We are IntechOpen, the world's leading publisher of Open Access books
Built by scientists, for scientists

6,900
Open access books available

185,000
International authors and editors

200M
Downloads

Our authors are among the

154
Countries delivered to

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Chapter

Thermal Stability Criteria of a Generic Quantum Black Hole

Aloke Kumar Sinha

Abstract

Thermodynamics of black holes were studied by Hawking, Bekenstein et al., considering black holes as classical spacetimes possessing a singular region hidden behind an event horizon. In this chapter, in contrast, we treat black hole from the perspective of a generic theory of quantum gravity, using certain assumptions which are consistent with loop quantum gravity (LQG). Using these assumptions and basic tenets of equilibrium statistical mechanics, we have derived criteria for thermal stability of black holes in any spacetime dimension with arbitrary number of charges ('hairs'), irrespective of whether classical or quantum. The derivation of these thermal stability criteria makes no explicit use of classical spacetime geometry at all. The only assumption is that the mass of the black hole is a function of its horizon area and all the 'hairs' (i.e. charge, angular momentum, any other types of hairs). We get a series of inequalities between derivatives of the mass function with respect to the area and other 'hairs' as the thermal stability criteria. These criteria are then tested in detail against various types of black holes in various dimensions. This permits us to predict the region of the parameter space of a given black hole in which it may be stable under Hawking radiation.

Keywords: black hole thermodynamics, thermal stability, saddle-point approximation, quantum gravity, multicharged black hole
PAC numbers: 04.70.-s, 04.70.Dy

1. Introduction

Semiclassical analysis has made the claim that non-extremal, asymptotically flat black holes are thermally unstable due to decay under Hawking radiation. Their instability is allegedly due to negativity of their specific heat [1, 2], as deduced from semiclassical mnemonics based on the classical metric. These black holes become hotter and hotter as they lose mass. This is a complete thermal runaway process. Note, however, that semiclassical analysis depends explicitly on the classical black hole metric and, as such, is inherently a 'case-by-case' analysis. This limitation implies that general results about thermal stability of black holes under Hawking decay cannot be obtained from such an analysis. For some asymptotically flat general relativistic black holes, semiclassical analysis has yielded the understanding that their specific heat, defined semiclassically from their metric, is negative, and hence the black holes must be thermally unstable under Hawking decay. However, there is little to glean from this approach which holds in general.

This interesting fact has motivated the study of thermal stability of black holes, from a perspective that is inspired by a definite proposal for *quantum* spacetime (like loop quantum gravity (LQG) [3, 4]) rather than on semiclassical assumptions. In the vicinity of a black hole horizon, gravity is very strong. So, a nonperturbative quantum theory of gravity is required to describe black holes from a quantum perspective. LQG is one of the promising candidates having this feature. A consistent understanding of the issue of *quantum* black hole entropy has been obtained through LQG [5, 6], where not only has the Bekenstein-Hawking area law been retrieved for macroscopic (astrophysical) black holes, but a whole slew of corrections to it, due to quantum spacetime fluctuations that have been derived as well [7, 8], with the leading correction being logarithmic in area with the coefficient $-3/2$. LQG plays only a motivational role in our work. Many of the assumptions, actually are made independently of LQG, are justified on the ground that LQG might provide situations where these assumptions are valid.

The implications of this quantum perspective on the thermal stability of black holes from decay due to Hawking radiation have therefore been an important aspect of black hole thermodynamics beyond semiclassical analysis and also somewhat beyond the strict equilibrium configurations that isolated horizons represent. Classically a black hole in general relativity is characterized by its mass (M), charge (Q) and angular momentum (J). Intuitively, therefore, we expect that thermal behaviour of black holes will depend on all of these parameters. For a given classical metric characterizing a black hole, the mass can be derived explicitly to be a function of the charge and angular momentum. However, the quantum spacetime perspective frees us from having to use classical formulae for this functional dependence of the mass. Instead, the assumption is simply this: the mass is a function of the horizon *area*, along with the charge and angular momentum.

The simplest case of vanishing charge and angular momentum has been investigated longer than a decade ago [9–11]. The obtained condition for thermal stability exactly matches with the condition, derived from semiclassical analysis. That condition has been derived from positivity of specific heat. This exact matching happens as the black holes have neither rotation nor charge. We are going to establish in this chapter that even if a black hole has at least one of those, the conditions for thermal stability are more elaborate. This is already obvious when one considers charged black holes ([12]). Therefore the conditions start to differ from classical ones. This is due to the fact that black holes are treated quantum mechanically. The earlier work has been generalized, via the idea of *thermal* holography ([13, 14]) and the saddle-point approximation to evaluate the canonical partition function corresponding to the horizon, retaining Gaussian thermal fluctuations. The consequence is a general criterion of thermal stability as an inequality connecting area derivatives of the mass and the microcanonical entropy. This inequality is nontrivial when the microcanonical entropy has corrections (of a particular algebraic sign) beyond the area law, as is the case for the loop quantum gravity calculation of the microcanonical entropy [15]. The generalized stability criterion indeed 'predicts' the thermal *instability* of asymptotically flat Reissner-Nordstrom black holes contrasted with the thermal *stability* of anti-de Sitter Reissner-Nordstrom black holes (for a range of cosmological constants).

In this chapter, this approach is generalized to quantum black holes carrying *both* charge and angular momentum. The inclusion of rotation poses challenges in the LQG formulation [16–19] of isolated horizons. However, the general understanding of nonradiant rotating isolated horizons has parallels in these assays. We do not review this body of work, but realize that the thermal stability behaviour of rotating radiant black holes may be *qualitatively different* from that of the nonrotating ones.

We have calculated the partition function for rotating charged black hole. Thereafter we have got several inequalities as criteria for thermal stability of such black hole. We interpret these criteria and show how they are related to various thermodynamical quantities. We also show how the stability criteria for nonrotating and neutral black holes can be derived from these seven conditions in appropriate limit.

Beyond the standard general relativity theory corresponding to $(3+1)$ dimensional spacetime, higher-dimensional theories of gravity are currently under extensive scrutiny. Consequently black hole solutions are also being considered in those theories [20]. Such black holes have additional charges (hairs) beyond the traditional hairs—electric charge and angular momentum[1]. A black hole can be completely designated by its charge (Q), mass or area (A) and angular momentum (J). Quantum mechanically, a black hole can have many extra hairs, i.e. many charges, which contribute to its mass [22–28]. Higher-dimensional black holes too have many new charges which contribute to its mass [29]. We also consider all such hairs of black holes, together. We generalize the analysis of thermal stability of $(3+1)$ dimensional charged rotating black holes, for black holes with arbitrary number of hairs in any spacetime dimension. We find that the process of generalization is reasonably straightforward, except for calculational complications.

2. Quantum algebra and black hole spectrum

Like for all quantum systems, an operator algebra of fundamental observables is required to have a proper quantum description of black holes. Classically, generic black holes are represented by four parameters (M, Q, J, A), with three of them being independent. So, this naturally raises the question of the choice of the trivalent subset of classical variables that are promoted to quantum operators. Hence that three will be the fundamental observables of the quantum theory of black hole, and the remaining variable would correspond to a secondary observable.

Now, it is not possible to have a rotating, charged black hole without any mass, i.e. $M = 0$ with $Q, J \neq 0$. In fact an uncharged, nonrotating black hole does have mass, i.e. $M \neq 0$ with $Q = J = 0$. Therefore, charge and angular momentum are additional structures that can be imposed on a black hole. Hence they are preferably fundamental observables in a quantum theory of black hole.

We can choose any one between area (A) and mass (M) as the third fundamental observable. We choose area (A) as the third fundamental observable. So, mass (M) becomes the secondary observable, i.e. $M = M(A, Q, J)$. So, the algebraic approach of black hole quantization gives $\widehat{Q}, \widehat{J}, \widehat{A}$ as quantum operators of fundamental observables and $\widehat{M}\left(\widehat{H}_b\right)$ as quantum operator of secondary observable. All these correspond to the isolated horizon of a black hole.

