Chapter

Improving Heat-Engine Performance by Employing Multiple Heat Reservoirs

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Abstract

The efficiencies of heat-engine operation employing various numbers (≥ 2) of heat reservoirs are investigated. Operation with the work output of the heat engines sequestered, as well as with it being totally frictionally dissipated, is discussed. We consider mainly heat engines whose efficiencies depend on ratios of a higher and lower temperature or on simple functions of such ratios but also provide brief comments concerning more general cases. We show that, if a hot reservoir supplies a heat engine whose waste heat is discharged *and* whose work output is totally frictionally dissipated into a cooler reservoir, which in turn supplies heat-engine operation that discharges waste heat into a still cooler reservoir, the total work output can exceed the heat input from the initial hot reservoir. This extra work output increases with increasing numbers (≥ 3) of reservoirs. We also show that this obtains within the restrictions of the First and Second Laws of Thermodynamics.

Keywords: First Law of Thermodynamics, Second Law of Thermodynamics, heat engines, work, heat, entropy, multiple heat reservoirs

1. Introduction

The efficiencies of heat-engine operation employing various numbers (≥ 2) of heat reservoirs are investigated. In Section 2, we discuss heat-engine operation with the work output of the heat engines sequestered. In Section 3, we discuss heat-engine operation with the work output of the heat engines being totally frictionally dissipated. We consider mainly heat engines whose efficiencies depend on ratios of a higher and lower temperature or on simple functions of such ratios. Examples include heat engines operating not only via the Carnot cycle [1–9] but also via the Ericsson, Stirling, air-standard Otto, and air-standard Brayton cycles [2–9], and endoreversible heat engines operating at maximum power output assuming Curzon-Ahlborn efficiency [10–12] (see also Ref. [4], Section 4-9). But we also provide brief comments concerning more general cases. Endoreversible heat-engine operation assumes irreversible heat flows directly proportional to temperature differences but otherwise reversible operation [10–12]. Although we do not employ them in this chapter, we note that generalizations of the Curzon-Ahlborn efficiency, and also various related efficiencies, have also been investigated [13–21]. In particular, we note that alternative results [21] to the Curzon-Ahlborn efficiency [10–12] (see also Ref. [4], Section 4-9) have been derived [21]. But for definiteness and for simplicity, in this chapter, we employ the standard Curzon-Ahlborn efficiency [10–12] (see also Ref. [4], Section 4-9) for cyclic heat engines operating at maximum power output.

We show that, if a hot reservoir supplies a heat engine whose waste heat is discharged *and* whose work output is totally frictionally dissipated into a cooler reservoir, which in turn supplies heat-engine operation that discharges waste heat into a still cooler reservoir, the total work output can exceed the heat input from the initial hot reservoir. This extra work output increases with increasing numbers (≥ 3) of reservoirs. We also show that this obtains within the restrictions of the First and Second Laws of Thermodynamics.

We fill in details and correct a few mistakes in an earlier, briefer, consideration of the efficiencies of heat-engine operation employing various numbers (≥ 3) of heat reservoirs [22]. We note that heat-engine operation employing various numbers (≥ 3) of heat reservoirs [22] should not be confused with recycling heat engines' frictionally dissipated work outputs into the hottest available reservoir [22–37], which is a *different* process that has been thoroughly investigated and discussed previously [22–37], and which we further investigate in another chapter [38] in this book.

We consider only cyclic heat engines. Noncyclic (necessarily one-time, single-use) heat engines are not limited by the Carnot bound and can in principle operate at unit (100%) efficiency. A simple example is the one-time expansion of a gas pushing a piston. Other examples include rockets: the piston (payload) is launched into space by a one-time power stroke (but typically most of the work output accelerates the exhaust gases, not the payload) and firearms: the piston (bullet) is accelerated by a one-time power stroke and then discarded (but some, typically less than with rockets, of the work output accelerates the exhaust gases resulting from combustion of the propellant). Even if the work output of a noncyclic engine could be frictionally dissipated and the resulting heat returned to the system, there would be, at best, restoration of temperature to its initial value but not restoration of the piston to its initial position. Hence the method investigated in this chapter is useless with respect to noncyclic heat engines.

General remarks, especially concerning entropy, are provided in Section 4. Concluding remarks are provided in Section 5.

2. Multiple-reservoir heat-engine efficiencies with work output sequestered

We designate the temperatures of the heat reservoirs via subscripts, with T_1 being the temperature of the initial, hottest, reservoir, T_2 the temperature of the second-hottest reservoir, T_3 the temperature of the third-hottest reservoir, etc., and T_n the temperature of the nth, coldest, reservoir.

Let a heat engine operate between two reservoirs, extracting heat Q_1 from a hot reservoir at temperature T_1 and rejecting waste heat to a cold reservoir at temperature T_2. If its efficiency is $\epsilon_{1\to 2}$, its work output is

$$W_{1\to 2} = Q_1 \epsilon_{1\to 2}. \tag{1}$$

It rejects waste heat $Q_1 - W_{1\to 2} = Q_1(1 - \epsilon_{1\to 2})$ to the reservoir at temperature T_2. If there is a third reservoir at temperature T_3 and $W_{1\to 2}$ is sequestered, that is, not frictionally dissipated, and if the efficiency of heat-engine operation between the second and third reservoirs is $\epsilon_{2\to 3}$, a heat engine can then perform additional work

$$W_{2\to 3} = Q_1(1 - \epsilon_{1\to 2})\epsilon_{2\to 3} \tag{2}$$

by employing the reservoir at temperature T_2 as a hot reservoir and the reservoir at temperature T_3 as a cold reservoir. All told it can do work:

$$\begin{aligned} W_{1\to2} + W_{2\to3} &= Q_1\epsilon_{1\to2} + Q_1(1-\epsilon_{1\to2})\epsilon_{2\to3} \\ &= Q_1(\epsilon_{1\to2} + \epsilon_{2\to3} - \epsilon_{1\to2}\epsilon_{2\to3}). \end{aligned} \tag{3}$$

By contrast, if the heat engine operates in a single step at efficiency $\epsilon_{1\to3}$, employing the reservoir at temperature T_1 as a hot reservoir and the reservoir at temperature T_3 as a cold reservoir, it can do work

$$W_{1\to3} = Q_1\epsilon_{1\to3}. \tag{4}$$

Anticipating that we will eventually deal with n heat reservoirs, let us consider efficiencies of the form

$$\epsilon_{i\to j} = 1 - \left(\frac{T_i}{T_j}\right)^x, \tag{5}$$

where i and j are positive integers in the respective ranges $1 \le i \le n-1$ and $i < j \le n$ and where x is a positive real number in the range $0 < x \le 1$. Applying Eqs. (3) and (5), $W_{1\to3} = W_{1\to2} + W_{2\to3}$, as we will now show. We have

$$\begin{aligned} W_{1\to2} + W_{2\to3} &= Q_1\left\{\begin{array}{c} \left[1-\left(\frac{T_2}{T_1}\right)^x\right] + \left[1-\left(\frac{T_3}{T_2}\right)^x\right] \\ -\left[1-\left(\frac{T_2}{T_1}\right)^x\right]\left[1-\left(\frac{T_3}{T_2}\right)^x\right] \end{array}\right\} \\ &= Q_1\left\{\begin{array}{c} 2-\left(\frac{T_2}{T_1}\right)^x-\left(\frac{T_3}{T_2}\right)^x \\ -\left[1-\left(\frac{T_2}{T_1}\right)^x-\left(\frac{T_3}{T_2}\right)^x+\left(\frac{T_2}{T_1}\right)^x\left(\frac{T_3}{T_2}\right)^x\right] \end{array}\right\} \\ &= Q_1\left[1-\left(\frac{T_2}{T_1}\right)^x\left(\frac{T_3}{T_2}\right)^x\right] \\ &= Q_1\left[1-\left(\frac{T_3}{T_1}\right)^x\right] = W_{1\to3.} \end{aligned} \tag{6}$$

