Chapter

Appearance of Catastrophes and Plasticity in Porous and Cracked Media

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Abstract

This chapter is devoted to study the properties of structured continuum, with specific surface and characteristic size of structure. This linear dimension means the absence of automatic transforming difference relations into differential equations. It is impossible to apply conservation laws at any point of the real structural body, because any closed points in vicinity of inner surface can represent both solid and liquid (gas) phases. We need use some representative minimal volume, which characterized the complicate body at hole. This approach leads to differential equations of motion of the infinite order. Solutions of them, along usual P and S waves, contain many waves with abnormally low velocities, which are not bounded below. It is shown that in such media, weak perturbations can increase or decrease without limit. The reason of the infinite order of differential equations is many degrees of freedom in such media. Catastrophes correspond to unstable solutions equations of motion. Plasticity begins in elastic state like continuous phenomenon, and there is a finite distance between the sliding lines on the contrary with classic plasticity, where distances between sliding lines are infinitely small.

Keywords: structure of pore space, porous and cracked media, instability, plasticity

1. Introduction

The main idea of continuous mechanics is that any volume is the representative one. It means that the integral of loadings, which concentrates on the surface and bounds mentioned volume, is equal to zero in statics or to inertial forces in dynamics. The evident disagreement that the surface forces and inertial ones apply to different points (inertial forces apply to center of gravity of volume) overcomes due to an assumption about infinite small sizes of the mentioned volume. This assumption gives us a possibility to equal the volume forces (divergence of the stress tensor), which was created by the internal stresses, and the inertial forces, according to the second Newton law. Mathematical technique is based on the Gauss theorem about relation between the field flux across surface and divergence of this field in the volume, which is bounded by closed surface. However, in the structured bodies, there is a fundamentally different situation. The representative volume must contain some set of elementary structures. Otherwise, a small volume will contain only one of the phases, for example, liquid in the pores or the solid skeleton without liquid, and will not characterize the properties of the structured body. The characteristic size of the structure leads to fact that the average distance is between one of the cracks to another and one pore to another given by the specific surface of the sample. It is necessary to connect the integral geometric properties of a medium with physical processes of such bodies deforming. On the contrary with a classic continuum of Cauchy and Poisson, the new continuum for structured or blocked media must contain many degrees of freedom. It is evident because elementary blocks may translate the motion by contact interactions, by rotations, and by group of particle's motion. It means that the energy contents not in first derivatives (strains) only. The potential energy contents in the second derivatives (curvatures) and other orders of ones. It means that the equation of motion of a blocked medium should contain many derivatives; in other words, the equation of motion may have been very high, probably, the infinite order. The static and dynamic processes in the classic continuum are divided by the Great Wall of China from each other. The equation of equilibrium never will pass in the equation of motion. However, it is evident that the dynamic processes often arise very slow and are quasi-static motions. It would be nice to destroy this mentioned wall by a newly structured continuum. It would be a good idea to destroy the abovementioned wall by means of justification of the newly structured continuum. The seismic emission, which causes due to static loading, maybe not a bad example of such phenomena, which are existed between statics and dynamics.

2. Equations of motion for structured media

In **Figure 1**, an element of the volume of structured body is shown, in which l_0 is the average distance between one pore and another. Earlier presented was the result about the relation between the specific surface and the average length between cracks and pores. There is a theorem of integral geometry, which relates the specific surface σ_0 and l_0, namely [1]

$$\sigma_0 l_0 = 4(1 - f) \tag{1}$$

where f is the porosity. Hence, if there is a specific surface of sample, there is automatically the average range of microstructure l_0.