It is physically obvious that both area and charge should be invariant under SO (3) rotations and area should also be invariant under U(1) gauge transformation. Now, angular momentum is the generator for rotation $(SO(3)$ Group), and charge is the generator of the U(1) global gauge group. These give

$$\left[\widehat{A}, \widehat{J}\right] = \left[\widehat{A},\ Q\right] = \left[\widehat{Q}, \widehat{J}\right] = 0 \tag{1}$$

[1] 'No hair' theorem [21–29] for black holes states that black hole cannot have any hair classically

Since $\widehat{M}\left(\widehat{H}_b\right)$ is a quantum operator of secondary observable $(M(A,J,Q))$, Eq. (1) can be extended as

$$\left[\widehat{A},\widehat{J}\right]=\left[\widehat{A},\widehat{Q}\right]=\left[\widehat{A},\widehat{M}\right]=\left[\widehat{Q},\widehat{J}\right]=\left[\widehat{M},\widehat{Q}\right]=\left[\widehat{J},\widehat{M}\right]=0 \tag{2}$$

Hence, $\widehat{A},\widehat{J},\widehat{Q}$ can have simultaneous eigenstate. This implies the fact that a black hole can have definite values of charge, area and angular momentum up to thermodynamical and quantum fluctuation. In fact these eigenvalues of $\widehat{A},\widehat{J},\widehat{Q}$ are precisely the values that are used in the classical metric of a black hole to express mass (M) as a function of them.

3. Thermal holography

Quantum black holes associated with an ambient thermal reservoir have been considered in the past [9–11, 13, 30]. In this approach key results of LQG like the discrete spectrum of the area operator [3, 4] have been used, and the main assumption was that the thermal equilibrium configuration is indeed an isolated horizon (IH) whose microcanonical entropy, including quantum spacetime fluctuations, has already been computed via LQG. The idea was to study the interplay between thermal and quantum fluctuations, and a criterion for thermal stability of such horizons has been obtained [11, 13, 14], using a 'thermal holographic' description involving a canonical ensemble and incorporating Gaussian thermal fluctuations. The generalization to horizons carrying charge has also been attempted, using a grand canonical ensemble, even though a somewhat ad hoc mass spectrum has been assumed [10].

Here, we attempt to generalize the thermal holography for nonrotating electrically charged quantum radiant horizons discussed in [12], to the situation when the horizon has both charge and angular momentum, without any ad hoc assumptions on the mass spectrum. Such a generalization completes the task set out in [9, 13] to include charge and angular momentum simultaneously in consideration of thermal stability of the horizon under Hawking radiation. A comparison with semiclassical thermal stability analysis of black holes [31] is made wherever possible.

3.1 Mass associated with horizon

Black holes at equilibrium are represented by isolated horizons, which are internal boundaries of spacetime. Hamiltonian evolution of this spacetime gives the first law associated with isolated horizon (b), assumed to be a null hypersurface with the properties of a 'one-way membrane' [16, 32]. The law is given as

$$\delta E_h^t=\frac{\kappa^t}{8\pi}\delta A_h+\Phi^t\delta Q_h+\Omega^t\delta J_h \tag{3}$$

where E_h^t is the energy function associated with the horizon; κ^t, Φ^t and Ω^t are, respectively, the surface gravity, electric potential and angular velocity of the horizon; and Q_h, A_h and J_h are, respectively, the charge, area and angular momentum of the horizon. The label 't' denotes the particular time evolution field (t^μ) associated with the spatial hypersurface chosen. κ^t, Φ^t and Ω^t are defined for this particular choice of time evolution vector field t^μ. The family of time evolution vector fields $[t^\mu]$ satisfying such first laws on the horizon are the permissible time evolution vector fields. These evolution vector fields also need to satisfy other boundary

conditions. Each of these time evolution vector fields associates an energy function with the horizon which is a function of area, charge and angular momentum, i.e. E_h^t is a function of A_h, Q_h and J_h.

The advantage of the isolated (and also the radiant or *dynamical*) horizon description is that one can associate with it a mass M_h^t, related to the ADM energy of the spacetime through the relation

$$E_{ADM}^t = M_h^t + H_{rad}^t \tag{4}$$

where H_{rad}^t is the Hamiltonian associated with spacetime between the horizon and asymptopia. It is the Hamiltonian of the covariant phase space, which is the space of various classes of solutions of the Einstein equations admitting internal boundaries. For stationary spacetimes the global time-like Killing field (ξ^μ) is the time evolution vector field. There is nothing between the internal boundary and asymptopia for stationary spacetimes; hence $H_{rad}^\xi = 0$. Actually, H_{rad}^ξ generates evolution along ξ^μ. So, for the stationary black hole, H_{rad}^ξ must vanish as a first-class constraint on the phase space [16, 33]. This gives $M_h^\xi = E_{ADM}^\xi$. This implies that for stationary black hole spacetimes, the ADM mass equals the energy of the black hole. Hence it is legitimate to identify E_h^ξ with the horizon mass M_h in the stationary case. The difference for an arbitrary nonstationary case is that $H_{rad}^t \neq 0$. Thus it can be called as the mass associated with the isolated horizon. So, an isolated horizon does not require stationarity, and therefore admits $H_{rad}^t \neq 0$, and hence admits a mass defined *locally* on the horizon, since the theory is topological and insensitive to small metric deformations.

Clearly, the horizon mass is *not* affected by boundary conditions at asymptopia. It is defined *locally* on the horizon without knowing the asymptotic structure at all. The asymptotic conditions only modify the energy associated with asymptopia and the bulk equation of motion (Einstein equations) [16, 34]. This Hamiltonian framework above is also applicable for both asymptotically flat and AdS spacetimes.

3.2 Quantum geometry

The boundary conditions of a classical spacetime with boundary determine the boundary degrees of freedom and their dynamics. For a quantum spacetime, fluctuations of the boundary degrees of freedom have a 'life' of their own [5, 6]. Therefore the Hilbert space of a quantum spacetime with boundary has the tensor product structure $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_v$, where the subscript b (v) denotes the boundary (bulk) component.

Thus, a generic quantum state $(|\Psi\rangle)$ can be expanded as

$$|\Psi\rangle = \sum_{b,v} C_{b,v} |\chi_b\rangle \otimes |\psi_v\rangle \tag{5}$$

where $|\chi_b\rangle$ is the boundary part of the full quantum state and $|\psi_v\rangle$ denotes the bulk component of the full quantum state.

The total Hamiltonian operator $(\widehat{H})$ acting on the generic state $(|\Psi\rangle)$ is given as

$$\widehat{H}|\Psi\rangle = \left(\widehat{H_b} \otimes I_v + I_b \otimes \widehat{H_v}\right)|\Psi\rangle \tag{6}$$

where, respectively, $I_b(I_v)$ is the identity operator on $\mathcal{H}_b(\mathcal{H}_v)$ and $\widehat{H_b}\left(\widehat{H_v}\right)$ is the Hamiltonian operator on $\mathcal{H}_b(\mathcal{H}_v)$.

In the presence of electric charge and rotation, $|\psi_v\rangle$ will be the composite bulk state. Hence, these bulk states are annihilated by the *full* bulk Hamiltonian, i.e.

$$\widehat{H_v}|\psi_v\rangle = 0 \tag{7}$$

This is the quantum version of the classical Hamiltonian constraint [4].

The charge operator ($\widehat{Q}$) for a black hole is defined as

$$\widehat{Q}|\Psi\rangle = \left(\widehat{Q}_b \otimes \widehat{I}_v + \widehat{I}_b \otimes \widehat{Q}_v\right)|\Psi\rangle \tag{8}$$

where, respectively, $\widehat{Q}_b$ and $\widehat{Q}_v$ are corresponding charge operators for the boundary states ($|\chi_b\rangle$) and the bulk states ($|\psi_v\rangle$).