We note that $x = 1$ for the Carnot, Ericsson, Stirling, air-standard Otto, and air-standard Brayton cycles [1–9] and $x = 1/2$ for endoreversible heat engines operating at Curzon-Ahlborn efficiency [10–12] (see also Ref. [4], Section 4-9). For all of these cycles, the temperature in the numerator is that of the coldest available reservoir for a given cycle [1–12]. For the Carnot, Ericsson, and Stirling cycles, and for endoreversible heat engines operating at Curzon-Ahlborn efficiency, the temperature in the denominator is that of the hottest available reservoir for a given cycle [1–12]. For the air-standard Otto and air-standard Brayton cycles, the temperature in the denominator is that at the end of the adiabatic-compression process but before the addition of heat from the hottest available reservoir (substituting, in air-standard cycles, for combustion of fuel) [2–9] in a given cycle. The Second Law of Thermodynamics forbids $x > 1$ if the temperature in the numerator is that of the coldest available reservoir for a given cycle and the temperature in the denominator is that of the hottest available reservoir for a given cycle, because then the Carnot efficiency would be exceeded. Since for the aforementioned heat engines, and indeed for any heat engine for which Eq. (5) is applicable, $W_{1\to3} = W_{1\to2} + W_{2\to3}$, this additivity of W obtains for any number of steps, that is, we have

$$W_{1\to n} = W_{1\to2} + W_{2\to3} + \ldots + W_{n-1\to n} = \sum_{j=1}^{n-1} W_{j\to j+1.} \tag{7}$$

For more complex efficiencies than those of Eq. (5), for example, those of the Diesel and dual cycles, which are functions of more than two temperatures, and also for some more complex efficiencies that are functions of two temperatures, the equality of Eq. (7) may not always obtain [3–9, 13–19]. But whether or not the equality of Eq. (7) obtains, the Second Law of Thermodynamics requires that, whichever reservoirs are employed, the efficiency with all work outputs sequestered, whether $W_{j\to j+1}/Q_j$ $(1\le j\le n-1)$, $W_{j\to j+k}/Q_j$ $(1\le j\le n-1$ and $1\le k\le n-j)$, or $W_{1\to n}/Q_1$, cannot exceed the Carnot limit.

3. Multiple-reservoir heat-engine efficiencies with work output totally frictionally dissipated

Let a heat engine operate between two reservoirs, extracting heat Q_1 from a hot reservoir at temperature T_1 and rejecting waste heat to a cold reservoir at temperature T_2. If its efficiency is $\epsilon_{1\to 2}$, its work output is

$$W_{1\to 2} = Q_1\epsilon_{1\to 2}. \tag{8}$$

It rejects waste heat $Q_1 - W_{1\to 2} = Q_1(1-\epsilon_{1\to 2})$ to a reservoir at temperature T_2. But now, in addition, we let the work output $W^D_{1\to 2} = Q_1\epsilon_{1\to 2}$ be totally frictionally dissipated and rejected into the reservoir at temperature T_2 (indicated via a superscript D). This is in fact by far the most common mode of heat-engine operation. With rare exceptions (e.g., a heat engine's work output being sequestered for a long time interval as gravitational potential energy in the construction of a building, or essentially permanently in the launching of a spacecraft), heat engines' work outputs are typically totally frictionally dissipated immediately or on short time scales (see Ref. [6], Chapter VI (especially Sections 54, 60, and 61); and Ref. [7], Sections 6.9–6.14 and 16.8). Indeed, this is true of almost all engines, heat engines or otherwise. The work outputs of all engines of vehicles (automobiles, trains, ships, submarines, aircraft, etc.) operating at constant speed, and of all factory and appliance engines operating at constant speed, are immediately and continually frictionally dissipated. The work output temporarily sequestered as kinetic energy when a vehicle accelerates, or when a factory or appliance engine is turned on, is frictionally dissipated a short time later when the vehicle decelerates, or when the factory or appliance engine is turned off.

If both the waste heat $Q_1 - W^D_{1\to 2} = Q_1(1-\epsilon_{1\to 2})$ has been rejected and the work output $W^D_{1\to 2} = Q_1\epsilon_{1\to 2}$ has been totally frictionally dissipated into the reservoir at temperature T_2, and there is a third reservoir at temperature T_3, a heat engine operating at efficiency $\epsilon_{2\to 3}$ can then perform additional work

$$W_{2\to 3} = Q_1\epsilon_{2\to 3} \tag{9}$$

by employing the reservoir at temperature T_2 as a hot reservoir and the reservoir at temperature T_3 as a cold reservoir. ($W^D_{2\to 3}$ may or may not be frictionally dissipated, so it only optionally carries the superscript D.) All told the total work output is

$$\begin{aligned} W^D_{1\to 3} &= W^D_{1\to 2} + W^D_{2\to 3} = Q_1\epsilon_{1\to 2} + Q_1\epsilon_{2\to 3} \\ &= Q_1(\epsilon_{1\to 2} + \epsilon_{2\to 3}). \end{aligned} \tag{10}$$

If $\epsilon_{i\to j} = 1 - \left(T_i/T_j\right)^x$, where i and j are positive integers in the respective ranges $1\le i\le n-1$ and $i<j\le n$, and where x is a positive real number in the range $0<x\le 1$, applying Eqs. (5) and (10), we have:

$$\begin{aligned} W^D_{1\to 3} = W^D_{1\to 2} + W^D_{2\to 3} &= Q_1\left\{\left[1-\left(\frac{T_2}{T_1}\right)^x\right]+\left[1-\left(\frac{T_3}{T_2}\right)^x\right]\right\} \\ &= Q_1\left[2-\left(\frac{T_2}{T_1}\right)^x-\left(\frac{T_3}{T_2}\right)^x\right]. \end{aligned} \tag{11}$$

We now maximize $W^D_{1\to 3}$ with respect to T_2:

$$\begin{aligned} \frac{dW^D_{1\to 3}}{dT_2} = 0 &\Rightarrow \frac{d}{dT_2}\left[2-\left(\frac{T_2}{T_1}\right)^x-\left(\frac{T_3}{T_2}\right)^x\right]=0 \\ &\Rightarrow \frac{d}{dT_2}\left(\frac{T_2}{T_1}+\frac{T_3}{T_2}\right)=0 \\ &\Rightarrow \frac{1}{T_1}-\frac{T_3}{T_2^2}=0 \\ &\Rightarrow T_{2,\mathrm{opt}}=(T_1T_3)^{1/2}. \end{aligned} \tag{12}$$

