applied voltage is being neglected while calculating static pull-in voltage. However, calculating the actual pull-in voltage in dynamic condition when a bias alternative voltage source exists is obligatory and different as that of static pull-in voltage. Hence, there is a need to calculate the pull-in voltage called as dynamic pull-in voltage, considering the dynamics of the micro-beam instead of static state only.

In calculating the dynamic pull-in voltage, inertia and dissipative elements along with the components storing the strain energy for elastic deformation play an influential role in a dynamic condition. Here, the dynamic pull-in behavior has been depicted and investigated by directly simulating the Eq. (1) using the well-known R-K method. A qualitative phase-plane analysis has been illustrated to capture the global behavior of the response trajectories. Hence, the dynamic pull-in voltage has been illustrated in the phase portrait, i.e., in the plane of velocity ∝ displacement for the every applied voltage. The voltage turns out to be the critical voltage, i.e., pull-in voltage until the trajectories lead to an intersection of the orbits with the origin. The voltage at which monoclinic orbits are passing through the saddle node or degenerate singularity point is known as dynamic pull-in voltage. It is however advised to go through the article [29] for the detail explanation of obtaining the dynamic pull-in voltage.

Beyond the critical voltage, the system is found to be dynamically unstable. It has been found that the dynamic pull-in voltage is well below the static pull-in voltage nearly 80–95% of static pull-in voltage depending upon geometric configurations and physical properties **Figure 3.**

For an applied voltage less than critical one, the trajectories exhibit closed periodic orbits with steady response amplitude lower than the dynamic pull-in deflection. Hence, a periodic solution initiated always from an initial guess for a

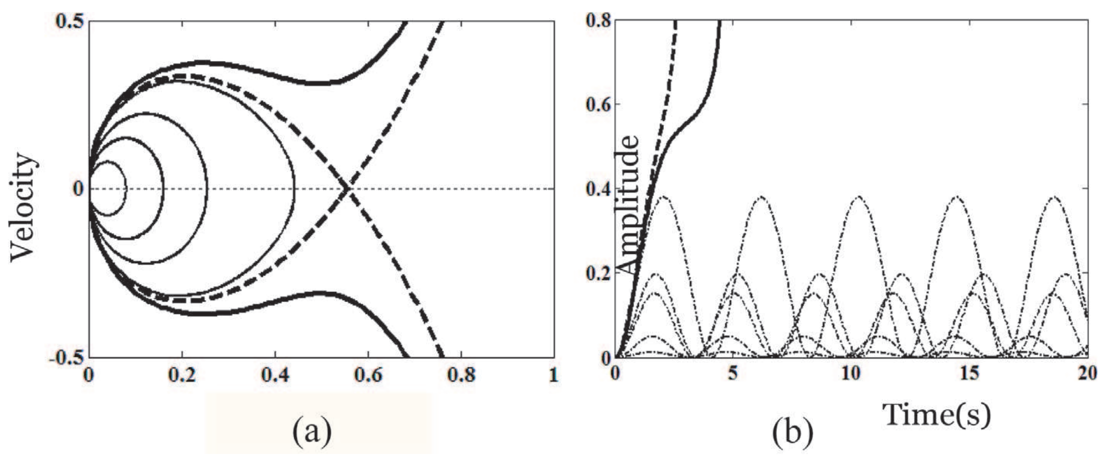

Figure 3.
(a) Dynamic pull-in voltage of undamped system for electrode length 45 μm [29]. (b) Time responses at pull-in voltage in [29].