Classically, the charge of a black hole is defined on the horizon, i.e. the internal boundary of the spacetime (e.g. one can see how charge can be properly defined for spacetimes admitting internal boundaries in Einstein-Maxwell or Einstein-Yang-Mills theories in [32]). There is *no* charge associated with the bulk black hole spacetime, i.e. $Q_v \approx 0$, which is basically the Gauss law constraint for electrodynamics. Hence, its quantum version is of the form:

$$\widehat{Q}_v|\psi_v\rangle = 0 \tag{9}$$

Like the charge operator, angular momentum operator ($\widehat{J}$) of a black hole can be defined as

$$\widehat{J}|\Psi\rangle = \left(\widehat{J}_b \otimes \widehat{I}_v + \widehat{I}_b \otimes \widehat{J}_v\right)|\Psi\rangle \tag{10}$$

where, respectively, $\widehat{J}_b$ and $\widehat{J}_v$ are corresponding angular momentum operators for the boundary states ($|\chi_b\rangle$) and the bulk states ($|\psi_v\rangle$).

A generic quantum bulk Hilbert space is invariant under local spacetime rotations, as a part of local Lorentz invariance. Angular momentum is the generator of spacetime rotation. Therefore it implies that bulk states are annihilated by angular momentum operator, i.e.

$$\widehat{J}_v|\psi_v\rangle = 0 \tag{11}$$

Hence Eqs. (7), (9) and (11) together give

$$\left[\widehat{H_v} - \beta\Phi\widehat{Q_v} - \beta\Omega\widehat{J_v}\right]|\psi_v\rangle = 0 \tag{12}$$

where β, Φ and Ω can be any function. But we will see that those will correspond to inverse temperature, electric potential and angular velocity, respectively, in afterwards.

3.3 Grand canonical partition function

We now consider a grand canonical ensemble of quantum spacetimes with horizons as boundaries, in contact with a heat bath, at some (inverse) temperature β. We will assume that this grand canonical ensemble of massive rotating charged black holes can exchange energy, angular momentum and charge with the heat bath. Therefore the grand canonical partition function is then given as

$$Z_G = Tr\left(exp\left(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J}\right)\right) \tag{13}$$

over all states. $\widehat{Q}$ is the charge operator for the black hole and Φ is the corresponding electrostatic potential. Similarly $\widehat{J}$ is the angular momentum operator for the black hole, and Ω is the corresponding angular velocity.

The above definition, together with Eqs. (5), (6), (8), (10) and (12), yields

$$\begin{aligned} Z_G &= \sum_{b,v}|C_{b,v}|^2\left\langle\psi_v|\otimes\left\langle\chi_b|exp\left(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J}\right)|\chi_b\right\rangle\otimes|\psi_v\right\rangle \\ &= \sum_{b,v}|C_{b,v}|^2\left\langle\psi_v|exp\left(-\beta\widehat{H}_v + \beta\Phi\widehat{Q}_v + \beta\Omega\widehat{J}_v\right)|\psi_v\right\rangle\left\langle\chi_b|exp\left(-\beta\widehat{H}_b + \beta\Phi\widehat{Q}_b + \beta\Omega\widehat{J}_b\right)|\chi_b\right\rangle \\ &= \sum_{b,v}|C_{b,v}|^2\langle\psi_v|\psi_v\rangle\left\langle\chi_b|exp\left(-\beta\widehat{H}_b + \beta\Phi\widehat{Q}_b + \beta\Omega\widehat{J}_b\right)|\chi_b\right\rangle \\ &= \sum_b\left(\sum_v|C_{b,v}|^2\langle\psi_v|\psi_v\rangle\right)\langle\chi_b|exp(-\beta\widehat{H}_b + \beta\Phi\widehat{Q}_b + \beta\Omega\widehat{J}_b|\chi_b\rangle \\ &= \sum_b|C_b|^2\left\langle\chi_b|exp\left(-\beta\widehat{H}_b + \beta\Phi\widehat{Q}_b + \beta\Omega\widehat{J}_b\right)|\chi_b\right\rangle \end{aligned} \tag{14}$$

assuming that the boundary states can be normalized through the squared norm $\sum_v|c_{vb}|^2\langle\psi_v|\psi_v\rangle = |C_b|^2$. This is analogous to the canonical ensemble scenario described in [13].

The partition function thus turns out to be completely determined by the boundary states (Z_{Gb}), i.e.

$$Z_G = Z_{Gb} = Tr_b\exp\left(-\beta\widehat{H_b} + \beta\Phi\widehat{Q_b} + \beta\Omega\widehat{J_b}\right) \tag{15}$$

In LQG, quantum black holes are represented by spin network, collection of graphs with links and vertices. For black holes with large area, the major contribution to the entropy comes from the lowermost spins. Hence, only spin 1/2 contribution for *all punctures* is taken into account which yields $A \sim N$ for a total of $N, N \gg 1$ punctures on the horizon. This leads to the equispaced area spectrum as an *approximation*. Of course the higher spins contribute, but their contribution is exponentially suppressed.

So, spectrum of the boundary Hamiltonian is a function of the discrete area spectrum. But the complete spectrum of the boundary Hamiltonian operator is still unknown in LQG. So, we will *assume* that the spectrum of the boundary Hamiltonian operator is also a function of the discrete charge spectrum and the discrete angular momentum spectrum associated with the horizon, respectively.[2] Quantum mechanically, the total charge of a black hole has to be proportional to some fundamental charge, i.e. the black hole is made of such charge particle. Hence the charge spectrum is taken to be equispaced due to quantization [17–19, 35, 36]. In fact angular momentum spectrum can also be considered as equispaced in the macroscopic spectrum limit of the black hole [37], in which we are ultimately interested.

It has already been shown in Subsection (II) that area, charge and angular momentum operators of a black hole commute among them. This implies that they

[2] Actually this second assumption follows from the discussion in Subsection (III.A) [16, 32] for spacetimes admitting weakly isolated horizons where there exists a mass function determined by the area and charge associated with the horizon. This is an extension of that assumption to the quantum domain.

are simultaneously diagonalizable. Therefore working in a basis in which area, charge and angular momentum operators are simultaneously diagonal, the partition function (15) can be written as

$$Z_G = \sum_{k,l,m} g(k,l,m) \quad \exp(-\beta(E(A_k, Q_l, J_m) - \Phi Q_l - \Omega J_m)) \tag{16}$$

where $g(k,l,m)$ is the degeneracy corresponding to the area eigenvalue A_k, the charge eigenvalue Q_l and the angular momentum eigenvalue J_m. k,l,m are the quantum numbers corresponding to eigenvalues of area, charge and angular momentum, respectively. In the macroscopic spectra limit of quantum isolated horizons, i.e. regime of the large area, charge and angular momentum eigenvalues $(k \gg 1, l \gg 1, m \gg 1)$, application of the Poisson resummation formula (9) gives

$$Z_G = \int dx\, dy\, dz\, g(A(x), Q(y), J(z)) \exp(-\beta(E(A(x), Q(y), J(z)) - \Phi Q(y) - \Omega J(z))) \tag{17}$$

where x, y, z are, respectively, the continuum limit of k,l,m, respectively.

Now, A, Q and J are, respectively, functions of x, y and z alone. Therefore we have

$$dx = \frac{dA}{A_x}, \quad dy = \frac{dQ}{Q_y}, \quad dz = \frac{dJ}{J_z}$$

where $A_x \equiv \frac{dA}{dx}$ and so on.

So, the partition function, in terms of area, charge and angular momentum as free variables, can be written as follows:

$$Z_G = \int dA \quad dQ \quad dJ \quad \exp[S(A) - \beta(E(A,Q,J) - \Phi Q - \Omega J)], \tag{18}$$

where, following [38], the *microcanonical* entropy of the horizon is defined by $\exp S(A) \equiv \frac{g(A(x),Q(y),J(z))}{\frac{dA\,dQ\,dJ}{dx\,dy\,dz}}$ and is a function of horizon area (A) alone, as has been established within LQG [5, 6, 15] .