Thus, the optimum value T_2,opt of T_2, which maximizes $W^D_{1\to 3}$, is the geometric mean of T_1 and T_3. Applying Eqs. (11) and (12), the maximum value $W^D_{1\to 3,\max}$ of $W^D_{1\to 3}$ is

$$\begin{aligned} W^D_{1\to 3,\max} &= Q_1\left\{2-\left[\frac{(T_1T_3)^{1/2}}{T_1}\right]^x-\left[\frac{T_3}{(T_1T_3)^{1/2}}\right]^x\right\} \\ &= Q_1\left[2-\left(\frac{T_3}{T_1}\right)^{x/2}-\left(\frac{T_3}{T_1}\right)^{x/2}\right] \\ &= Q_1\left[2-2\left(\frac{T_3}{T_1}\right)^{x/2}\right] \\ &= 2Q_1\left[1-\left(\frac{T_3}{T_1}\right)^{x/2}\right]. \end{aligned} \tag{13}$$

Note that

$$W^D_{1\to 3,\max} > Q_1 \ \text{ if } \ \left(\frac{T_3}{T_1}\right)^{x/2} < \frac{1}{2} \Leftrightarrow \frac{T_3}{T_1} < \frac{1}{2^{2/x}}. \tag{14}$$

This obtains if $T_3/T_1 < 1/4$ for $x = 1$ and if $T_3/T_1 < 1/16$ for $x = 1/2$. Also, comparing the last line of Eq. (6) with Eq. (13), we find for the maximum extra work $W^{D,\mathrm{extra}}_{1\to 3,\max}$:

$$\begin{aligned} W^{D,\mathrm{extra}}_{1\to 3,\max} &= W^D_{1\to 3,\max} - W_{1\to 3} \\ &= 2Q_1\left[1-\left(\frac{T_3}{T_1}\right)^{x/2}\right]-Q_1\left[1-\left(\frac{T_3}{T_1}\right)^x\right] \\ &= Q_1\left\{2\left[1-\left(\frac{T_3}{T_1}\right)^{x/2}\right]-\left[1-\left(\frac{T_3}{T_1}\right)^x\right]\right\} \\ &= Q_1\left[2-2\left(\frac{T_3}{T_1}\right)^{x/2}-1+\left(\frac{T_3}{T_1}\right)^x\right] \\ &= Q_1\left[1+\left(\frac{T_3}{T_1}\right)^x-2\left(\frac{T_3}{T_1}\right)^{x/2}\right] \geq 0. \end{aligned} \tag{15}$$

It is easily shown that $W^{D,\text{extra}}_{1\to 3,\max} \geq 0$, with the equality obtaining if and only if $\frac{T_3}{T_1} = 1 \Rightarrow W^{D}_{1\to 3,\max} = W_{1\to 3} = 0 \Rightarrow W^{D}_{1\to 3,\max} - W_{1\to 3} = W^{D,\text{extra}}_{1\to 3,\max} = 0$. For, denoting the ratio $\left(\frac{T_3}{T_1}\right)^{x/2}$ as r and setting $dW^{D,\text{extra}}_{1\to 3,\max}/dr = 0$ yields

$$\begin{aligned}\frac{dW^{D,\text{extra}}_{1\to 3,\max}}{dr} = 0 &\Rightarrow \frac{d}{dr}\left(r^2 - 2r\right) = 0\\ &\Rightarrow 2r - 2 = 0\\ &\Rightarrow r = 1.\end{aligned} \tag{16}$$

Thus $W^{D}_{1\to 3,\max}$ is minimized at 0 if $r = \left(\frac{T_3}{T_1}\right)^{x/2} = 1 \Rightarrow \frac{T_3}{T_1} = 1$. For all $\frac{T_3}{T_1} < 1, \;\; W^{D}_{1\to 3,\text{extra}} > 0$. Moreover, applying Eqs. (5), (13), and (15), note that

$$\begin{aligned}&\lim_{T_3/T_1\to 0} W^{D}_{1\to 3,\max} = 2Q_1 = 2\lim_{T_3/T_1\to 0} W_{1\to 3}\\ \Rightarrow\; &\lim_{T_3/T_1\to 0} W^{D,\text{extra}}_{1\to 3,\max} = 2Q_1 - Q_1 = Q_1 = \lim_{T_3/T_1\to 0} W_{1\to 3}.\end{aligned} \tag{17}$$

Now consider heat-engine operation employing four heat reservoirs, with all work totally frictionally dissipated (except possibly at the last step; thus, $W^{D}_{3\to 4}$ only optionally carries the superscript D). Thus we have

$$\begin{aligned}W^{D}_{1\to 4} &= W^{D}_{1\to 2} + W^{D}_{2\to 3} + W^{D}_{3\to 4} = Q_1\epsilon_{1\to 2} + Q_1\epsilon_{2\to 3} + Q_1\epsilon_{3\to 4}\\ &= Q_1(\epsilon_{1\to 2} + \epsilon_{2\to 3} + \epsilon_{3\to 4}).\end{aligned} \tag{18}$$

If $\epsilon_{i\to j} = 1 - \left(T_i/T_j\right)^x$, where i and j are positive integers in the respective ranges $1 \leq i \leq n-1$ and $i < j \leq n$, and where x is a positive real number in the range $0 < x \leq 1$, applying Eqs. (5) and (18), we have:

$$\begin{aligned}W^{D}_{1\to 4} &= W^{D}_{1\to 2} + W^{D}_{2\to 3} + W^{D}_{3\to 4}\\ &= Q_1\left\{\left[1 - \left(\frac{T_2}{T_1}\right)^x\right] + \left[1 - \left(\frac{T_3}{T_2}\right)^x\right] + \left[1 - \left(\frac{T_4}{T_3}\right)^x\right]\right\}\\ &= Q_1\left[3 - \left(\frac{T_2}{T_1}\right)^x - \left(\frac{T_3}{T_2}\right)^x - \left(\frac{T_4}{T_3}\right)^x\right].\end{aligned} \tag{19}$$

We wish to maximize $W^{D}_{1\to 4}$. Based on Eq. (12) and the associated discussions, the optimum value $T_{j,\text{opt}}$ of T_j of reservoir j ($1 < j < n \Leftrightarrow 2 \leq j \leq n-1$), which maximizes $W^{D}_{j-1\to j+1}$, is the geometric mean of T_{j-1} and T_{j+1}. Thus we have

$$T_{2,\text{opt}} = \left(T_1 T_{3,\text{opt}}\right)^{1/2} \tag{20}$$

and

$$T_{3,\text{opt}} = \left(T_{2,\text{opt}} T_4\right)^{1/2}. \tag{21}$$

Applying Eqs. (20) and (21), we obtain

$$\frac{T_{2,\text{opt}}}{T_1} = \frac{\left(T_1 T_{3,\text{opt}}\right)^{1/2}}{T_1} = \left(\frac{T_{3,\text{opt}}}{T_1}\right)^{1/2} \tag{22}$$

and

$$\frac{T_4}{T_{3,\text{opt}}} = \frac{T_4}{\left(T_{2,\text{opt}} T_4\right)^{1/2}} = \left(\frac{T_4}{T_{2,\text{opt}}}\right)^{1/2}. \tag{23}$$