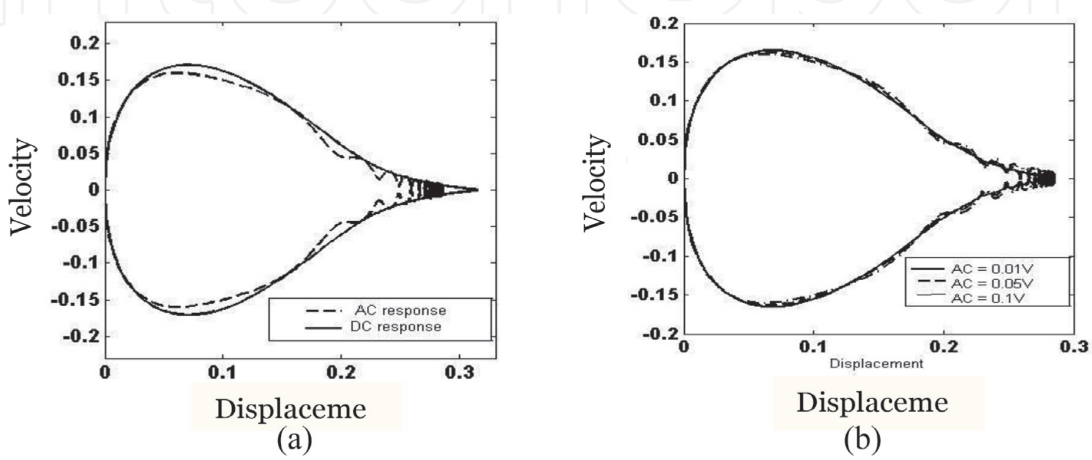

Figure 4.
(a) Dynamic pull-in voltage under DC and combined actuation. (b) Dynamic response at pull-in voltage under combined actuation at various voltages of AC actuation.

certain voltage leads to a closed trajectory as time approaches infinity. Hence, for every applied voltage V < VDPI (voltage in dynamic pull-in), the system shows an isolated stable closed trajectory. For any voltages greater than the dynamic pull-in voltage, the closed loop curves merge into a single curve, thus leading to an unstable domain. An effective DC contribution considering both DC and AC components has been depicted in the phase plane as shown in **Figure 4**, while the presence of AC component leads to a distortion in the solution trajectories. The presence of electrostatic actuation combining both DC and AC voltages system undergoes pseudo-dynamic pull-in, which is expected to all voltage-input combinations exceeding the predicted margin. However, this phenomenon generally holds true for small AC component, while inconsistency observes for a larger AC voltages.

3. Non-linear analysis

A comprehensive knowledge of non-linear dynamics in MEMS resonator is of great importance for the optimum design and operational stability. Thus, an understanding of the conditions to explore the non-linear phenomena arise; e.g., multiple-solutions; bifurcation can be implemented for further predicting chaotic responses in micro/nano-resonators. In this section, system's non-linear response at three distinct resonant conditions in the parametric space has been discussed. The non-linear phenomena have been reported here with the understanding of its instability via bifurcation. The characteristics form of the system non-linearity and electrostatic pressures and their effects on the system stability along with the effect of input voltages offer great flexibility toward designing the resonant sensors and filters. In order to obtain a detail understanding about the non-linear phenomena in MEMS systems, one may go through the articles [1, 5, 7, 11, 12, 19, 25, 30].

3.1 Problem description

Adopting the Galerkin's techniques and replacing $w = \Phi(x)X(\tau)$, where $\Phi(x)$ is admissible function obtained by satisfying the boundary conditions only and with similar procedures used in [25], the partial governing equation is then discretized into non-autonomous, time-dependent equation of motion with considering viscous damping effect. Expanding the non-linear electrostatic force developed due to applied voltage by Taylor series, one may obtain the following non-autonomous equation of motion.

$$\begin{aligned} &\ddot{X} + X + a\dot{X} + bX^3 + c\dot{X}^2X + d\ddot{X}X^2 \\ &= F\cos\Omega\tau + GX\cos\Omega\tau + KX^2\cos\Omega\tau \end{aligned} \tag{2}$$

Here, X is the non-dimensional displacement function or time modulation, while τ and Ω are the non-dimensional time and frequency, respectively. The expression for the co-efficient of non-autonomous $(a - d, F - G, K)$ is expressed in [25]. The equation of motion is further reduced to another form of micro-system neglecting effect of higher-order electrostatic distribution pressure, and mid-plane stretching effect.

$$\ddot{X} + X + a\dot{X} + bX^3 + c\dot{X}^2X + d\ddot{X}X^2 = F\cos\Omega\tau. \tag{3}$$

A huge number of researchers still consider either simple lumped-spring-mass model or Euler-Bernoulli beam theory with small air-gap assumption to carry out the theoretical and experimental investigation of dynamic performance of MEMS devices considering the mid-plane stretching effect.

