Chapter

On the Nonuniqueness of the Hamiltonian for Systems with One Degree of Freedom

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Abstract

The alternative Hamiltonians for systems with one degree of freedom are solved directly from the Hamilton's equations. These new Hamiltonians produce the same equation of motion with the standard one (called the Newtonian Hamiltonian). Furthermore, new Hamiltonians come with an extra-parameter, which can be used to recover the standard Hamiltonian.

Keywords: Hamiltonian, Lagrangian, nonuniqueness, variational principle, inverse problem of calculus of variations

1. Introduction

It was well known that the Lagrangian possesses the nonuniqueness property. It means that the constant can be added or multiplied into the Lagrangian: $L_N(\dot{x}, x) \to \alpha L_N(\dot{x}, x) + \beta$

Furthermore, the total derivative term can also be added to the Lagrangian without alternating the equation of motion: $L_N(\dot{x}, x) \to \alpha L_N(\dot{x}, x) + \beta + df/dt$,

where $= f(x, t)$. This fact can be seen immediately from the variational principle with the action functional:

$$S[x] = \int_0^T dt \left(\alpha L_N(\dot{x}, x) + \beta + \frac{df}{dt} \right) = \int_0^T dt (\alpha L_N(\dot{x}, x) + \beta) + f(T) - f(0) \tag{1}$$

Obviously, the last two terms contribute only at the boundary. Then the variation $x \to x + \delta x$ on the action and $\delta S[x] = 0$, with conditions $\delta x(0) = \delta x(T) = 0$, give us the same Euler-Lagrange equation:

$$\frac{\partial}{\partial x} L_N(\dot{x}, x) - \frac{d}{dt} \frac{\partial}{\partial \dot{x}} L_N(\dot{x}, x) = 0 \tag{2}$$

The standard Lagrangian takes the form

$$L_N(\dot{x}, x) = T(\dot{x}) - V(x) \tag{3}$$

where $T(\dot{x})$ is the kinetic energy and $V(x)$ is the potential energy of the system. For a system with one degree of freedom, the kinetic energy is $T(\dot{x}) = m\dot{x}^2/2$. The equation of motion associated with the Lagrangian Eq. (1) is

$$\ddot{x} = -(1/m)dV(x)/dx \overset{\text{def}}{=} Q \tag{4}$$

Recently, it has been found that actually there is an alternative form of the Lagrangian called the multiplicative form [1–3]: $L(\dot{x}, x) = F(\dot{x})G(x)$, where F and G are to be determined. Putting this new Lagrangian into the Euler–Lagrange Eq. (2), we obtained

$$L_\lambda(\dot{x}, x) = m\lambda^2 \left[e^{-\frac{E(x,\dot{x})}{m\lambda^2}} + \frac{\dot{x}}{\lambda^2} \int_0^{\dot{x}} e^{-\frac{E(x,\dot{q})}{m\lambda^2}} d\dot{q} \right] \tag{5}$$

where $E(x, \dot{x}) = m\dot{x}^2/2 + V(x)$ is the energy function and $m\lambda^2$ is in the energy unit. We find that under the limit λ which is very large $\lim_{\lambda\to\infty}(L_\lambda(\dot{x}, x) - m\lambda^2) = L_N(\dot{x}, x)$, we recover the standard Lagrangian. The derivation of Eq. (5) can be found in the Appendix. Interestingly, this new Lagrangian can be treated as a generating function producing an infinite hierarchy of the Lagrangian:

$$L_\lambda(\dot{x}, x) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{-1}{m\lambda^2} \right)^{j-1} L_j(\dot{x}, x) \tag{6}$$

where

$$L_j(\dot{x}, x) = \sum_{k=0}^{j} \left(\frac{j! T^{j-k} V^k}{(j-k)! k! (2j - (2k+1))} \right) \tag{7}$$

These new Lagrangians $L_j(\dot{x}, x)$, however, produce the same equation of motion. Equations (6) and (7) provide an alternative way to modify the Lagrangian Eq. (3).

The problem studied in [1–3] that is actually related to the inverse problem of calculus of variations in the one-dimensional case. The well-known result can be dated back to the work of Sonin [4] and Douglas [5].