Applying Eqs. (20)–(23), we obtain

$$\begin{aligned}
\frac{T_{3,\text{opt}}}{T_{2,\text{opt}}} &= \frac{T_{3,\text{opt}}}{\left(T_1 T_{3,\text{opt}}\right)^{1/2}} = \left(\frac{T_{3,\text{opt}}}{T_1}\right)^{1/2} \\
&= \frac{\left(T_{2,\text{opt}} T_4\right)^{1/2}}{T_{2,\text{opt}}} = \left(\frac{T_4}{T_{2,\text{opt}}}\right)^{1/2} \\
&\Rightarrow \left(\frac{T_{3,\text{opt}}}{T_1}\right)^{1/2} = \left(\frac{T_4}{T_{2,\text{opt}}}\right)^{1/2} \\
&\Rightarrow \frac{T_{2,\text{opt}}}{T_1} = \frac{T_{3,\text{opt}}}{T_{2,\text{opt}}} = \frac{T_4}{T_{3,\text{opt}}}.
\end{aligned} \tag{24}$$

Applying Eqs. (22)–(24), we obtain

$$\begin{aligned}
\frac{T_4}{T_1} &= \frac{T_2}{T_1}\frac{T_3}{T_2}\frac{T_4}{T_3} \text{ in general} \\
&= \frac{T_{2,\text{opt}}}{T_1}\frac{T_{3,\text{opt}}}{T_{2,\text{opt}}}\frac{T_4}{T_{3,\text{opt}}} \text{ in particular} \\
&= \left(\frac{T_{2,\text{opt}}}{T_1}\right)^3 = \left(\frac{T_{3,\text{opt}}}{T_{2,\text{opt}}}\right)^3 = \left(\frac{T_4}{T_{3,\text{opt}}}\right)^3 \\
&\Rightarrow \frac{T_{2,\text{opt}}}{T_1} = \frac{T_{3,\text{opt}}}{T_{2,\text{opt}}} = \frac{T_4}{T_{3,\text{opt}}} = \left(\frac{T_4}{T_1}\right)^{1/3}.
\end{aligned} \tag{25}$$

Applying Eqs. (19) and (25), we obtain

$$\begin{aligned}
W^D_{1\to 4,\max} &= Q_1\left[3 - 3\left(\frac{T_4}{T_1}\right)^{x/3}\right] \\
&= 3Q_1\left[1 - \left(\frac{T_4}{T_1}\right)^{x/3}\right].
\end{aligned} \tag{26}$$

We now slightly modify Eqs. (14)–(17) to apply for our four-reservoir system. We obtain

$$W^D_{1\to 4,\max} > Q_1 \text{ if } \left(\frac{T_4}{T_1}\right)^{x/3} < \frac{2}{3} \Leftrightarrow \frac{T_4}{T_1} < \left(\frac{2}{3}\right)^{3/x}. \tag{27}$$

This obtains if $T_4/T_1 < (2/3)^3 = 8/27$ for $x = 1$ and if $T_4/T_1 < (2/3)^6 = 64/729$ for $x = 1/2$. Also, applying Eqs. (5) and (26),

$$\begin{aligned} W^{D,\text{extra}}_{1\to4,\max} &= W^{D}_{1\to4,\max} - W_{1\to4} \\ &= 3Q_1\left[1-\left(\frac{T_4}{T_1}\right)^{x/3}\right] - Q_1\left[1-\left(\frac{T_4}{T_1}\right)^{x}\right] \\ &= Q_1\left\{3\left[1-\left(\frac{T_4}{T_1}\right)^{x/3}\right] - \left[1-\left(\frac{T_4}{T_1}\right)^{x}\right]\right\} \\ &= Q_1\left[3-3\left(\frac{T_4}{T_1}\right)^{x/3} - 1 + \left(\frac{T_4}{T_1}\right)^{x}\right] \\ &= Q_1\left[2+\left(\frac{T_4}{T_1}\right)^{x} - 3\left(\frac{T_4}{T_1}\right)^{x/3}\right] \geq 0. \end{aligned} \tag{28}$$

It is easily shown that $W^{D,\text{extra}}_{1\to4,\max} \geq 0$, with the equality obtaining if and only if $\frac{T_4}{T_1} = 1 \Rightarrow W^{D}_{1\to4,\max} = W_{1\to4} = 0 \Rightarrow W^{D}_{1\to4,\max} - W_{1\to4} = W^{D,\text{extra}}_{1\to4,\max} = 0$. For, denoting the ratio $\left(\frac{T_4}{T_1}\right)^{x/3}$ as r and setting $dW^{D,\text{extra}}_{1\to4,\max}/dr = 0$ yields

$$\begin{aligned} \frac{dW^{D,\text{extra}}_{1\to4,\max}}{dr} = 0 &\Rightarrow \frac{d}{dr}\left(r^3 - 3r\right) = 0 \\ &\Rightarrow 3r^2 - 3 = 0 \\ &\Rightarrow r^2 = 1 \\ &\Rightarrow r = 1. \end{aligned} \tag{29}$$

Thus $W^{D,\text{extra}}_{1\to4,\max}$ is minimized at 0 if $r = \left(\frac{T_4}{T_1}\right)^{x/3} = 1 \Rightarrow \frac{T_4}{T_1} = 1$. For all $\frac{T_4}{T_1} < 1,\ \ W^{D,\text{extra}}_{1\to4,\max} > 0$. Moreover, applying Eqs. (5), (26), and (28), note that

$$\begin{aligned} &\lim_{T_4/T_1\to0} W^{D}_{1\to4,\max} = 3Q_1 = 3\lim_{T_4/T_1\to0} W_{1\to4} \\ \Rightarrow\ &\lim_{T_4/T_1\to0} W^{D,\text{extra}}_{1\to4,\max} = 3Q_1 - Q_1 = 2Q_1 = 2\lim_{T_4/T_1\to0} W_{1\to4}. \end{aligned} \tag{30}$$

Comparing Eqs. (13)–(17) with Eqs. (26)–(30), note the larger values in Eqs. (26), (28), and (30) than in Eqs. (13), (15), and (17), respectively, and the easier fulfillment of the inequality in Eq. (27) than in Eq. (14) (concerning the latter point: $8/27 > 1/4$ and $64/729 > 1/16$).