$$\ddot{X} + X + a\dot{X} + bX^3 = F\cos\Omega\tau. \tag{4}$$

3.2 Bifurcation and stability

Equation of motions [Eqs. (2)–(4)] for various MEMS devices comprise linear and non-linear terms, direct forced, parametric term, and non-linear parametric terms due to non-linear electrostatic actuation. Since, the temporal equation of motion holds non-linear terms; it is difficult to find closed form solution. Hence, one may go for approximate solution by using the perturbation method. Here, method of multiple scales as explained in [25, 29–33] is used to obtain the set of algebraic equations turning into non-autonomous equations of motion for three resonance conditions, viz. primary resonance, parametric resonance condition, and third-order sub-harmonic conditions are being expressed under steady state conditions. The procedures used to derive the reduced order equation are similar to those explained in [25, 29–33]. Based on numerical values of the coefficients of the damping, forcing, and non-linear terms, they are one order less than the coefficients of the linear terms, which have a value of unity in this case and as result, in the following technique, co-efficient are expressed as $a = 2\varepsilon\varsigma$, $b = \varepsilon b$, $c = \varepsilon c$, $d = \varepsilon d$, $F = \varepsilon F$, $G = \varepsilon G$, and $K = \varepsilon K$ for sake of simplicity. By using method of multiple scales with the procedure as explained in [25, 30, 32, 33], substituting $T_n = \varepsilon^n\tau$, $n = 0, 1, 2, 3\cdots$ and displacement $X(\tau;\varepsilon) = X_0(T_0, T_1) + \varepsilon X_1(T_0, T_1) + O(\varepsilon^2)$ in Eq. (2) and equating the coefficients of like powers of ε, one may obtain the following expressions:

Order

$$\varepsilon^0 : {D_0}^2 X_0 + X_0 = 0, \tag{5}$$

Order

$$\begin{aligned}\varepsilon^1 &: {D_0}^2 X_1 + X_1 \\ &= -2D_0D_1X_0 - 2i\zeta X_0 - bX_0^3 - c\left(D_0^2X_0\right)X_0 - d(D_0X_0)^2X_0^2 + F\cos\Omega T_0 \\ &\quad + GX_0\cos\Omega T_0 + KX_0^2\cos\Omega T_0.\end{aligned} \tag{6}$$

General solutions of Eq. (5) can be written as

$$X_0 = A(T_1)\exp(iT_0) + \overline{A}(T_1)\exp(-iT_0). \tag{7}$$

After substituting Eq. (7) into Eq. (8), we have

$$\begin{aligned}&{D_0}^2X_1 + X_1 = -2iD_1A\exp(iT_0) - 2i\zeta A\exp(iT_0) - 3(b-c+d)A^2\overline{A}\exp(iT_0) - \\ &(b-c+d)A^3\exp(3iT_0) + \frac{F}{2}\exp(i\Omega T_0) + \frac{G}{2}A\exp i(\Omega+1)T_0 + \frac{G}{2}\overline{A}\exp i(\Omega-1)T_0 + \\ &\frac{K}{2}A^2\exp i(\Omega+2)T_0 + \frac{K}{2}A^2\exp i(\Omega-2)T_0 + \frac{K}{2}A\overline{A}\exp(i\Omega T_0) + cc.\end{aligned} \tag{8}$$

Any solution from the above equation may lead to an unbounded solution due to the existence of small divisor and secular terms in the equation. The terms associated e^{iT_0} or $\approx e^{i\Omega T_0}$, $\approx e^{i(\Omega-1)T_0}$, $\approx e^{i(\Omega-2)T_0}$ are known as small divisor and secular terms. These terms are required to be eliminated to obtain any bounded solution.

It may be observed that these terms exist when $\Omega \approx 1$, $\Omega \approx 3$, or $\Omega \approx 2$. In the following sub-sections, three resonance conditions, i.e., primary resonance, parametric resonance condition, and third-order sub-harmonic conditions have been briefly discussed. A details derivation and explanation is being carried out in [25].