Theorem (Sonin): For every function Q, there exists a solution (g, L) of the equation:

$$g(Q - \ddot{x}) = \frac{\partial}{\partial x} L(\dot{x}, x) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} L(\dot{x}, x). \text{where} g = \frac{\partial^2}{\partial \dot{x}^2} L(\dot{x}, x) \neq 0 \tag{8}$$

What we did in [1–3] is that we went further to show that actually Eq. (8) admits infinite solutions.

In the present chapter, we will construct the Hamiltonian hierarchy for the system with one degree of freedom. In Section 2, the multiplicative Hamiltonian will be solved directly from Hamilton's equations. In Section 3, the physical meaning of the parameter λ will be discussed. In Section 4, the redundancy of the Hamiltonians and Lagrangians will be explained. In the last section, a summary will be delivered.

2. The multiplicative Hamiltonian

To obtain the Hamiltonian, we may use the Legendre transformation:

$$H_N(p, x) = p\dot{x} - L_N(\dot{x}, x) \tag{9}$$

where $p = \partial L/\partial \dot{x} = m\dot{x}$ is the momentum variable. The standard form of the Hamiltonian is

$$H_N(p,x) = \frac{p^2}{2m} + V(x) \tag{10}$$

which is nothing but the total energy of the system. The action is then

$$S[p,x] = \int_0^T dt(p\dot{x} - H_N(p,x)) \tag{11}$$

With the variations $x \to x + \delta x$ and $p \to p + \delta p$, with conditions $\delta x(0) = \delta x(T) = 0$, the least action principle $\delta S[x] = 0$ gives us

$$\dot{x} = \frac{\partial}{\partial p} H_N(p,x), \dot{p} = -\frac{\partial}{\partial x} H_N(p,x) \tag{12}$$

which are known as a set of Hamilton's equations.

We now introduce a new Hamiltonian, called the multiplicative Hamiltonian, in a form

$$\mathrm{H}(p,x) = K(p)W(x), \tag{13}$$

where $K(p)$ and $W(x)$ are to be determined. Equations (12) and (3) give us a new equation:

$$0 = \frac{1}{m}\frac{\partial}{\partial x} H_N(p,x) + \dot{p}\frac{\partial^2}{\partial p^2} H_N(p,x) + \frac{p}{m}\frac{\partial^2}{\partial x \partial p} H_N(p,x) \tag{14}$$

Replacing H_N by H and inserting Eqs. (13) into (14), we obtain

$$0 = \frac{d^2K}{dp^2} + \frac{1}{m\dot{p}W}\frac{dW}{dx}\left(p\frac{dK}{dp} + K\right) \tag{15}$$

Now we define

$$A \stackrel{\text{def}}{=} \frac{1}{m\dot{p}W}\frac{dW}{dx} \tag{16}$$

where A is a constant to be determined. Equation (16) can be immediately solved and result in

$$W(x) = ae^{-mAV(x)} \tag{17}$$

where a is a constant of integration. Substituting Eq. (17) into Eq. (15), we find that the function $K(p)$ is in the form

$$K(p) = be^{-\frac{Ap^2}{2}} \tag{18}$$

where b is another constant. Then the multiplicative Hamiltonian Eq. (13) becomes

$$H(p,x) = ce^{-\frac{Ap^2}{2} - mAV(x)} \tag{19}$$

where $c = ab$. If we now choose $c = -m\lambda^2$ and $A = 1/m^2\lambda^2$, the Hamiltonian Eq. (19) becomes

$$H_\lambda(p,x) = -m\lambda^2 e^{-\frac{H_N(p,x)}{m\lambda^2}} \tag{20}$$

Inserting Eq. (20) into Eq. (14), we find that

$$\begin{gathered} -\frac{dV(x)}{dx}\left(-\frac{p^2}{m^2\lambda^2}+1\right) = \dot{p}\left(-\frac{p^2}{m^2\lambda^2}+1\right) \\ -\frac{dV(x)}{dx} = \dot{p} \end{gathered} \tag{21}$$

which is the equation of motion of the system. Then this new Hamiltonian Eq. (20) gives us the same equation of motion as Eq. (10).