Generalizing Eqs. (20)–(30) for an n-reservoir system (n = any positive integer ≥ 4), we obtain:

$$T_{j+1} = \left(T_j T_{j+2}\right)^{1/2}, \tag{31}$$

where j is any positive integer in the range $1 \leq j \leq n-2$ and

$$T_{j+2} = \left(T_{j+1} T_{j+3}\right)^{1/2}, \tag{32}$$

where j is any positive integer in the range $1 \leq j \leq n-3$. The respective temperatures T_1 and T_n of the extreme (hottest and coldest) reservoirs are assumed to be fixed. The temperatures T_2 through T_{n-1} of all intermediate reservoirs are all assumed to be optimized in accordance with Eqs. (31) and (32). With that understood, for brevity and to avoid using different subscripts for the extreme and intermediate reservoirs, the subscript "opt" is omitted in Eqs. (31)–(35). Applying Eqs. (31) and (32), we obtain:

$$\frac{T_{j+1}}{T_j} = \frac{\left(T_j T_{j+2}\right)^{1/2}}{T_j} = \left(\frac{T_{j+2}}{T_j}\right)^{1/2} \tag{33}$$

and

$$\frac{T_{j+2}}{T_{j+1}} = \frac{T_{j+2}}{\left(T_j T_{j+2}\right)^{1/2}} = \left(\frac{T_{j+2}}{T_j}\right)^{1/2}. \tag{34}$$

Applying Eqs. (33) and (34), and recognizing that Eqs. (33) and (34) obtain for *all* values of j such that j is any positive integer in the range $1 \le j \le n-2$, we obtain:

$$\begin{gathered} \frac{T_{j+2}}{T_{j+1}} = \frac{T_{j+1}}{T_j} \\ \Rightarrow \frac{T_{j+2}}{T_j} = \frac{T_{j+1}}{T_j}\frac{T_{j+2}}{T_{j+1}} = \left(\frac{T_{j+1}}{T_j}\right)^2 \Leftrightarrow \frac{T_{j+1}}{T_j} = \left(\frac{T_{j+2}}{T_j}\right)^{1/2} \\ \Leftrightarrow \frac{T_n}{T_1} = \left(\frac{T_{j+1}}{T_j}\right)^{n-1} \Leftrightarrow \frac{T_{j+1}}{T_j} = \left(\frac{T_n}{T_1}\right)^{1/(n-1)}. \end{gathered} \tag{35}$$

The first two lines of Eq. (35) obtain for all values of j such that j is any positive integer in the range $1 \le j \le n-2$, and the third line of Eq. (35) obtain for all values of j such that j is any positive integer in the range $1 \le j \le n-1$. The first two lines of Eq. (35) pertain to any three adjacent heat reservoirs, and hence 2 appears in the exponents of the second line thereof; the third line of Eq. (35) pertains to all n heat reservoirs, and hence $n-1$ appears in the exponents thereof. The second and third lines of Eq. (35) mutually justify each other: the third line of Eq. (35) *must* obtain because the second line thereof obtains for *all* values of j; and, conversely, given that the third line of Eq. (35) obtains, the second line thereof *must* obtain for *all* values of j.

If, as per Eq. (5), $\epsilon_{i\to j} = 1 - \left(T_i/T_j\right)^x$, where i and j are positive integers in the respective ranges $1 \le i \le n-1$ and $i < j \le n$, and where x is a positive real number in the range $0 < x \le 1$, then, applying Eqs. (5) and (31)–(35), we now generalize Eqs. (13)–(17) and (26)–(30), as well as the associated discussions, to apply for our n-reservoir system. We obtain:

$$W^D_{1\to n,\max} = (n-1)\,Q_1\left[1 - \left(\frac{T_n}{T_1}\right)^{x/(n-1)}\right], \tag{36}$$

$$W^D_{1\to n,\max} > Q_1 \text{ if } \left(\frac{T_n}{T_1}\right)^{x/(n-1)} < \frac{n-2}{n-1} \Leftrightarrow \frac{T_n}{T_1} < \left(\frac{n-2}{n-1}\right)^{(n-1)/x}, \tag{37}$$

and

$$
\begin{aligned}
W_{1\to n,\max}^{D,\text{extra}} &= W_{1\to n,\max}^{D} - W_{1\to n} \\
&= (n-1)\,Q_1\left[1-\left(\frac{T_n}{T_1}\right)^{x/(n-1)}\right] - Q_1\left[1-\left(\frac{T_n}{T_1}\right)^{x}\right] \\
&= Q_1\left\{(n-1)\left[1-\left(\frac{T_n}{T_1}\right)^{x/(n-1)}\right] - \left[1-\left(\frac{T_n}{T_1}\right)^{x}\right]\right\} \\
&= Q_1\left[n-1-(n-1)\left(\frac{T_n}{T_1}\right)^{x/(n-1)} - 1 + \left(\frac{T_n}{T_1}\right)^{x}\right] \\
&= Q_1\left[n-2+\left(\frac{T_n}{T_1}\right)^{x} - (n-1)\left(\frac{T_n}{T_1}\right)^{x/(n-1)}\right] \geq 0.
\end{aligned} \tag{38}
$$

It is easily shown that $W_{1\to n,\max}^{D,\text{extra}} \geq 0$, with the equality obtaining if and only if $\frac{T_n}{T_1} = 1 \Rightarrow W_{1\to n,\max}^{D} = W_{1\to n} = 0 \Rightarrow W_{1\to n,\max}^{D} - W_{1\to n} = W_{1\to n,\max}^{D,\text{extra}} = 0$. For, denoting the ratio $\left(\frac{T_n}{T_1}\right)^{x/(n-1)}$ as r and setting $dW_{1\to n,\max}^{D,\text{extra}}/dr = 0$ yields

$$
\begin{aligned}
\frac{dW_{1\to n,\max}^{D,\text{extra}}}{dr} = 0 &\Rightarrow \frac{d}{dr}\left[r^{n-1} - (n-1)r\right] = 0 \\
&\Rightarrow (n-1)r^{n-2} - (n-1) = 0 \\
&\Rightarrow r^{n-2} = 1 \\
&\Rightarrow r = 1.
\end{aligned} \tag{39}
$$

Thus $W_{1\to n,\max}^{D,\text{extra}}$ is minimized at 0 if $r = \left(\frac{T_n}{T_1}\right)^{x/(n-1)} = 1 \Rightarrow \frac{T_n}{T_1} = 1$. For all $\frac{T_n}{T_1} < 1$, $W_{1\to n,\max}^{D,\text{extra}} > 0$. Moreover, applying Eqs. (5), (36), and (38), note that

$$
\begin{aligned}
&\lim_{T_n/T_1\to 0,\, n \text{ fixed}} W_{1\to n,\max}^{D} = (n-1)Q_1 = (n-1)\lim_{T_n/T_1\to 0,\, n \text{ fixed}} W_{1\to n} \\
\Rightarrow &\lim_{T_n/T_1\to 0,\, n \text{ fixed}} W_{1\to n,\max}^{D,\text{extra}} = \lim_{T_n/T_1\to 0,\, n \text{ fixed}} \left(W_{1\to n,\max}^{D} - W_{1\to n}\right) \\
&= (n-1)Q_1 - Q_1 = (n-2)Q_1 = (n-2)\lim_{T_n/T_1\to 0,\, n \text{ fixed}} W_{1\to n}.
\end{aligned} \tag{40}
$$

Note that the values in Eqs. (36), (38), and (40) increase monotonically with increasing n and that the fulfillment of the inequality in Eq. (37) becomes monotonically easier with increasing n. Equation (40) is valid not only for Carnot efficiency ($x = 1$) but even for Curzon-Ahlborn efficiency ($x = 1/2$), indeed for any x finitely greater than 0 in the range $0 < x \leq 1$, because $\left(\frac{T_n}{T_1}\right)^{x/(n-1)} \to 0 \Leftrightarrow 1 - \left(\frac{T_n}{T_1}\right)^{x/(n-1)} \to 1$ in the limit $T_n/T_1 \to 0$, albeit ever more slowly with decreasing x.