4. Stability against Gaussian fluctuations

4.1 Saddle-point approximation

The equilibrium configuration of a black hole is given by the saddle point $(\overline{A}, \overline{Q}, \overline{J})$ in the three-dimensional space of integration over area, charge and angular momentum. The idea now is to examine the grand canonical partition function for fluctuations $a = (A - \overline{A}), q = (Q - \overline{Q}), j = (J - \overline{J})$ around the saddle point, in order to determine the stability of the equilibrium isolated horizon under Hawking radiation. We restrict our attention to Gaussian fluctuations, as per common practice in equilibrium statistical mechanics, with the motivation towards extremizing the free energy for the most probable configuration. Taylor expanding Eq. (18) about the saddle point yields

$$Z_G = \exp\left[S(\overline{A}) - \beta M(\overline{A}, \overline{Q}, \overline{J}) + \beta\Phi\overline{Q} + \beta\Omega\overline{J}\right]$$
$$\times \int da\, dq\, dj\, \exp\{-\frac{\beta}{2}[\left(M_{AA} - \frac{S_{AA}}{\beta}\right)a^2 + (M_{QQ})q^2 + (2M_{AQ})aq \qquad (19)$$
$$+(M_{JJ})j^2 + (2M_{AJ})aj + (2M_{QJ})qj]\},$$

where $M(\overline{A}, \overline{Q}, \overline{J})$ is the mass of equilibrium isolated horizon. Here $M_{AQ} \equiv \frac{\partial^2 M}{\partial A \partial Q}\Big|_{(\overline{A},\overline{Q},\overline{J})}$, etc.

We assume, just like in LQG, observables used here are self-adjoint operators over the boundary Hilbert space, and hence their eigenvalues are real [3]. It suffices therefore to restrict integrations over the spectra of these operators to the real axes.

Now, in the saddle-point approximation, the coefficients of terms linear in a, q, j vanish by definition of the saddle point. These imply that

$$\beta = \frac{S_A}{M_A}, \Phi = M_Q, \Omega = M_J \qquad (20)$$

Of course these derivatives are calculated at the saddle point.

4.2 Quantum correction of black hole entropy

Note that in the stability criteria derived in the last section, first- and second-order derivatives of the microcanonical entropy of the horizon at equilibrium play a crucial role, in making some of the criteria nontrivial. Thus, corrections to the microcanonical entropy beyond the Bekenstein-Hawking area law, arising due to quantum spacetime fluctuations, might play a role of some significance. It has been shown that [15] the microcanonical entropy for *macroscopic* isolated horizons has the form:

$$S = S_{BH} - \frac{3}{2}\log S_{BH} + O(S_{BH}^{-1}) \qquad (21)$$

$$S_{BH} = \frac{A}{4A_P}, A_P \equiv \text{Planck area}, A \equiv \text{black hole area} \qquad (22)$$

In Ref. [15] the above formula was derived for nonrotating, uncharged black holes in (3 + 1) spacetime dimension. But it has already been shown that the above formula equally holds in the case of black holes with charge [33]. Actually black hole entropy depends on the degrees of freedom on its horizon. It is purely a geometrical property of the isolated horizon. Adding charge to the black hole does not alter this geometry at all. In fact it is also shown that results from analysis for isolated horizons with charge is similar to that with angular momentum, except for certain technical issues [33, 35, 39]. Therefore the above formula will be taken to be valid for charged, rotating black holes as well.

4.3 Stability criteria

Convergence of the integral (19) implies that the Hessian matrix (H) has to be positive definite, where

$$H = \begin{pmatrix} \beta M_{AA}(\overline{A},\overline{Q},\overline{J}) - S_{AA}(\overline{A}) & \beta M_{AQ}(\overline{A},\overline{Q},\overline{J}) & \beta M_{AJ}(\overline{A},\overline{Q},\overline{J}).1in \\ \beta M_{AQ}(\overline{A},\overline{Q},\overline{J}) & \beta M_{QQ}(\overline{A},\overline{Q},\overline{J}) & \beta M_{JQ}(\overline{A},\overline{Q},\overline{J}).1in \\ \beta M_{AJ}(\overline{A},\overline{Q},\overline{J}) & \beta M_{JQ}(\overline{A},\overline{Q},\overline{J}) & \beta M_{JJ}(\overline{A},\overline{Q},\overline{J}) \end{pmatrix} \tag{23}$$

The necessary and sufficient conditions for a real symmetric square matrix to be positive definite are that determinants of all principal square submatrices and the determinant of the full matrix are positive [40–42]. This condition leads to the following 'stability criteria:

$$M_{AA}(\overline{A},\overline{Q},\overline{J}) - \frac{S_{AA}(\overline{A})}{\beta} > 0 \tag{24}$$

$$M_{QQ}(\overline{A},\overline{Q},\overline{J}) > 0 \tag{25}$$

$$M_{JJ}(\overline{A},\overline{Q},\overline{J}) > 0 \tag{26}$$

$$M_{QQ}(\overline{A},\overline{Q},\overline{J})M_{JJ}(\overline{A},\overline{Q},\overline{J}) - \left(M_{JQ}(\overline{A},\overline{Q},\overline{J})\right)^2 > 0 \tag{27}$$

$$M_{JJ}(\overline{A},\overline{Q},\overline{J})\left(M_{AA}(\overline{A},\overline{Q},\overline{J}) - \frac{S_{AA}(\overline{A})}{\beta}\right) - \left(M_{AJ}(\overline{A},\overline{Q},\overline{J})\right)^2 > 0 \tag{28}$$

$$M_{QQ}(\overline{A},\overline{Q},\overline{J})\left(M_{AA}(\overline{A},\overline{Q},\overline{J}) - \frac{S_{AA}(\overline{A})}{\beta}\right) - \left(M_{AQ}(\overline{A},\overline{Q},\overline{J})\right)^2 > 0 \tag{29}$$

$$\begin{aligned} &\left[\left(M_{AA}(\overline{A},\overline{Q},\overline{J}) - \frac{S_{AA}(\overline{A})}{\beta}\right)\left(M_{QQ}(\overline{A},\overline{Q},\overline{J})M_{JJ}(\overline{A},\overline{Q},\overline{J}) - \left(M_{JQ}(\overline{A},\overline{Q},\overline{J})\right)^2\right)\right. \\ &-M_{AQ}(\overline{A},\overline{Q},\overline{J})\left(M_{AQ}(\overline{A},\overline{Q},\overline{J})M_{JJ}(\overline{A},\overline{Q},\overline{J}) - M_{JQ}(\overline{A},\overline{Q},\overline{J})M_{AJ}(\overline{A},\overline{Q},\overline{J})\right) \\ &\left.+M_{AJ}(\overline{A},\overline{Q},\overline{J})\left(M_{AQ}(\overline{A},\overline{Q},\overline{J})M_{JQ}(\overline{A},\overline{Q},\overline{J}) - M_{QQ}(\overline{A},\overline{Q},\overline{J})M_{AJ}(\overline{A},\overline{Q},\overline{J})\right)\right] > 0 \end{aligned} \tag{30}$$

Of course, (inverse) temperature β is assumed to be positive for a stable configuration.

Now, the temperature is defined as $T \equiv \frac{1}{\beta} = \frac{M_A}{S_A}$ (from Eq. (20)).

Eqs. (21) and (22) together yield

$$S_A = \frac{1}{4A_P} - \frac{3}{2A} \tag{31}$$

This is positive for macroscopic black holes ($A >> A_P$). So, positivity of M_A implies the positivity of β for macroscopic black holes. The relation $T = \frac{M_A}{S_A}$ implies that

$$\frac{dT}{dA} = \frac{\beta M_A}{(S_A)^2}\left(M_{AA} - \frac{S_{AA}}{\beta}\right) \tag{32}$$

So, the positivity of the quantity $\left(M_{AA} - \frac{S_{AA}}{\beta}\right)$, which is a stability criteria (24), means the positivity of $\frac{dT}{dA}$. In words, a stable black hole becomes hotter as it grows

in size. If this is violated, as, for example, in the case of the standard Schwarzschild black hole ([9]), thermal instability is inevitable.