For the case $m\lambda^2 \gg H_N(p,x)$, we find that the multiplicative Hamiltonian

$$H_\lambda(p,x) \approx -m\lambda^2 + H_N(p,x) \tag{22}$$

gives back the standard Hamiltonian. The constant $-m\lambda^2$ does not alter the equation of the motion of the system.

We find that the multiplicative Hamiltonian Eq. (20) can also be directly obtained from the Legendre transformation:

$$H_\lambda(p,x) = p_\lambda \dot{x} - L_\lambda(\dot{x},x) \tag{23}$$

where

$$p_\lambda = \frac{\partial}{\partial \dot{x}} L_\lambda(\dot{x},x) = \frac{1}{\lambda^2}\int_0^p e^{-\frac{\zeta^2}{2m^2\lambda^2}} \frac{d\zeta}{m} \tag{24}$$

Inserting Eqs. (24) and (5) into Eq. (23), we obtain

$$\begin{aligned} H_\lambda(p,x) = m\lambda^2 &\left[\left(\frac{1}{\lambda^2}\int_0^p e^{-\frac{\zeta^2}{2m^2\lambda^2}}\frac{d\zeta}{m}\right)\frac{p}{m}\right. \\ &\left. -m\lambda^2\left(e^{-\frac{p^2}{2m\lambda^2}} + \frac{p}{m^2\lambda^2}\int_0^p e^{-\frac{\zeta^2}{2m^2\lambda^2}}\frac{d\zeta}{m}\right)\right] e^{-\frac{V(x)}{m\lambda^2}} \\ &= -m\lambda^2 e^{-\frac{H_N(p,x)}{m\lambda^2}} \end{aligned} \tag{25}$$

which is identical to Eq. (20).

Furthermore, we can rewrite the multiplicative Hamiltonian Eq. (20) in terms of the series:

$$H_\lambda(p,x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{-1}{m\lambda^2}\right)^{j-1} H_j(p,x) \tag{26}$$

where $H_j(p,x) \equiv H_N^j = (p^2/2m + V(x))^j$. It is not difficult to see that $H_j(p,x)$ produces exactly the equation of motion Eq. (21).

From the structure of Eqs. (26) and (6), it must be a hierarchy of the Legendre transformation. To establish such hierarchy, we start to rewrite the momentum Eq. (24) in the form

$$p_\lambda = \frac{1}{\lambda^2}\int_0^p e^{-\frac{\zeta^2}{2m^2\lambda^2}}\frac{d\zeta}{m} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-1}{m\lambda^2}\right)^{j-1} p_j \tag{27}$$

where

$$p_j(p,x) = j!\left[p_{j-1}V(x) + \frac{p^{2j-1}}{(j-1)\left(2^{j-1}\right)(2j-1)m^{j-1}}\right], j \geq 1 \text{ and } p = m\dot{x} \tag{28}$$

Then the Legendre transformation Eq. (23) becomes

$$0 = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-1}{m\lambda^2}\right)^{j-1}\left[L_j(\dot{x},x) - p_j\dot{x} + H_j(p,x)\right] \tag{29}$$

Eq. (29) holds if

$$L_j(\dot{x},x) = p_j\dot{x} + H_j(p,x) \tag{30}$$

which are the Legendre transformations for each pair of the Hamiltonian $H_j(p,x)$ and Lagrangian $L_j(\dot{x},x)$ in the hierarchy.

Next, we consider the total derivative $dH_j(p,x) = d\left(p_j\dot{x}\right) - dL_j(\dot{x},x)$ resulting in

$$dx\left(\frac{\partial H_j}{\partial x} + \dot{p}\frac{\partial p_j}{\partial p}\right) + dp\left(\frac{\partial H_j}{\partial p} - \dot{x}\frac{\partial p_j}{\partial p}\right) = 0 \tag{31}$$

Eq. (31) holds if

$$\frac{\partial H_j}{\partial x} = -\dot{p}\frac{\partial p_j}{\partial p}, \frac{\partial H_j}{\partial p} = \dot{x}\frac{\partial p_j}{\partial p} \tag{32}$$

Eq. (32) can be considered as the modified Hamilton's equations for each $H_j(p,x)$ in the hierarchy. Obviously, for $j = 1$, we retrieve the standard Hamilton's Eq. (12), since $p_1 = p = m\dot{x}$.