By contrast, even granting Carnot efficiency ($x = 1$) [22]:

$$
\lim_{n\to\infty,\, T_n/T_1 \text{ fixed}} W_{1\to n,\max}^{D} = Q_1 \ln\frac{T_1}{T_n} = \left(\lim_{T_n/T_1\to 0,\, n \text{ fixed}} W_{1\to n}\right)\ln\frac{T_1}{T_n}. \tag{41}
$$

Note the *linear* divergence of $W^D_{1\to n,\max}$ in the limit $T_n/T_1 \to 0$ with n fixed as per Eq. (40) even *not* assuming Carnot efficiency, as contrasted with the paltry *logarithmic* divergence of $W^D_{1\to n,\max}$ in the limit $n \to \infty$ with T_n/T_1 fixed even granting Carnot efficiency as per the derivation [22] of Eq. (41).

But we note that the temperature of the cosmic background radiation is only 2.7 K, while the most refractory materials remain solid at temperatures slightly exceeding 2700 K. This provides a temperature ratio of $T_1/T_n \approx 10^3 \Leftrightarrow T_n/T_1 \approx 10^{-3}$. Could even larger values of T_1/T_n be possible, at least in principle? Perhaps, maybe, if frictional dissipation of work into heat might somehow be possible into a gaseous hot reservoir at temperatures exceeding the melting point or even the critical temperature (the maximum boiling point at any pressure) of even the most refractory material. Yet even with the paltry *logarithmic* divergence of $W^D_{1\to n,\max}$ in the limit $n \to \infty$ with T_1/T_n fixed as per Eq. (41) and even with a temperature ratio of $T_1/T_n \approx 10^3 \Leftrightarrow T_n/T_1 \approx 10^{-3}$, assuming Carnot efficiency by Eq. (41) $W^D_{1\to n,\max}/Q_1 \approx \ln 10^3 \approx 7$. Hence by Eq. (41) an advanced civilization employing 7 concentric Dyson spheres [39, 40] can procure 7 times as much work output (to the nearest whole number) as its host star's total energy output. Actually the limit $n \to \infty$ with T_1/T_n fixed is not sufficiently closely approached to apply Eq. (41): we should instead apply Eq. (36). Applying Eq. (36) and assuming Carnot efficiency with $T_1/T_n \approx 10^3 \Leftrightarrow T_n/T_1 \approx 10^{-3}$, $W^D_{1\to n,\max}/Q_1 \approx 4$. Hence by Eq. (36) an advanced civilization employing 4 concentric Dyson spheres [39, 40] can procure 4 times as much work output (to the nearest whole number) as its host star's total energy output.

4. General remarks, especially concerning entropy

It is important to emphasize that the super-unity cyclic-heat-engine efficiencies $W^D_{1\to n,\max}/Q_1$ that can obtain with work output totally frictionally dissipated (if $n \ge 3$) are consistent with both the First and Second Laws of Thermodynamics. The two laws are *not* violated because, if the work output of a heat engine is frictionally dissipated as heat into a cooler reservoir, both laws allow this heat to be partially converted to work again if another, still cooler, reservoir is available.

In this Section 4 we do not restrict heat-engine efficiencies to the form given by Equation (5), nor necessarily assume efficiencies of the same form at each step $j \to j+1$ or $j \to j+k$ $(1 \le k \le n-j)$. The validity of this Section 4 requires only that the efficiency with all work sequestered, or at any one given step $j \to j+1$ whether work is sequestered or not, be within the Carnot limit, in accordance with the Second Law.

The extra work that is made available via frictional dissipation into cooler reservoirs is paid for by an extra increase in entropy. Consider the work available via heat-engine operation between reservoir j at temperature T_j and reservoir $j+2$ at temperature T_{j+2} without versus with frictional dissipation into reservoir $j+1$ at temperature $T_{j+1}(T_j > T_{j+1} > T_{j+2})$. Without frictional dissipation a heat engine performs work

$$W_{j\to j+1} = Q_j \epsilon_{j\to j+1} \tag{42}$$

by employing the reservoir at temperature T_j as a hot reservoir and the reservoir at temperature T_{j+1} as a cold reservoir. It rejects waste heat $Q_j - W_{j\to j+1} = Q_j(1-\epsilon_{j\to j+1})$ to the reservoir at temperature T_{j+1}. If a third reservoir at temperature T_{j+2} and $W_{j\to j+1}$ is sequestered, that is, not frictionally dissipated, a heat engine can then perform additional work:

$$W_{j+1\to j+2} = Q_j(1-\epsilon_{j\to j+1})\epsilon_{j+1\to j+2} \tag{43}$$

by employing the reservoir at temperature T_{j+1} as a hot reservoir and the reservoir at temperature T_{j+2} as a cold reservoir. All told it can do work:

$$\begin{aligned} W_{j\to j+2} &= W_{j\to j+1} + W_{j+1\to j+2} = Q_j\epsilon_{j\to j+1} + Q_j(1-\epsilon_{j\to j+1})\epsilon_{j+1\to j+2} \\ &= Q_j(\epsilon_{j\to j+1} + \epsilon_{j+1\to j+2} - \epsilon_{j\to j+1}\epsilon_{j+1\to j+2}). \end{aligned} \tag{44}$$

With total frictional dissipation of $W_{j\to j+1}$ into reservoir $j+1$ at temperature T_{j+1}, we still have

$$W^D_{j\to j+1} = W_{j\to j+1} = Q_1\epsilon_{j\to j+1}. \tag{45}$$

But now we let the work output $W^D_{j\to j=1} = Q_1\epsilon_{j+1\to j+2}$ be totally frictionally dissipated into the reservoir at temperature T_{j+1} (indicated via a superscript D). If there is a third reservoir at temperature T_{j+2}, a heat engine can then perform additional work:

$$W^D_{j+1\to j+2} = Q_1\epsilon_{j+1\to j+2}. \tag{46}$$

All told it can do work:

$$\begin{aligned} W^D_{j\to j+2} &= W^D_{j\to j+1} + W^D_{j+1\to j+2} = Q_j\epsilon_{j\to j+1} + Q_j\epsilon_{j+1\to j+2} \\ &= Q_j(\epsilon_{j\to j+1} + \epsilon_{j+1\to j+2}). \end{aligned} \tag{47}$$

The extra work

$$\begin{aligned} W^D_{\text{extra}} &= W^D_{j+1\to j+2} \\ &= Q_j\epsilon_{j\to j+1}\epsilon_{j+1\to j+2} \\ &= W_{j\to j+1}\epsilon_{j+1\to j+2} \\ &= W^D_{j\to j+1}\epsilon_{j+1\to j+2} \end{aligned} \tag{48}$$

is paid for by the extra increase in entropy owing to frictional dissipation into extra heat Q^D_{extra} of the work output as per Eqs. (42) and (45)

$$Q^D_{\text{extra}} = W_{j\to j+1} = W^D_{j\to j+1} = Q_j\epsilon_{j\to j+1} \tag{49}$$

into reservoir $j+1$ at temperature T_{j+1}. This extra increase in entropy is

$$\Delta S^D_{\text{extra}} = \frac{Q^D_{\text{extra}}}{T_{j+1}} = \frac{Q_j\epsilon_{j\to j+1}}{T_{j+1}} = \frac{W_{j\to j+1}}{T_{j+1}} = \frac{W^D_{j\to j+1}}{T_{j+1}} = \frac{W^D_{\text{extra}}}{\epsilon_{j+1\to j+2}T_{j+1}}. \tag{50}$$

[In the last four steps of Eq. (50), we applied Eqs. (42), (45), (48), and (49).] Thus

$$W^D_{\text{extra}} = T_{j+1}\Delta S^D_{\text{extra}}\epsilon_{j+1\to j+2}. \tag{51}$$

In no case do we assume an efficiency with all work sequestered, or at any one given step $j \to j + 1$ whether work is sequestered or not, exceeding the Carnot efficiency, and hence we are within the restrictions of the Second Law. (The First Law, of course, puts no restrictions whatsoever on the recycling of energy, except that it is conserved—and we *never* violate conservation of energy.)