Eq. (20) implies that $M_{QQ} = \frac{d\Phi}{dQ}$. So, positivity of M_{QQ} is an artefact of the fact that accumulation of charge increases the electric potential of the black hole. This is a feature of a stable black hole (25).

Similarly, Eq. (20) shows that $M_{JJ} = \frac{d\Omega}{dJ}$. So, positivity of M_{JJ} implies that the gathering of angular momentum helps the black hole to rotate faster. Hence this is the case with stable black holes (26).

The convexity property of the entropy follows from the condition of convergence of partition function under Gaussian fluctuations [9, 31, 38]. The thermal stability is related to the convexity property of entropy. Hence, the above conditions are correctly the conditions for thermal stability. For chargeless, nonrotating horizons, Eq. (24) reproduces the thermal stability criterion and condition of positive specific heat (i.e. variation of black hole mass with temperature) given in ([13]), as expected. Actually for a chargeless, nonrotating black hole, both the mass and the temperature are functions of the horizon area (A) only. So, the specific heat (C) of the black hole is given as

$$C \equiv \frac{dM}{dT} = \frac{(S_A)^2}{(\beta M_{AA} - S_{AA})} \tag{33}$$

For charged, nonrotating black holes, Eqs. (24), (25) and (29) describe the stability, in perfect agreement with [12], while (24), (26) and (28) describe the thermal stability criteria for uncharged rotating radiant horizons. The new feature for black holes with both charge and angular momentum is that not only does the specific heat have to be positive for stability, but the charge and the angular momentum play important roles as well.

5. Thermal stability of higher-dimensional black holes with arbitrary hairs

5.1 Thermal holography

In this section, we present a generalization of thermal holography for rotating electrically charged quantum radiant horizons discussed in [43], to the situation when the horizon has arbitrary number of hairs [44]. This section of the chapter will of course have substantial overlap with some of the appropriate previous sections of this chapter, so for brevity we focus on the novel aspects here.

5.1.1 Mass associated with horizon

Isolated horizons (b) represent black holes at equilibrium. These isolated horizons are the internal boundaries of spacetime. The first law associated with isolated horizon (b) comes from the Hamiltonian evolution of this spacetime. The law is given as

$$\delta E_h^t = \frac{\kappa^t}{8\pi}\delta A_h + P_i^t \delta C_h^i \tag{34}$$

Here, Einstein summation convention is used, i.e. summation over repeated indices i from 1 to n (=total number of hairs) is implied. E_h^t is the energy function

associated with the horizon. κ^t and P_i^t are, respectively, the surface gravity associated with the area of horizon (A_h) and the potential corresponding to the charge (hair) C_h^i. For example, if C_h^i is the angular momentum (J_h), then P_i^t will be the angular velocity (Ω^t). The label 't' denotes the particular time evolution field (t^μ) associated with the spatial hypersurface chosen. E_h^t is assumed here to be a function of A_h and all C_h^i.

The advantage of the isolated (and also the radiant or *dynamical*) horizon description is that one can associate with it a mass M_h^t, related to the ADM energy of the spacetime through the relation

$$E_{ADM}^t = M_h^t + E_{rad}^t \tag{35}$$

where E_{rad}^t is the energy associated with spacetime between the horizon and asymptopia. An isolated horizon admits $E_{rad}^t \neq 0$, and hence a mass is defined *locally* on the horizon.

5.1.2 Quantum algebra and quantum geometry

We consider a quantum black hole with n charges (hairs) $C^1, \ldots, C^n$. These charges are independent of each other. Therefore, respectively, the corresponding operators $\widehat{C^1}, \ldots, \widehat{C^n}$ are also independent of each other, i.e.

$$\left[\widehat{C^i}, \widehat{C^j}\right] = 0, \quad for \quad i \neq j \tag{36}$$

These charges are intrinsic to the black holes and independent of the horizon area (A) of the black hole, if we choose the mass (M) of the black hole to be a dependent variable which depends on the horizon area and the charges. This implies

$$\left[\widehat{A}, \widehat{C^i}\right] = 0, \quad \forall i = 1(1)n \tag{37}$$

where $\widehat{A}$ is the area operator of the black hole.

Choosing mass (M) to be the dependent variable implies that mass (M) is a function of area (A) and all the charges ($C^1, \ldots, C^n$), i.e. $M = M\left(A, C^1, \ldots, C^n\right)$. This gives

$$\left[\widehat{M}, \widehat{C^i}\right] = 0, \quad \left[\widehat{M}, \widehat{A}\right] = 0 \quad \forall i = 1(1)n \tag{38}$$

where $\widehat{M}$ is the mass operator of the black hole. Eqs. (36)–(38) together imply that a black hole with a given mass can simultaneously be an eigenstate of its area operator ($\widehat{A}$) and all the charge operators ($\widehat{C^1}, \ldots, \widehat{C^n}$). So, we can consider that a black hole of given mass M has specified area A and specified charges $C^1, \ldots, C^n$.

The Hilbert space of a generic quantum spacetime is given as $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_v$, where $b(v)$ denotes the boundary (bulk) space. A generic quantum state is thus given as

$$|\Psi\rangle = \sum_{b,v} C_{b,v} |\chi_b\rangle \otimes |\psi_v\rangle \tag{39}$$

Now, the full Hamiltonian operator $(\widehat{H})$, operating on $\mathcal{H}$, is given by

$$\widehat{H}|\Psi\rangle = \left(\widehat{H_b} \otimes I_v + I_b \otimes \widehat{H_v}\right)|\Psi\rangle \tag{40}$$

where, respectively, $I_b(I_v)$ is the identity operator on $\mathcal{H}_b(\mathcal{H}_v)$ and $\widehat{H_b}\left(\widehat{H_v}\right)$ is the Hamiltonian operator on $\mathcal{H}_b(\mathcal{H}_v)$.

Now, the bulk Hamiltonian operator annihilates bulk physical states:

$$\widehat{H_v}|\psi_v\rangle = 0 \tag{41}$$

The charge operators $\widehat{C^i}$s are each an infinitesimal generator of a continuous transformation on the bulk Hilbert space. For example, electric charge operator $(\widehat{Q})$ is the generator of local $U(1)$ transformation, and angular momentum operator $(\widehat{J})$ is the generator of local spatial rotation. So, the assumption that bulk quantum spacetime is invariant under all these transformations implies that bulk spacetime is free of any charge (hair). This gives

$$\widehat{C_v^i}|\psi_v\rangle = 0 \tag{42}$$

where $\widehat{C_v^i}$ is the bulk charge operator corresponding to the charge C^i.

So Eqs. (41) and (42) together produce

$$\left[\widehat{H_v} - \beta P_i \widehat{C_v^i}\right]|\psi_v\rangle = 0. \tag{43}$$

where β can be any function, but we treat it as inverse temperature of the black hole afterwards.