From the structure of the multiplicative Hamiltonian Eq. (20), it seems to suggest that the exponential of the function, defined on phase space, is always a solution of the Eq. (14). Then we now introduce an ansatz form of the Hamiltonian as

$$H_{a,b}(p,x) = be^{aZ(p,x)} \tag{33}$$

where a and b are constants to be determined. Substituting Eq. (33) into Eq. (14), we obtain

$$0 = \frac{1}{m}\frac{\partial Z}{\partial x} + \dot{p}\frac{\partial^2 Z}{\partial p^2} + \frac{p}{m}\frac{\partial^2 Z}{\partial x \partial p} + a\left[\dot{p}\left(\frac{\partial Z}{\partial p}\right)^2 + \frac{p}{m}\frac{\partial Z}{\partial p}\frac{\partial Z}{\partial x}\right] \tag{34}$$

We find that if we take $H_N(p,x) = Z(p,x)$ to be the standard Hamiltonian, the first three terms in Eq. (34) give us back Eq. (14). Then the last bracket must vanish and gives us an extra-relation:

$$0 = \dot{p}\left(\frac{\partial H_N}{\partial p}\right)^2 + \frac{p}{m}\frac{\partial H_N}{\partial p}\frac{\partial H_N}{\partial x} \tag{35}$$

or

$$0 = \dot{p}\frac{\partial H_N}{\partial p} + \frac{p}{m}\frac{\partial H_N}{\partial x} \tag{36}$$

We immediately see that actually Eq. (36) is a consequence of the conservation of the energy of the system:

$$0 = \frac{dH_N}{dt} = \frac{\partial H_N}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial H_N}{\partial x}\frac{\partial x}{\partial t} = \dot{p}\frac{\partial H_N}{\partial p} + \frac{p}{m}\frac{\partial H_N}{\partial x} \tag{37}$$

Then what we have here is another equation that can be used to determine for the Hamiltonian subject to the equation of motion Eq. (21). To see this, we may start with the standard form of the Hamiltonian $H_N(p,x) = T(p) + V(x)$, where $T(p)$ is a function of the momentum and to be determined. Inserting the Hamiltonian into Eq. (36), we obtain

$$0 = \dot{p}\frac{dT}{dp} + \frac{p}{m}\frac{dV}{dx} \tag{38}$$

Using Eqs. (21) and (38), it can be rewritten in the form

$$0 = \dot{p}\left(\frac{dT}{dp} - \frac{p}{m}\right) \tag{39}$$

Since $\dot{p} \neq 0$, it means that the term inside the bracket must be zero and

$$\int dT = \int \frac{p}{m} dp \rightarrow T(p) = \frac{p^2}{2m} + C \tag{40}$$

where C is a constant which can be chosen to be zero. So we successfully solved the standard Hamiltonian.

Next, we put $Z(p,x) = K(p)W(x)$ which is in the multiplicative form Eq. (13) into Eq. (36), and we obtain

$$0 = W\left[\dot{p}\left(W\frac{dK}{dp}\right) + \frac{p}{m}\left(K\frac{dW}{dx}\right)\right] \tag{41}$$

or

$$\frac{m}{Kp}\frac{dK}{dp} = \frac{1}{W}\frac{dW}{dV} \tag{42}$$

We see that both sides of Eq. (42) are independent to each other. Then Eq. (42) holds if both sides equal to a constant β. We have now for the left-hand side

$$\frac{m}{Kp}\frac{\partial K}{\partial p}=\beta$$

$$\int\frac{dK}{K}=\beta\int\frac{p}{m}dp\rightarrow K(p)=C_1e^{\beta T(p)} \tag{43}$$

where C_1 is a constant to be determined. Next, we consider the right-hand side

$$\frac{1}{W}\frac{dW}{dV}=\beta$$

$$\int\frac{dW}{W}=\beta\int dV\rightarrow W(x)=C_2e^{\beta V(x)} \tag{44}$$

where C_2 is a constant to be determined. Then finally the function $Z(p,x)$ becomes

$$Z(p,x)=C_1C_2e^{\beta H_N(p,x)} \tag{45}$$

where $H_N(p,x)$ is the standard Hamiltonian. If we now choose $C_1C_2=-m\lambda^2$ and $\beta=-1/m\lambda^2$, the function $Z(p,x)$ is exactly the same with Eq. (20).