We note that, while frictional dissipation of work into intermediate reservoirs can yield extra work W^D_{extra} in heat-engine operation (albeit at the expense of $\Delta S^D_{\text{extra}}$), it seems to be of no help in reverse, that is, refrigerator or heat pump, operation. For, in refrigerator or heat pump operation, with an intermediate reservoir $j+1$ at temperature T_{j+1}, $Q_{j+2} + W_{j+2\to j+1} = Q_{j+1}$, $Q_{j+1} + W_{j+1\to j} = Q_j$, hence $Q_{j+2} + W_{j+2\to j+1} + W_{j+1\to j} = Q_{j+2} + W_{j+2\to j} = Q_j$. Without an intermediate reservoir $j+1$ at temperature T_{j+1}, $Q_{j+2} + W_{j+2\to j} = Q_j$. The bottom line $Q_{j+2} + W_{j+2\to j} = Q_j$ is identical with or without an intermediate reservoir $j+1$ at temperature T_{j+1}. With or without the intermediate reservoir $j+1$ at temperature T_{j+1}, *all* of the energy must end up as Q_j; thus, there is *none* left over to be frictionally dissipated. Hence the presence or absence of this intermediate reservoir makes no difference with respect to reverse, that is, refrigerator or heat pump, operation: See Ref. [1], Section 20-3; Ref. [2], Section 5.12 and Problem 5.22; Ref. [3], Sections 4.3, 4.4, and 4.7 (especially Section 4.7); Ref. [4], Sections 4-4, 4-5, and 4-6 (especially Section 4-6); Ref. [5], Sections 5-7-2, 6-2-2, 6-9-2, and 6-9-3, and Chapter 17; Ref. [6], Chapter XXI; Ref. [7], Sections 6.7, 6.8, 7.3, and 7.4); and Ref. [9], pp. 233–236 and Problems 1, 2, 4, 6, and 7 of Chapter 8. [Problem 2 of Chapter 8 in Ref. [9] considers absorption refrigeration, wherein the entire energy output is into an intermediate-temperature (most typically ambient-temperature) reservoir, and hence for which also there is *no* energy left over to be frictionally dissipated.]

5. Conclusion

We investigated the increased heat-engine efficiencies obtained via operation employing increasing numbers (≥ 3) of heat reservoirs and with work output totally frictionally dissipated into all reservoirs except the first, hottest, one at temperature T_1 and (possibly) also the last, coldest, one at temperature T_n. We emphasize again that our results are consistent with both the First and Second Laws of Thermodynamics. The two laws are *not* violated because, if the work output of a heat engine is frictionally dissipated as heat into a cooler reservoir, both laws allow this heat to be partially converted to work again if another, still cooler, reservoir is available.

We do, however, challenge an *over*statement of the Second Law that is sometimes made, namely, that energy can do work only once. Energy can indeed do work more than once, because the Second Law does not forbid recycling of energy, so long as total entropy does not decrease as a result. This criterion of non-decrease of total entropy *is* obeyed, as per Section 4. In no case do we assume an efficiency with all work sequestered, or at any one given step $j \to j + 1$ whether work is sequestered or not, exceeding the Carnot efficiency, and hence we are within the restrictions of the Second Law. (The First Law, of course, puts no restrictions whatsoever on the recycling of energy, except that it is conserved—and we *never* violate conservation of energy).

While in this chapter we do not challenge the First or Second Laws of Thermodynamics, we should note that there have been many challenges to the Second Law, especially in recent years [41–46]. By contrast, the First Law has been questioned only in cosmological contexts [47–49] and with respect to fleeting violations thereof associated with the energy-time uncertainty principle [50, 51]. But there are contrasting viewpoints [50, 51] concerning the latter issue.

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Conflict of interest

The author declares no conflict of interest.

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References

[1] Walker J, Halliday D, Resnick R. Fundamentals of Physics. 11th extended ed. Hoboken, NJ: John Wiley & Sons; 2018, Chapters 18 and 20 (especially Sections 20-2 and 20-3)

[2] Reif F. Fundamentals of Statistical and Thermal Physics. New York: McGraw-Hill; 1965. (reissued: Long Grove, IL: Waveland Press; 2009), Sections 5-11 ad 5-12, and Problems 5.22 through 5.26

[3] Callen HC. Thermodynamics: New York: John Wiley & Sons; 1960, Chapter 4

[4] Callen HC. Thermodynamics and an Introduction to Thermostatistics. 2nd ed. New York: John Wiley & Sons; 1985, Chapter 4

[5] Wark K, Richards DE. Thermodynamics. 6th ed. Boston, MA: WCB/McGraw-Hill; 1999, Chapters 6, 8, 9, and 15–17

[6] Faries VM. Applied Thermodynamics. Revised Ed. New York, NY: MacMillan; 1949, Chapters V–VIII, XVII, and XIX

[7] Zemansky, MW, Dittman RH. Adapted by Chattopadhyay AK. Heat and Thermodynamics, 8th ed. Chennai, India: McGraw Hill Education (India); 2011. (Seventeenth reprint 2018), Chapters 6 and 7

[8] Baierlein R. Thermal Physics. Cambridge, UK: Cambridge University Press; 1999, Chapters 2 and 3

[9] Kittel C, Kroemer H. Thermal Physics. 2nd ed. San Francisco, CA: W. H. Freeman and Company; 1980, Chapter 8

[10] Curzon FL, Ahlborn B. Efficiency of a Carnot engine at maximum power output. American Journal of Physics. 1975;**43**:22-24. DOI: 10.1119/1.10023

[11] Vaudrey A, Lanzetta F, Feidt M. Reitlinger and the origins of the efficiency at maximum power output for heat engines. Journal of Non-Equilibrium Thermodynamics. 2014;**39**: 199-203. DOI: 10.1515/jnet-2014-0018

[12] Endoreversible thermodynamics [Online]. Available from: https://www.wikipedia.org/ [Accessed: 16 March 2020]

[13] De Vos A. Efficiency of some heat engines at maximum-power conditions. American Journal of Physics. 1985;**53**: 570-573. DOI: 10.1119/1.14240

[14] Gordon JM. Maximum power-point characteristics of heat engines as a general thermodynamic problem. American Journal of Physics. 1989;**57**: 1136-1142. DOI: 10.1119/1.16130