3.2.1 Primary resonance condition

3.2.1.1 Reduced order model

Here, the resonance condition occurs when the frequency of applied voltage becomes equal to that of one of the natural frequencies, i.e., fundamental natural frequency. Following reduced equations are obtained as given below replacing $X = a(T_0)e^{i\phi(T_0)}$. A detail explanation about to obtain these reduced equations is being carried out in [25].

$$8a' = -8\zeta a + 4F\sin\phi + Ka^2\sin\phi, \tag{9}$$

$$8a\,\phi' = 8a\sigma - (3b - 3c + d)(\simeq\mu)a^3 + 4F\cos\phi + 3Ka^2\cos\phi. \tag{10}$$

Here, system only exhibits only non-trivial responses, i.e., $a \neq 0$ obtained from (Eqs. 9–10). Dynamic responses are being determined by solving the set of algebraic equations obtained by converting differential equations into set of algebraic equations under steady state conditions, i.e., $a' = 0$, and $\phi' = 0$. Here, stability of the steady state responses has been analyzed by investigating eigenvalues of the Jacobian matrix, which has been obtained by perturbing the algebraic equations with $a = a_o + a_1$ and $\gamma = \gamma_0 + \gamma_1$, where a_0, γ_0 are the singular points.

3.2.1.2 Results and discussions

Here, the condition at which the resonator has been excited with a frequency of the applied alternative voltage nearly equal to the fundamental frequency of the resonator is being discussed. In this vibrating state, the amplitude of vibration is always found to be a non-zero solution, while both stable and unstable non-trivial solutions are being observed. The study of bifurcation is being carried out on to see the losses the stability of system when a parameter passes through a critical value called a bifurcation point. The sudden change in amplitude undergoes catastrophic failure of the system. The graphical illustration of the vibration amplitude with varying the system control parameter has been constructed.

Figure 5a represents a typical frequency response curves for a specific air-gap $(\simeq\mu)$ between resonator and stationary. It is noteworthy that amplitude of vibration becomes increasing with increase in forcing frequency. The non-linear mode of operation possesses most likely hardening when the effective structural non-linearity becomes $\mu = +2.0$. In this condition, restoring forces due to geometric non-linearity overshadows the inertia effect of the device that leads to hardening spring effect. When the effective structural non-linearity becomes $\mu = -2.0$, vice versa effect is observed. For sweeping up the frequency moving toward E from point D, system undergoes a sudden jump down when the frequency reaches to its critical value regarded as saddle-node fixed bifurcation point. Similarly, a sudden upward jump in the amplitude from lower to higher amplitude undergoes for a sweeping down frequency. With further increase in frequency, the amplitude of vibratory motion decreases and follows the path DA. Hence, with decrease in frequency leads to the lower amplitude of responses from a higher value continuously.

With the experimental investigation, it has been noted that these jump phenomena may lead to mechanical crack across the width of the beam. The growth of the crack may further propagate with the experiences of subsequent jump up and down in response amplitude (**Figures 6** and 7).

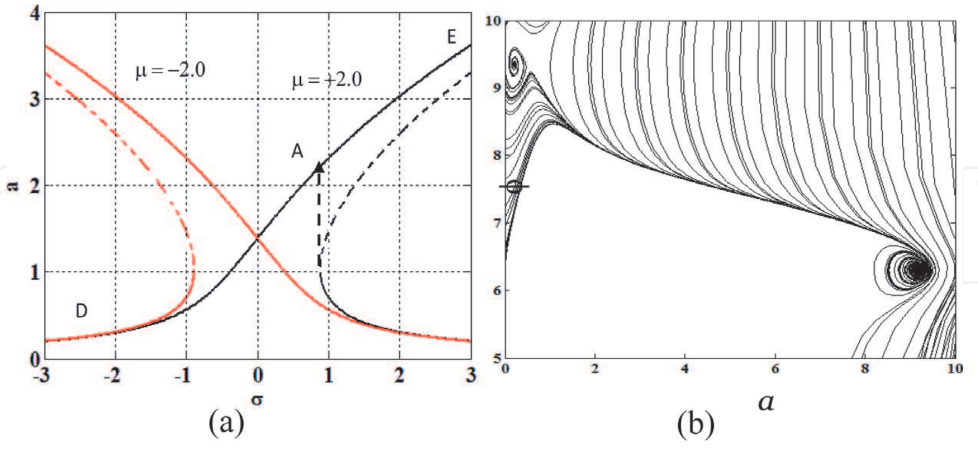

Figure 5.
(a) Frequency response curves for $\varsigma = 0.1, F = 1.2, K = 0.12$. *(b) Basins of attraction at bi-stable point [25].*