We see that with Eq. (36) the Hamiltonian can be easily determined. Here we come with the conclusion that in every function $Q'=-\frac{dV}{dx}$, there exist infinite Hamiltonians of equation

$$Q'-\dot{p}=\dot{p}\frac{\partial H}{\partial p}+\frac{p}{m}\frac{\partial H}{\partial x} \tag{46}$$

The existence of solutions of Eq. (46) implies that actually we can do an inverse problem of the Hamiltonian for the systems with one degree of freedom.

Remark: The perspective on nonuniqueness of Hamiltonian, as well as Lagrangian, here in the present work is quite different from those in Aubry-Mather theory [6, 7] (see also [8]). What they had been investigating is the modification of the Tonelli Lagrangian $L_\eta := L-\hat{\eta}$, where mechanical Lagrangian $L_N(\dot{x},x)=T(\dot{x})-V(x)$ is one of Tonelli Lagrangians. Here $\hat{\eta}=<\eta(x),\dot{x}>:TM\rightarrow\mathbb{R}$ and $\eta(x)$ is a closed 1-from on the manifold M. This means that $\int dtL_\eta$ and $\int dtL$ will have the same extremals and therefore the same Euler-Lagrange evolution, since $\delta\int dt\hat{\eta}=0$. Thus for a fixed L, the extreamise of the action will depend only on the de Rham cohomology class $c=[\eta]\in H^1(M,\mathbb{R})$. Then we have a family of modified Lagrangians, parameterized over $H^1(M,\mathbb{R})$. With the modified Tonelli Lagrangian L_η, one can easily find the associated Hamiltonian $H_\eta(x,p)=H(x,\eta(x)+p)$, where the momentum is altered: $p\rightarrow p+\eta(x)$. Then we also have a family of modified Hamiltonians, parameterized over $H^1(M,\mathbb{R})$. To make all this more transparent, we better go with a simplest example. Consider the modified Lagrangian $L_\epsilon := L_N+\epsilon\dot{x}$, where ϵ is a constant. We find that a new action differs from an old action by a constant depending on the endpoints, $\int_a^b dtL_\epsilon=\int_a^b dtL_N+\epsilon(x(b)-x(a))$, and they give exactly the same Euler-Lagrange equation (see also Eq. (1)). With this new Lagrangian L_ϵ, we can directly obtain the Hamiltonian $H_\epsilon(x,p)=H_N(x,p+\epsilon)$.

3. Harmonic oscillator

In this section, we give an explicit example, e.g., the harmonic example, and also give the physical interpretation of the parameter λ. The standard Hamiltonian for the harmonic oscillator reads

$$H(p,x) = \frac{p^2}{2m} + \frac{kx^2}{2} \tag{47}$$

Then the multiplicative Hamiltonian for the harmonic oscillator is

$$H_\lambda(p,x) = -m\lambda^2 e^{-\frac{1}{m\lambda^2}\left(\frac{p^2}{2m}+\frac{kx^2}{2}\right)} \tag{48}$$

Now we introduce $\eta = (x,p)$, and then we consider

$$\frac{d\eta}{dt_\lambda} = J\frac{\partial H_\lambda}{\partial \eta} \text{ where } \frac{\partial}{\partial \eta} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial p \end{pmatrix} \tag{49}$$

where t_λ is a time variable associated with the multiplicative Hamiltonian and J is the symplectic matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{50}$$

Inserting Eq. (48) into Eq. (49), we obtain

$$\frac{d\eta}{dt_\lambda} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-1}{m\lambda^2}\right)^{k-1} J\frac{\partial H^k}{\partial \eta} \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{d\eta}{dt_k} \tag{51}$$

where

$$\frac{d}{dt_k} = \frac{E^{k-1}}{(k-1)!\left(m\lambda^2\right)^{k-1}}\frac{d}{dt} \tag{52}$$

where $E = T + V$ is the energy function and t is the standard time variable associated with the Hamiltonian Eq. (47). Equation (51) suggests that the λ-flow is comprised of infinite different flows on the same trajectory on the phase space (see **Figure 1**).