[15] Gordon JM. Observations on efficiency of heat engines operating at maximum power. American Journal of Physics. 1990;**58**:370-375. DOI: 10.1119/1.16175

[16] Schmiedl T, Seifert U. Efficiency at maximum power: An analytically solvable model for stochastic heat engines. Europhysics Letters. 2008;**81**: 20003. DOI: 10.1209/0295-5075/81/20003

[17] Tu ZC. Efficiency at maximum power of Feynman's ratchet as an engine. Journal of Physics A. 2008;**41**: 312003. DOI: 10.1088/1751-8113/41/31/312003

[18] Leff HS. Thermal efficiency at maximum work output: New results for old heat engines. American Journal of Physics. 1987;**55**:602-610. DOI: 10.1119/1.15071

[19] Ouerdane H, Apertet Y, Goupil C, Lecoeur P. Continuity and boundary conditions in thermodynamics: From Carnot's efficiency to efficiencies at maximum power. European Physical

Journal - Special Topics. 2015;**55**: 839-862. DOI: 10.1140/epjst/e2015-02431-x

[20] Parrando JMR, Ouerdane H, et al. Debate. Continuity and boundary conditions in thermodynamics: From Carnot's efficiency to efficiencies at maximum power. European Physical Journal - Special Topics. 2015;**224**: 862-864

[21] Apertet Y, Ouerdane H, Goupil C, Lecoeur Ph. True nature of the Curzon-Ahlborn efficiency. Physical Review E. 2017;**96**:022119. DOI: 10.1103/PhysRevE.96.022119

[22] Denur J. The apparent "super-Carnot" efficiency of hurricanes: Nature's steam engine versus the steam locomotive. American Journal of Physics. 2011;**79**:631-643. DOI: 10.1119/13534841 (especially Section VI)

[23] Emanuel K. Divine Wind. Oxford, UK: Oxford University Press; 2005 (especially Chapter 10)

[24] Emanuel K. Hurricanes: Tempests in a greenhouse. Physics Today. 2006; **59**(8):74-75. DOI: 10.1063/1.2349743

[25] Emanuel K. Tropical cyclones. Annual Review of Earth and Planetary Sciences. 2003;**31**:75-104. DOI: 10.1146/annurev.earth.31.100901.141259

[26] Emanuel K. Thermodynamic control of hurricane intensity. Nature. 1999;**401**:665-669. DOI: 10.1038./44326

[27] Bister M, Emanuel KA. Dissipative heating and hurricane intensity. Meteorology and Atmospheric Physics. 1998;**65**:223-230

[28] Zhang DL, Altshuler E. The effects of dissipative heating on hurricane intensity. Monthly Weather Review. 1999;**127**:3032-3038

[29] Emanuel K. Response of tropical cyclone activity to climate change: Theoretical basis. In: Murmane RJ, Liu K-B, editors. Hurricanes and Typhoons: Past, Present, and Future. New York: Columbia University Press; 2004, pp. 395–407

[30] Emanuel KA, Speer K, Rotunno R, Srivastava R, Molina M. Hypercanes: A possible link in global extinction scenarios. Journal of Geophysical Research-Atmospheres. 1995;**100**: 13755-13765. DOI: 10.1029/95JD01368

[31] Emanuel K, Callagham J, Otto PA. A hypothesis for redevelopment of warm-core cyclones over northern Australia. Monthly Weather Review. 2008;**136**: 3863-3872. DOI: 10.1175/2008MWR2409.1

[32] Kieu C. Revisiting dissipative heating in tropical cyclone maximum potential intensity. Quarterly Journal of the Royal Meteorological Society. 2015; **141**:2497-2504. DOI: 10.1002/qj.2534

[33] Apertet Y, Ouerdane H, Goupil C, Lecoeur P. Efficiency at maximum power of thermally coupled heat engines. Physical Review E. 2012;**85**: 041144. DOI: 10.1103/PhysRevE.85041144

[34] Makarieva AM, Gorshkov VC, Li B-L, Nobre AD. A critique of some modern applications of the Carnot heat engine concept: The dissipative engine cannot exist. Proceedings of the Royal Society A. 2010;**466**:1893-1902. DOI: 10.1098/rspa.2009.0581

[35] Bejan A. Thermodynamics of heating. Proceedings of the Royal Society A. 2019;**475**:20180820. DOI: 10.1098/rspa.2018.0820

[36] Bister M, Renno N, Pauluis O, Emanuel K. Comment on Makarieva et al. 'A critique of some modern applications of the Carnot heat engine concept: The dissipative engine cannot exist'. Proceedings of the Royal Society A. 2011;**467**:1-6. DOI: 10.1098/rspa.2010.0087

[37] Ozawa H, Shimokawa S. Thermodynamics of a tropical cyclone: Generation and dissipation of mechanical energy in a self-driven convection system. Tellus A. 2015;**67**: 24216. DOI: 10.3402/tellusa.v67.24216. 15 pages

[38] Denur J. Improving heat-engine performance via high-temperature recharge. In: Vizureanu P, Academic editor. Applied Thermodynamics and Energy Engineering. London, UK: IntechOpen; 2019

[39] Dyson Sphere. Available from: https://www.wikipedia.org/ [Accessed: 16 March 2020]

[40] Dyson Spheres in Popular Culture. Available from: https://www.wikipedia.org/ [Accessed: 16 March 2020]

[41] Sheehan DP, editor. Quantum limits to the second law. In: AIP Conference Proceedings Volume 643; Melville, NY: American Institute of Physics; 2002

[42] Nikulov AV, Sheehan DP, editors. Special issue: Quantum limits to the second law of thermodynamics. Entropy 2004;**6**(1)

[43] Čápek V, Sheehan DP. Challenges to the Second Law of Thermodynamics: Theory and Experiment. Dordrecht, The Netherlands: Springer; 2005

[44] Sheehan DP, editor. Special issue: The second law of thermodynamics: Foundations and status. Foundations of Physics. 2007;**37**(12)

[45] Sheehan DP, editor. Second law of thermodynamics: Status and challenges. In AIP Conference Proceedings Volume 1411; Melville, NY: American Institute of Physics; 2011

[46] Sheehan DP, editor. Special issue: Limits to the second law of thermodynamics: Experiment and theory. Entropy. 2017;**19**

[47] Harrison ER. Mining energy in an expanding universe. The Astrophysical Journal. 1995;**446**:63-66

[48] Sheehan DP, Kriss VG. Energy Emission by Quantum Systems in an Expanding FRW Metric [Online]. Available from: arXiv:astroph/0411299v1 [Accessed: 16 March 2020]

[49] Parry R. Extracting Energy from the Expanding Universe: Can we Avoid the Heat Death? Honours Physics 2015. Sydney, Australia: Sydney Institute for Astronomy, School of Physics, The University of Sydney; 2015

[50] Griffiths DJ, Schroeter DF. Introduction to Quantum Mechanics. 3rd ed. Cambridge, UK: Cambridge University Press; 2018, Section 3.5.3

[51] Hagmann MJ. Distribution of times for barrier traversal caused by energy fluctuations. Journal of Applied Physics. 1993;**74**:7302-7305