5.2 Grand canonical partition function

We now consider the black hole with the contact of a heat bath, at some (inverse) temperature β, with which it can exchange energy, charge, angular momentum and all quantum hairs. The grand canonical partition function of the black hole is given as

$$Z_G = Tr\left(exp\left(-\beta\widehat{H} + \beta P_i \widehat{C^i}\right)\right) \tag{44}$$

where the trace is taken over all states. This definition, together with Eqs. (39) and (43) yield

$$\begin{aligned} Z_G &= \sum_{b,v} |C_{b,v}|^2 \langle\psi_v|\psi_v\rangle \left\langle \chi_b | exp\left(-\beta\widehat{H_b} + \beta P_i \widehat{C^i}\right)|\chi_b\right\rangle \\ &= \sum_{b} |C_b|^2 \left\langle \chi_b | exp\left(-\beta\widehat{H_b} + \beta P_i \widehat{C^i}\right)|\chi_b\right\rangle, \end{aligned} \tag{45}$$

assuming that the boundary states are normalized. The partition function thus turns out to be completely determined by the boundary states (Z_{Gb}), i.e.

$$Z = Z_{Gb} = Tr_b \exp\left(-\beta\widehat{H} + \beta P_i \widehat{C^i}\right)$$
$$= \sum_{l,k_1,\ldots,k_n} g(l,k_1,\ldots,k_n) \quad \exp\left(-\beta\left(E\left(A_l, C^1_{k_1},\ldots,C^n_{k_n}\right) - \sum_{i=1}^{n} P_i C^i_{k_i}\right)\right). \tag{46}$$

where $g(l,k_1,\ldots,k_n)$ is the degeneracy corresponding to energy $E\left(A_l, C^1_{k_1},\ldots,C^n_{k_n}\right)$ and l,k_i are the quantum numbers corresponding to area and charge C^i, respectively. Here, the spectrum of the boundary Hamiltonian operator is assumed to be a function of area and all other charges of the boundary, considered here to be the horizon. Following [12], it is further assumed that these 'hairs' have a discrete spectrum. In the macroscopic limit of the black hole, they all have large eigenvalues, i.e. $(l,k_i >> 1)$, so that application of the Poisson resummation formula (9) gives

$$Z_G = \int dx \left(\prod_{i=1}^{n}\int dy_i\right) g(A(x), C^1(y_1),\ldots,C^n(y_n)) \exp\left(-\beta\left(E(A(x), C^1(y_1),\ldots,C^n(y_n))\right.\right.$$
$$\left.\left. - \sum_{i=1}^{n} P_i C^i(y_i)\right)\right). \tag{47}$$

where x, y_i are, respectively, the continuum limits of l,k_i, respectively.

Following [12], we now assume that the macroscopic spectrum of the area and all charges are linear in their arguments, so that a change of variables gives, with constant Jacobian, the result

$$Z_G = \int dA \left(\prod_{i=1}^{n}\int dC^i\right) \exp\left(S(A)\right.$$
$$\left. -\beta\left(E\left(A, C^1,\ldots,C^n\right) - P_i C^i\right)\right), \tag{48}$$

where, following [38], the *microcanonical* entropy of the horizon is defined by

$$\exp S(A) \equiv \frac{g\left(A(x), C(y_1),\ldots,C(y_n)\right)}{\frac{dA}{dx}\frac{dC^1}{dy_1}\cdots\frac{dC^n}{dy_n}} \tag{49}$$

5.3 Saddle-point approximation

The equilibrium configuration of black hole is given by the saddle point $(\overline{A}, \overline{C}^1, , \overline{C}^n)$ in the $(n+1)$ dimensional space of integration over area and n charges. This configuration is identified with an isolated horizon, as already mentioned. The idea now is to examine the grand canonical partition function for fluctuations $a = \left(A - \overline{A}\right)$ and $c^i = \left(C^i - \overline{C}^i\right)$ around the saddle point, in order to determine the stability of the equilibrium isolated horizon under Hawking radiation. We restrict our attention to Gaussian fluctuations. Taylor expanding Eq. (48) about the saddle point yields

$$Z_G = \exp\left[S(\overline{A}) - \beta M\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right) + \beta P_i \overline{C}^i\right] \times \int dA \quad \left(\prod_{i=1}^{n} \int dC^i\right) \exp\{-\frac{1}{2}[(\beta M_{AA} - S_{AA})a^2 + 2\sum_{i=1}^{n} \beta M_{AC^i} a c^i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta M_{C^i C^j} c^i c^j]\}, \tag{50}$$

where $M\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right)$ is the mass of equilibrium isolated horizon.

Here $M_{AC^i} \equiv \partial^2 M / \partial A \partial C^i \big|_{\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right)}$, etc.

Now, in the saddle-point approximation, the coefficients of terms linear in a, c^i vanish by definition of the saddle point. These imply that

$$\beta = \frac{S_A}{M_A}, \quad P_i = M_{C^i} \tag{51}$$

Of course these are evaluated at the saddle point.

5.4 Stability criteria

Convergence of the integral (50) implies that the Hessian matrix (H) has to be positive definite, where

$$H = \begin{pmatrix} \beta M_{AA} - S_{AA} & \beta M_{AC^1} & \beta M_{AC^2} & \ldots\ \ldots\ \ldots & \beta M_{AC^n} \\ \beta M_{AC^1} & \beta M_{C^1C^1} & \beta M_{C^1C^2} & \ldots\ \ldots\ \ldots & \beta M_{C^1C^n} \\ \beta M_{AC^2} & \beta M_{C^2C^1} & \beta M_{C^2C^2} & \ldots\ \ldots\ \ldots & \beta M_{C^2C^n} \\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots\ \ldots & \ldots\ \ldots \\ \beta M_{AC^n} & \beta M_{C^nC^1} & \beta M_{C^nC^2} & \ldots\ \ldots\ \ldots & \beta M_{C^nC^n} \end{pmatrix} \tag{52}$$

Here, all the derivatives are calculated at the saddle point. Hence the stability criteria, i.e. the criteria for positive definiteness of Hessian matrix, are given as

$$D_1 > 0, D_2 > 0, \ldots, D_{n+1} > 0 \tag{53}$$

where

$$D_1 = \beta M_{AA} - S_{AA}, \quad D_2 = \begin{vmatrix} \beta M_{AA} - S_{AA} & \beta M_{AC^1} \\ \beta M_{AC^1} & \beta M_{C^1C^1} \end{vmatrix},$$
$$D_3 = \begin{vmatrix} \beta M_{AA} - S_{AA} & \beta M_{AC^1} & \beta M_{AC^2} \\ \beta M_{AC^1} & \beta M_{C^1C^1} & \beta M_{C^1C^2} \\ \beta M_{AC^2} & \beta M_{C^2C^1} & \beta M_{C^2C^2} \end{vmatrix}, \ldots, D_{n+1} = |H| \tag{54}$$

where $|H|$ = determinant of the Hessian matrix H.

Of course, (inverse) temperature β is assumed to be positive for a stable configuration. We again find that temperature must increase with horizon area, inherent in the positivity of the quantity $(\beta M_{AA} - S_{AA})$.

The convexity property of the entropy follows from the condition of convergence of partition function under Gaussian fluctuations [9, 31, 38]. The thermal stability is related to the convexity property of entropy. Hence, the above conditions are correctly the conditions for thermal stability. For rotating charged horizons, Eqs. (53) and (54) reproduce the thermal stability criterion with $n = 2$, i.e. $D_1 > 0, D_2 > 0, D_3 > 0$ with the identification that charge of the black hole (Q)=C^1 and angular momentum of the black hole (J)=C^2. It can be easily checked that these three conditions correctly reproduce the earlier ([43]) seven conditions of thermal stability of charged rotating black holes. Eqs. (53) and (54) necessarily tell us that thermal stability of black hole is a consequence of the interplay among all the charges of the black hole.

Now, we are going to show that Eqs. (53) and (54) correctly produce the criteria of stability for charged rotating black holes (24)–(30), taking $n = 2$.