This means that we can choose any Hamiltonian in the hierarchy to work with. The physics of the system remains the same but with a different time scale. Then we may say that the parameter λ plays a role of scaling in the Hamiltonian flow on the phase space. From Eq. (52), we see that as $m\lambda^2 \to \infty$, only the standard flow survives, and of course we retrieve the standard evolution $t_1 = t$ of the system on phase space.

Next we consider the standard Lagrangian of the harmonic oscillator

$$L_N(\dot{x},x) = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} \tag{53}$$

and the multiplicative Lagrangian is

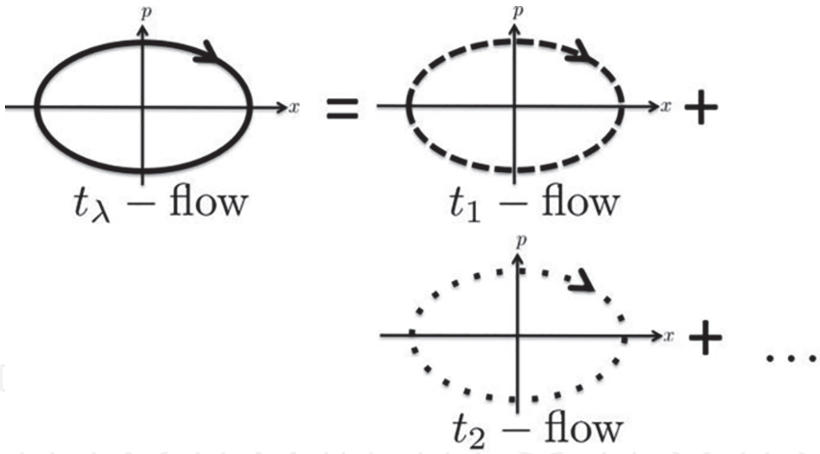

Figure 1.
Differential flows on the same trajectory on the phase space.

$$L_\lambda(\dot{x},x) = m\lambda^2\left[e^{-\frac{E(x,\dot{x})}{m\lambda^2}} + \frac{\dot{x}}{\lambda^2}\int_0^{\dot{x}} e^{-\frac{E(x,\dot{q})}{m\lambda^2}}d\dot{q}\right] \tag{54}$$

where $E(x,\dot{x}) = m\dot{x}^2/2 + kx^2/2$ is the energy function. We know that Lagrangian Eq. (54) can be rewritten in the form

$$L_\lambda(\dot{x},x) = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-1}{m\lambda^2}\right)L_j(\dot{x},x) \tag{55}$$

where

$$L_j(\dot{x},x) = \sum_{k=0}^{j}\left(\frac{j!\left(m\dot{x}^2/2\right)^{j-k}\left(kx^2/2\right)^k}{(j-k)!k!(2j-(2k+1))}\right) \tag{56}$$

The action of the system is given by

$$S[x] = \int_0^T dtL_\lambda(\dot{x},x) = \sum_{j=1}^{\infty}\int_0^T dt_jL_j(\dot{x},x) \tag{57}$$

where

$$dt_j = \frac{1}{j!}\left(\frac{1}{m\lambda^2}\right)^{k-1}dt \tag{58}$$

The variation $x \to x + \delta x$ with conditions $\delta x(0) = \delta x(T) = 0$ results in

$$\delta S[x] = \sum_{j=1}^{\infty}\int_0^T dt_j\left(\frac{\partial L_j}{\partial x} - \frac{d}{dt_j}\frac{\partial L_j}{\partial x_j}\right) \tag{59}$$