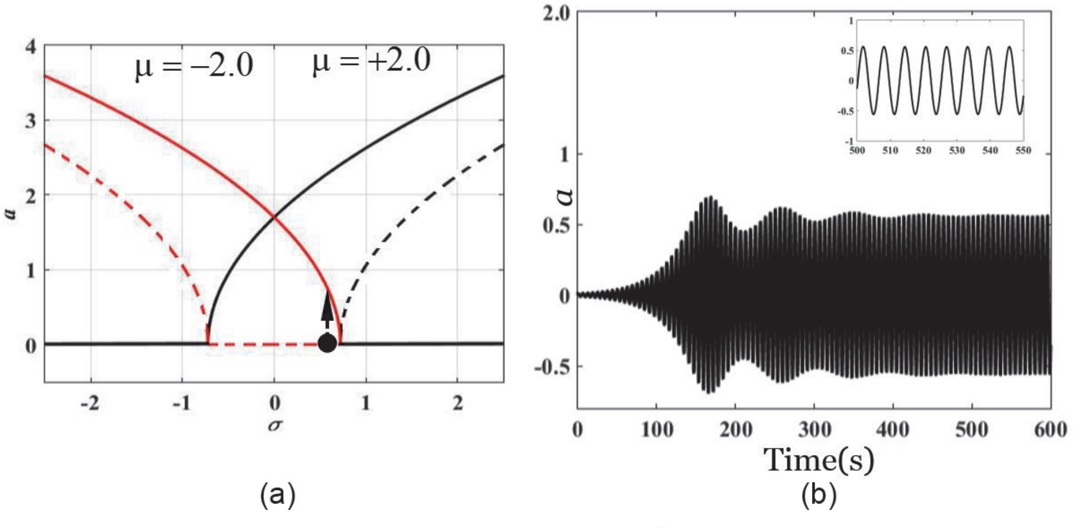

Figure 6.
(a) Frequency characteristics curves for $\varsigma = 0.1, H = 1.5$. *(b) Time histories at unstable point [25].*

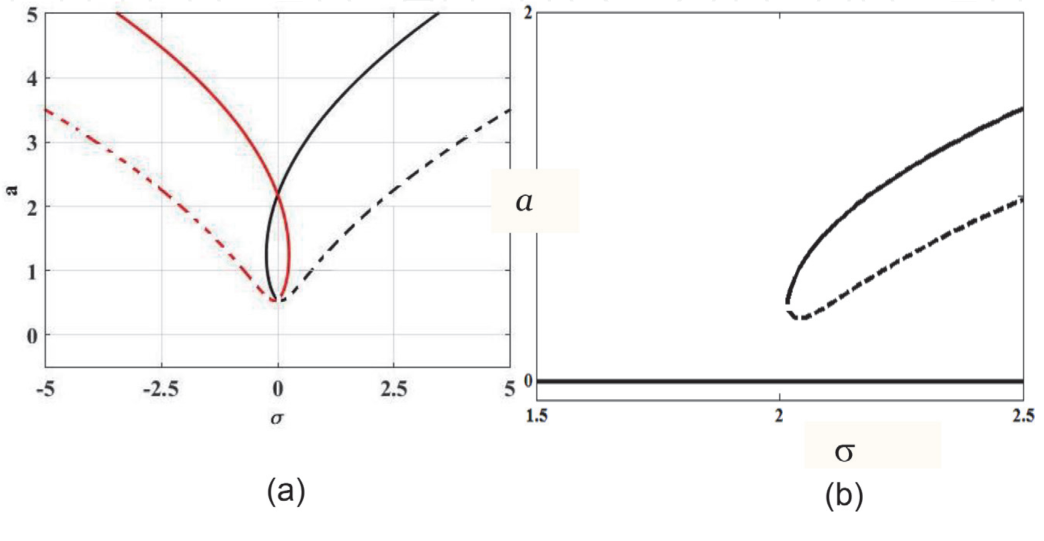

Figure 7.
(a) Frequency characteristics curve for $\varsigma = 0.1$, *and* $K = 1.5$. *(b) Frequency characteristics curve for* $d/l = 0.2, \varsigma = 0.2$ *and* $K = 1.0$.

It is observed that multiple solutions exist at some frequencies in the entire frequency range from 0.5 to 1.5. Being existence of multiple solutions, it is desirable to check whether all solutions are found to be stable or unstable or mixed solution. For a specific initial condition, trajectories have been drawn in the plane of amplitude and phase as time goes infinity. It is being observed that the system possesses the condition of bi-stability at some regions. Thus, in this region, wrong selection of initial conditions mostly results the wrong output response. It is thus keyed to opt out an appropriate condition for a specific solution that can prevail physically by the system.