Consider the following integral,

$$I = \int\int\int dxdydz \exp\left(-\left(ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fzx\right)\right)$$

Define $U \equiv \left(ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fzx\right)$

Now, we can rewrite the argument of the exponential (U) part as

$$\begin{aligned} U = a\left(x + \frac{d}{a}y + \frac{f}{a}z\right)^2 + \frac{\left(ab - d^2\right)}{a}\left(y + \frac{e - df/a}{\left(\left(ab - d^2\right)/a\right)}z\right)^2 \\ + \left(\frac{ac - f^2}{a} - \frac{(e - df/a)^2}{\left(\left(ab - d^2\right)/a\right)}\right)z^2 \end{aligned} \tag{55}$$

Considering the notations, given in Eqs. (53) and (54), we can write

$$U = D_1\left(x + \frac{d}{a}y + \frac{f}{a}z\right)^2 + \frac{D_2}{D_1}\left(y + \frac{e - df/a}{\left(\left(ab - d^2\right)/a\right)}z\right)^2 + \frac{D_3}{D_2}z^2 \tag{56}$$

where

$$\begin{aligned} \mathrm{D_1} &= \mathrm{a} \\ \mathrm{D_2} &= \left(\mathrm{ab} - \mathrm{d}^2\right) \\ \frac{\mathrm{D_3}}{\mathrm{D_2}} &= \frac{\left(\mathrm{abc} - \mathrm{cd}^2 - \mathrm{bf}^2 - \mathrm{ae}^2 + 2\mathrm{dfe}\right)}{\left(\mathrm{ab} - \mathrm{d}^2\right)} \\ &= \frac{\mathrm{ac} - \mathrm{f}^2}{\mathrm{a}} - \frac{(\mathrm{e} - \mathrm{df}/\mathrm{a})^2}{\left(\left(\mathrm{ab} - \mathrm{d}^2\right)/\mathrm{a}\right)} \end{aligned} \tag{57}$$

So, we have

$$I = \int\int\int dxdydz exp\left(-\left(D_1\left(x + \frac{d}{a}y + \frac{f}{a}z\right)^2 + \frac{D_2}{D_1}\left(y + \frac{e - df/a}{\left(\left(ab - d^2\right)/a\right)}z\right)^2 + \frac{D_3}{D_2}z^2\right)\right) \tag{58}$$

Consider the following change of variables:

$$\begin{pmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{a} & \frac{f}{a} \\ 0 & 1 & \frac{e - df/a}{\left((ab - d^2)/a\right)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{59}$$

Therefore Eqs. (58) and (59) together give

$$I = \int\int\int d\underline{x} d\,\underline{y} d\underline{z} \quad A \cdot exp\left(-\left(D_1\underline{x}^2 + \frac{D_2}{D_1}\underline{y}^2 + \frac{D_3}{D_2}\underline{z}^2\right)\right) \tag{60}$$

where A is the Jacobian of the transformation matrix, i.e.

$$\begin{aligned} A &= \begin{vmatrix} 1 & \frac{d}{a} & \frac{f}{a} \\ 0 & 1 & \frac{e - df/a}{\left((ab - d^2)/a\right)} \\ 0 & 0 & 1 \end{vmatrix} \\ &= 1 \end{aligned} \tag{61}$$

$$I = \int\int\int d\underline{x} d\,\underline{y} d\underline{z} \quad exp\left(-\left(D_1\underline{x}^2 + \frac{D_2}{D_1}\underline{y}^2 + \frac{D_3}{D_2}\underline{z}^2\right)\right) \tag{62}$$

This expression explicitly shows that I will be converging if and only if $D_1 > 0, D_2 > 0, D_3 > 0$. This is what we have claimed in this section as the condition for thermal stability of rotating charged black holes.

From the expression (57), we get:

1) If $D_1 > 0, D_2 > 0$, then $b > 0$.

2) If $D_1 > 0, D_2 > 0, D_3 > 0$, then $\left(ac - f^2\right) > 0$. Consequently, $c > 0$.

3) The expression of D_3 can be rearranged as

$$\begin{aligned} D_3 &= \left(abc - cd^2 - bf^2 - ae^2 + 2dfe\right) \\ &= \left(ab^2c - bcd^2 - b^2f^2 - abe^2 + 2bdfe\right)/b \\ &= \left(\left(ab - d^2\right)\left(bc - e^2\right) - \left(bf - ed\right)^2\right)/b \end{aligned} \tag{63}$$

So, the positivity of b, $D_2\left(= ab - d^2\right)$ and D_3 implies that $\left(bc - e^2\right) > 0$. Therefore these conditions for thermal stability described by $D_1 > 0, D_2 > 0, D_3 > 0$ are same as those described by inequalities (24–30) for rotating charged black holes.

For an n dimensional matrix, the total number of submatrices including the whole matrix (N_s) is given as

$$\begin{aligned} N_s &= n_{C_1} + n_{C_2} + \ldots\ldots\ldots + n_{C_n} \\ &= 2^n - 1 \end{aligned} \tag{64}$$

Now, any generic quadratic expression of n variables can be rearranged by redefining variables as an quadratic expression without any cross term, i.e. of the form $\sum_{i=1}^{n} a_i x_i^2$.

Thus if we consider the positivity of determinants of the all submatrices of Hessian (including itself), then we have to check $\left(2^{n+1}-1\right)$ conditions for testing thermal stability of a black hole with n charges. On the other hand, if we follow the different procedure set in this chapter, then $(n+1)$ conditions have to be checked for testing thermal stability of a black hole with n charges. Obviously $\left(2^{n+1}-1\right)$ is greater than $(n+1)$ for $n \geq 1$, i.e. black hole with at least one charge. But there are certain advantages of checking these additional criteria, i.e. positivity of submatrices of Hessian matrix. This is very useful for studying 'Quasi Stable' black holes, especially for studying the fluctuations of charges for such black holes [45]. But this is beyond the scope of this chapter. In fact the issue of thermal fluctuations for stable black holes is also interesting [46]. A stable black hole has to satisfy all the stability criteria. So, a simple inequality, i.e. determinant of a submatrix of Hessian matrix, may be negative. This can be easily checked, and the corresponding black hole is concluded to be unstable under Hawking radiation.

6. Discussions

The novelty of our approach is that it is purely based on quantum aspects of spacetime. *Classical metric* has not been used anywhere in the analysis. The construction of the partition function is based on LQG, e.g. the use of Chern-Simons states, the splitting up of the total Hilbert space, etc. and also on the Hamiltonian formulation of spacetimes admitting weakly isolated horizons. The entropy correction also follows from the quantum theory.

In this analysis of thermal stability of black holes, two physically reasonable assumptions are made. In classical Hamiltonian GR, total Hamiltonian vanishes. So, it is considered that the total quantum Hamiltonian operator annihilates the bulk states of quantum matter coupled spacetime. A similar argument follows for the assumption of the quantum constraint on the volume charge operator. These two assumptions may be considered to be one due to their fundamental similarity, and they ultimately give rise to a single quantum constraint.

In Section (III.C), a second assumption is made regarding the eigenvalue spectrum of the energy of the black hole. The classical mass associated with the horizon is a function of horizon area, charge and angular momentum. These horizon area, charge and angular momentum are the functions of the local fields on the horizon. So, quantization of the classical horizon area, charge and angular momentum will definitely lead to a well-defined boundary Hamiltonian operator. The existence of a quantum boundary Hamiltonian operator, acting on the boundary Hilbert space of the black hole, is an assumption as the exact form of such a Hamiltonian operator is still unknown. But the fact that its eigenvalue spectrum is a function of eigenvalue spectra of the area, charge and angular momentum operators are an obvious assumption, as it is bound to happen if such a boundary Hamiltonian operator exists. It follows from the classical analogue—the mass associated with the horizon must be a function of the horizon area, charge and angular momentum for a consistent Hamiltonian evolution.

In this chapter, we have derived the criteria for thermal stability of charged rotating black holes, for horizon areas that are largely relative to the Planck area (in these dimensions). We also generalize it for black holes with arbitrary hairs in any

spacetime dimension. Like earlier, results of LQG and equilibrium statistical mechanics of the grand canonical ensemble are sufficient for our analysis. The only assumption is that the mass of the black hole is a function of its horizon area and all the hairs. The obtained stability criteria can be applied to check the thermal stability of any black hole whose mass is given as function of its charges, and in fact this has been done [43] for various black holes as well.