where $x_j = dx/dt_j$. Least action principle $\delta S[x] = 0$ gives infinite Euler-Lagrange equations

$$0 = \frac{\partial L_j}{\partial x} - \frac{d}{dt_j}\frac{\partial L_j}{\partial x_j}, j = 1, 2, 3 \tag{60}$$

which produce the equation of motions

$$\frac{d^2x}{dt_j^2} = -\frac{kx}{m} \tag{61}$$

associated with different time variables. Again in this case, we have the same structure of equation of motion for each Lagrangian in hierarchy but with a different time scale. From Eq. (58), we see that as $m\lambda^2 \to \infty$, only the standard flow survives, and of course we retrieve the standard evolution $t_1 = t$ of the system. Then the parameter λ also plays the role of scaling in the Lagrangian structure.

4. Redundancy

From previous sections, we see that there are many forms of the Hamiltonian that you can work with. One may start with the assumption that any new Hamiltonian is written as a function of the standard Hamiltonian H_N: $H = f(H_N)$. Inserting this new Hamiltonian into Hamilton's equations, we obtain

$$f'(E)\frac{\partial H}{\partial x} = -\frac{\partial p}{\partial \tau},\ f'(E)\frac{\partial H}{\partial p} = \frac{\partial x}{\partial \tau} \tag{62}$$

where $f'(E) = df(H_N)/H_N$ with fixing $H_N = E$ and $t = f'(H_N)\tau$ is the rescaling of time parameter. This result agrees with what we have in Section 3, rescaling the time evolution of the system. However, there are some major different features as follows. The first thing is that our new Hamiltonians contain a parameter λ, since the explicit forms of the Hamiltonian are obtained. With this parameter, it makes our rescaling much more interesting with the fact that the rescaling time variables depend on also the parameter (see Eq. (52)). Then it means that we know how to move from one scale to another scale and of course we know how to obtain the standard time evolution by playing with the limit of the parameter λ. Without explicit form of the new Hamiltonian, which contains a parameter, we cannot see this fine detail of family of rescaling time variables, since there is only a fixed parameter E. The second thing is that actually the new Hamiltonian Eq. (20), which is a function of the standard Hamiltonian, can be obtained from the Lagrangian Eq. (5) by means of Legendre transformation. What we have seen is that Lagrangian Eq. (5) is nontrivial and is not a function of the standard Lagrangian. Again this new Lagrangian contains a parameter λ, the same with the one in the new Hamiltonian. With this parameter, the Lagrangian hierarchy Eq. (7) is obtained. What we have here is a family of nontrivial Lagrangians to work with, producing the same equation of motion, as a consequence of nonuniqueness property. An importance thing is that there is no way you can guess the form of this family of Lagrangian without our mechanism in the appendix. This means that the Hamiltonian in the form $H = f(H_N)$ cannot deliver all these fine details. The explicit form of the Hamiltonian Eq. (20) allows us to study in more detail and is definitely richer than the standard one.

5. Summary

We show that actually there exist infinite Hamiltonian functions for the systems with one degree of freedom. We may conclude that there exists the reverse engineering of the calculus of variation on phase space (see Eq. (44)). Furthermore, the solution of Eq. (44) exists not only as one but infinite. Interesting fact here is that these new Hamiltonians come with the extra-parameter called λ. We give the interpretation that the term $m\lambda^2$ involves the time scaling of the system. This means that we can pick any Hamiltonian or Lagrangian to study the system, but the evolution will be in different scales.

In the case of many degrees of freedom, the problem turns out to be very difficult. Even in the case of two degrees of freedom, the problem is already hard to solve from scratch. We may start with an anzast form of the Lagrangian: $L(\dot{x}, \dot{y}, x, y) = F(\dot{x}, \dot{y})G(x, y)$. This difficulty can be seen from the fact that we have to solve a non-separable coupled equation. A mathematical trig or further assumptions might be needed for solving $F(\dot{x}, \dot{y})$ and $G(x, y)$. The investigation is now monitored.