3.3 Principal parametric resonance ($\Theta \approx 2$)

3.3.1 Reduced order model

Here, the resonance condition occurs when the frequency of applied voltage becomes twice the natural frequencies, i.e., fundamental natural frequency. Following reduced equations are obtained as given below:

$$4a' = -4\zeta a + Ha \sin\phi, \tag{11}$$

$$8a\,\phi' = 8a\sigma - (3b - 3c + d)(\simeq\mu)a^3 + 4H\cos\phi. \tag{12}$$

Here, system possesses both trivial $a = 0$ and non-trivial $a \neq 0$ responses determined by solving the reduced non-linear algebraic equations at steady state condition by using Newton's method, simultaneously. The stability of the steady state responses has also here obtained by replacing a with $a_o + a_1$, γ with $\gamma_0 + \gamma_1$, respectively, and then investigating the eigenvalues of the resulting Jacobian matrix (J).

3.3.1.1 Results and discussions

The electrostatically actuated micro- beam is vibrating with a frequency of the applied voltage nearly equal to the twice the fundamental frequency of the resonator. Unlike, primary resonance case, here, the system possesses both trivial and non-trivial solutions. Here, the vibration amplitude may vary from zero to non-zero value and vice-versa depending upon the state of vibration being considered. Depending upon the selected values of control parameters, the trivial and non-trivial solutions are noticed as stable and unstable for a specific frequency of the AC voltages. Sub-critical pitchfork bifurcation leads to sudden change in amplitude. This discontinuity in amplitude results catastrophic failure of the system.

Approximate solutions obtained by using the method of multiple scales have been compared with those found by numerically solving the temporal Eq. (2). Time response clearly shows that the trajectory initiated from the unstable trivial response finally moves to stable non-trivial fixed point response. Responses mostly obtained by solving the temporal equation of motion are being in good agreement with those findings by perturbation technique.

3.4 Sub-harmonic resonance case ($\Theta \approx 3$)

Similarly, one may have following reduced equations when the frequency of applied voltage is nearly equal to that of three times of natural frequency

$$8a' = -8\zeta a + Ka^2 \sin\phi, \tag{13}$$

$$8a\phi' = 8a\sigma - (3b - 3c + d)(\simeq\mu)a^3 + 3Ka^2\cos\phi. \qquad (14)$$

Similar to the previous resonance case, here also system possesses both trivial $a = 0$ and non-trivial $a \neq 0$ responses obtained by solving the equations obtained after setting a' and γ' equal to zero using similar Newton's method for different system parameters. Similar procedure one may follow to find out the stability of the steady state response of this case by investigating the nature of the equilibrium points.

This resonance takes place when the frequency of applied voltage is nearly equal to thrice the fundamental frequency of the resonator. Here, the amplitude of vibration may shift from non-zero to zero value depending on the initial operating point. In this resonance case, trivial solutions are found to be stable for any frequency and control parameters. The loss of stability of the system depends on the position of critical point and selection of control parameters, while the system can bring down to stable condition by simply choosing the frequency and other system parameters, appropriately. The bifurcation present here is known as addle-node bifurcation point. The jump length is found to be increased with increase in control parameters. Similarly, it has been observed that jump length will increase with increase in both forcing parameter and damping.

4. Conclusions

The investigation on stability and bifurcation analysis of a highly non-linear electrically driven MEMS resonator along with pull-in behavior has been established. A non-linear mathematical model has been briefly described accounting off mid-plane stretching and non-linear electrostatic pressure under both DC and AC actuation. A short description of perturbation method to study the steady state responses has been highlighted. The pull-in results and consequences of non-linear effects on dynamics responses have been reported. Non-linear phenomenon has been studied to highlight the possible undesirable catastrophic failure at the unstable critical points, i.e., bifurcation. Basins of attractions that postulate a unique response in multi-region state for a specific initial condition has been demonstrated. This chapter enables a significant understanding about the locus of instability in micro-cantilever-based resonator when subjected to DC and AC potentials. A theoretical understanding for controlling the systems and optimizing their operation is being reported here.

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