IntechOpen

IntechOpen

Author details

Aloke Kumar Sinha
Haldia Government College, West Bengal, India

*Address all correspondence to: akshooghly@gmail.com

IntechOpen

© 2019 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. CC BY

References

[1] Hawking SW, Page DN. Thermodynamics of black holes in anti-de sitter space. Communications in Mathematical Physics. 1983;**87**:577

[2] Davis PCW. The thermodynamic theory of black holes. Proceedings of the Royal Society. 1977;**A353**:499

[3] Rovelli C. Quantum Gravity, Cambridge Monographs on Mathematical Physics. UK: Cambridge University Press; 2004

[4] Thiemann T. Modern Canonical Quantum General Relativity, Cambridge Monographs on Mathematical Physics. UK: Cambridge University Press; 2007

[5] Ashtekar A, Baez J, Corichi A, Krasnov K. Quantum geometry and black hole entropy. Physical Review Letters. 1998;**80**:904; e-Print arXiv: gr-qc/9710007

[6] Ashtekar A, Baez J, Krasnov K. Quantum geometry of isolated horizons and black hole entropy. Advances in Theoretical and Mathematical Physics. 2000;**4**:1; e-Print arXiv : gr-qc/ 0005126

[7] Kaul RK, Majumdar P. Quantum black hole entropy. Physics Letters. 1998;**B439**:267; e-Print arXiv : gr-qc/ 9801080

[8] Basu R, Kaul RK, Majumdar P. Entropy of isolated horizons revisited. Physics Review. 2010;**D82**:024007; e-Print arXiv : 0907.0846v3

[9] Chatterjee A, Majumdar P. Universal canonical black hole entropy. Physical Review Letters. 2004;**92**:141031; e-Print arXiv : gr-qc/0309026

[10] Chatterjee A, Majumdar P. Mass and charge fluctuations and black hole entropy. Physics Review. 2005;**D71**: 024003; e-Print arXiv : gr-qc/0409097

[11] Chatterjee A, Majumdar P. Universal criterion for black hole stability. Physics Review. 2005;**D72**: 044005; e-Print arXiv : gr-qc/0504064

[12] Majhi A, Majumdar P. Charged quantum black holes : Thermal stability criterion. Classical and Quantum Gravity. 2012;**29**:135013; e-Print arXiv : 1108.4670

[13] Majumdar P. Generalized hawking-page phase transition. Classical and Quantum Gravity. 2007;**24**:1747; e-Print arXiv : gr-qc/ 0701014

[14] Majumdar P. Holography, Gauge-gravity connection and black hole entropy. International Journal of Modern Physics A. 2009;**A24**:3414; e-Print arXiv :0903.5080

[15] Kaul RK, Majumdar P. Logarithmic correction to the bekenstein-hawking entropy. Physical Review Letters. 2000;**84**:5255; e-Print arXiv : gr-qc/ 0002040

[16] Ashtekar A, Krishnan B. Isolated and dynamical horizons and their applications. Living Reviews in Relativity. 2004;**7**:10; e-Print arXiv : gr-qc/ 0407042

[17] Kaul RK, Rama SK. Black hole entropy from spin one punctures. Physical Review D. 2003;**68**:024001, e-Print arxiv: gr-qc/0301128v1

[18] Frodden E, Perez A, Pranzetti D, Roken C. Modelling black holes with angular momentum in loop quantum gravity. General Relativity and Gravitation. 2014;**46**:1828, e-Print arxiv: 1212.5166

[19] Achour JB, Noui K, Perez A. Analytic continuation of the rotating black hole state counting. JHEP 1608. 2016:149, e-Print arxiv: 1607.02380

[20] Emparan R, Reall HS. Black holes in higher dimensions. Living Reviews in Relativity. 2008;**11**:6

[21] Carter B. Axisymmetric black hole has only two degrees of freedom. Physical Review Letters. 1971;**26**(6): 331-333

[22] Coleman S, Preslill J, Wilczek F. Growing hair on black holes. Physical Review Letters. 1975;**67**:1978

[23] Bizon P. Colored black holes. Physical Review Letters. 1990;**64**:2844

[24] Krauss LM, Wilczek F. Discrete gauge symmetry in continuum theories. Physical Review Letters. 1989;**62**:1221

[25] Bowick MJ, Giddings SB, Harvey JA, Horowitz GT, Strominger A. Axionic black holes and an aharonov-bohm effect for strings. Physical Review Letters. 1988;**61**:2823

[26] Lee K, Nair VP, Weinberg EJ. The fate of magnetically charged black holes. Physical Review D. 1992;**45**:2751

[27] Preskill J. Quantum hair. Physica Scripta. 1991;**T36**:258-264

[28] Coleman S, Preslill J, Wilczek F. Dynamical effect of quantum hair. Modern Physics Letters A. 1991;**06**:1631

[29] Myers RC, Perry MJ. Black holes in higher dimensional space-times. Annals of Physics. 1986;**172**:304-347

[30] Das S, Majumdar P, Bhaduri RK. General logarithmic corrections to black hole entropy. Classical and Quantum Gravity. 2002;**19**:2355-2368, e-Print arxiv:hep-th/0111001

[31] R. Monteiro. Classical and thermodynamic stability of black holes. 2010. e-Print arxiv:hep-th 1006.5358

[32] Ashtekar A, Fairhurst S, Krishnan B. Isolated horizons: Hamiltonian evolution and the first law. Physics Review, D62. 2000:104025; e-Print arxiv:gr-qc/0005083v3

[33] Ashtekar A, Beetle C, Fairhurst S. Mechanics of isolated horizons. Classical and Quantum Gravity. 2000;**17**:253-298, e-Print arXiv:gr-qc/9907068

[34] Ashtekar A, Pawlowski T, Broeck C. Mechanics of higher-dimensional black holes in asymptotically anti-de sitter space-times. Classical and Quantum Gravity. 2007;**24**:625-644; e-Print arXiv:gr-qc/0611049v2

[35] Ashtekar A, Beetle C, Lewandowski J. Mechanics of rotating isolated horizons. PRD 64. 2001: 044016, e-print arXiv:gr-qc/0103026

[36] Majhi A, Majumdar P. Quantum hairs and entropy of quantum isolated horizon from chern-simons theory. Classical and Quantum Gravity. 2014; **31**:195003; e-Print arXiv: 1301.4553

[37] Gour G, M AJ. Medved. Kerr black hole as a quantum rotator. Classical and Quantum Gravity. 2003:**20**;2261-2274. e-Print arXiv:gr-qc/0211089v1

[38] Landau LD, Lifschitz EM. Statistical Physics. Oxford, England: Pergamon Press; 1980

[39] Ashtekar A, Engle J, Broeck CVD. Quantum horizons and black hole entropy: Inclusion of distortion and rotation. Classical and Quantum Gravity. 2005;**22**:L27-L34. e-Print arxiv: gr-qc/0412003v1

[40] Meyer CD. Matrix Analysis and Applied Linear Algebra. Society for Industrial and Applied Mathematics; Har/Cdr edition. Philadelphia: University City Science Center; 1 June 2000

[41] Bhatia R. Positive Definite Matrices. USA: Princeton University Press; 2015. ISBN: 9780691168258

[42] Pinkus A. Totally Positive Matrices. Cambridge University Press; 2009

[43] Sinha AK, Majumdar P. Thermal stability of charged rotating quantum black holes. Modern Physics Letters A. 2017;**32**(37):1750208; e-Print Arxiv: 1512.04181v2

[44] Sinha AK. Thermal stability of black holes with arbitrary hairs. Modern Physics Letters A. 2018;**33**:1850031; e-Print arxiv:1702.08175

[45] Sinha AK. Quasi stable black holes and their implications. Classical and Quantum Gravity. 2018;**36**(2019): 035003

[46] Sinha AK. Thermal fluctuations and correlations among hairs of a stable quantum black hole: Some examples. Modern Physics Letters A. 2018:33, 1850190