Furthermore, promoting the Hamiltonian Eq. (20) to be a quantum operator in the context of Schrodinger's equation is also an interesting problem. This seems to suggest that an alternative form of the wave function for a considering system is possibly obtained. This can be seen as a result from that fact that with new Hamiltonian operator, we need to solve a different eigenvalue equation, and of course a new appropriate eigenstate is needed. From the Lagrangian point of view, extension to the quantum realm in the context of Feynman path integrals is quite natural to address. However, this problem is not easy to deal with since the Lagrangian multiplication is not in the quadratic form. Then a common procedure for deriving the propagator is no longer applicable. Further study is on our program of investigation.

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Notes

The content in this chapter is collected from a series of papers [1–3].

Appendix

In this section, we will demonstrate how to solve the multiplicative Lagrangian Eq. (5). We introduce here again the Lagrangian $L(\dot{x}, x) = F(\dot{x})G(x)$, where F and G are to be determined. Inserting the Lagrangian into the action and performing the variation $x \to x + \delta x$, with conditions $\delta x(0) = \delta x(T) = 0$, we obtain

$$\delta S[x] = \int_0^T dt \left(-\frac{d}{dt}\left(G\frac{dF}{d\dot{x}} \right) + F\frac{dG}{dx} \right) \delta x \tag{63}$$

The least action principle states that the system will follow the path which $\delta S = 0$ resulting in

$$-\frac{d}{dt}\left(G\frac{dF}{d\dot{x}}\right)+F\frac{dG}{dx}=0 \tag{64}$$

Eq. (64) can be rewritten in the form

$$\frac{d^2F}{d\dot{x}^2}-\frac{1}{\ddot{x}G}\frac{dG}{dx}\left(F-\dot{x}\frac{dF}{d\dot{x}}\right)=0 \tag{65}$$

Using equation of motion, we observe that the coefficient of the second term depends only x variable. Then we may set

$$\frac{1}{\ddot{x}G}\frac{dG}{dx}\overset{\text{def}}{=}A\rightarrow\frac{1}{G}\frac{dG}{dx}=-\frac{A}{m}\frac{dV}{dx} \tag{66}$$

We find that it is not difficult to see that the function G that satisfies Eq. (66) is

$$G(x)=\alpha_1 e^{-AV(x)/m} \tag{67}$$

where α_1 is a constant to be determined. Inserting Eq. (66) into Eq. (65), we obtain

$$\frac{d^2F}{d\dot{x}^2}-A\left(F-\dot{x}\frac{dF}{d\dot{x}}\right)=0 \tag{68}$$

and the solution F is given by

$$F(\dot{x})=\alpha_2\dot{x}-\alpha_3\left(e^{-A\dot{x}^2/2}+\dot{x}A\int_0^{\dot{x}}dve^{-Av^2/2}\right) \tag{69}$$

where α_2 and α_3 are constants. Then the multiplicative Lagrangian is

$$L(\dot{x},x)=\left[k_1\dot{x}-k_2\left(e^{-A\dot{x}^2/2}+\dot{x}A\int_0^{\dot{x}}dve^{-Av^2/2}\right)\right]e^{-AV(x)/m} \tag{70}$$

where $k_1=\alpha_1\alpha_2$ and $k_2=\alpha_1\alpha_3$ are new constants to be determined. We find that if we choose $k_1=0, A=1/\lambda^2$ which is a unit of inverse velocity squared and $k_2=-m\lambda^2$ which is in energy unit, Lagrangian Eq. (70) can be simplified to

$$\lim_{\lambda\rightarrow\infty}\left(L_\lambda(\dot{x},x)-m\lambda^2\right)=\frac{m\dot{x}^2}{2}-V(x)=L_N(\dot{x},x) \tag{71}$$

the standard Lagrangian at the limit λ approaching to infinity. Therefore, the Lagrangian Eq. (70) is now written in the form

$$L_\lambda(\dot{x},x)=m\lambda^2\left(e^{-\dot{x}^2/2\lambda^2}+\frac{\dot{x}}{\lambda^2}\int_0^{\dot{x}}dve^{-v^2/2\lambda^2}\right)e^{-V(x)/m\lambda^2} \tag{72}$$

which can be considered as the one-parameter extended class of the standard Lagrangian.

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