The distinction between classic and structured continuums is clear, see **Figure 1**. In the volume, which is inside into surface C, there is equation of equilibrium, because all forces delete to each other. In the volume, which is inside into surface D, there is equilibrium, because forces do not compensate to each other (on the one

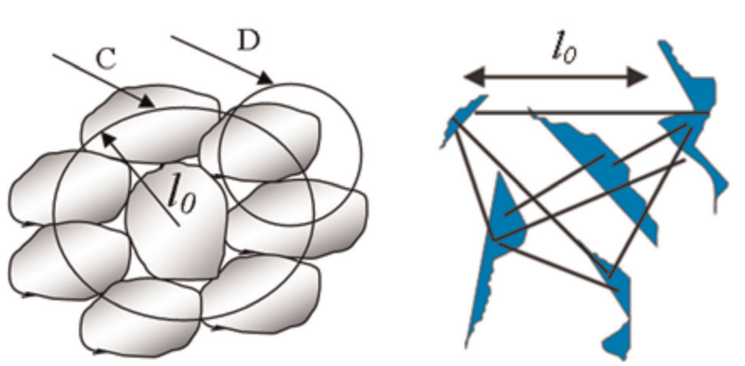


Figure 1.
Representative element of structured body for granular medium (left) and average distance l_o from one crack to another (right). On surface C, the equation of equilibrium is complied, and on surface D, it is not satisfied.

side of grain, we have forces; and on the other part of boundary surface D, we have no forces).

The idea of creation of the new model of space is as follows: consider some finite volume of the body (a sphere on a figure with radius l_0). Surface forces act on a sphere of radius l_0, while inertial forces applied at the center of the structure. There is no way for the volume element to tend to zero and to match the points of application of surface forces and inertial forces, as in the classical continuum. Therefore, since we must consider the representative finite volume, we have a problem of different positions of surface and inertial forces.

We need to translate the surface forces to the center of the structure by a special operator, and after this, it is possible to apply the law of conservation for some structural image continuum and to act as in a typical classical model of space. The main feature of this approach is to fill all the space, including the pores and cracks by field force. Because of it, we have a continuous image of a very complex media and a possibility to apply the physical laws into an image of the media.

The one-dimensional operator of field translation from point x into point $x \pm l_0$ is given by the symbolic formula [1]

$$u(x \pm l_0) = \exp\left(l_0 D_x\right) \tag{2}$$

The operator is $D_x = \frac{\partial}{\partial x}$. The difference operator $\Delta_1(x)$ is a difference between two translation operators

$$\begin{aligned} \Delta_1 &= \frac{1}{l_0}\left[u\left(x + \frac{l_0}{2}\right) - u\left(x - \frac{l_0}{2}\right)\right] \\ &= \frac{u(x)}{l_0}\left[\exp\left(\frac{l_0}{2}D_x\right) - \exp\left(-\frac{l_0}{2}D_x\right)\right] = u(x)\frac{\sinh\left(\frac{l_0}{2}D_x\right)}{\left(\frac{l_0}{2}\right)} \end{aligned} \tag{3}$$

This is a first difference for finite distance between two points. The second difference may be represented as quadrate of the first difference,

$$\Delta_2 = u(x)\frac{\sinh\left(\frac{l_0}{2}D_x\right)^2}{\left(\frac{l_0}{2}\right)^2} \tag{4}$$

The formally expansion in Taylor's series gives a finite increment of field. This expansion contains the infinite number of derivatives with different powers of l_0. The factor l_0 relates with the specific surface of the sample. The three-dimensional operator of field's translation for some cube with length of l_0 may be constructed as follows:

$$P[u(x)] = \frac{u(x)}{6}\left[\cosh\left(\frac{l_0}{2}D_x\right) + \cosh\left(\frac{l_0}{2}D_y\right) + \cosh\left(\frac{l_0}{2}D_z\right)\right] \tag{5}$$

The analogous operator of translation for some spheres is given by expression

$$P\left(l_0D_x; l_0D_y; l_0D_z\right) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \exp\left[l_0\left(D_x sin\theta cos\varphi + D_y sin\theta sin\varphi + D_z cos\theta\right)\right] sin\theta d\theta d\varphi \tag{6}$$

Because there is a Poisson formula [2]

$$\int_0^{2\pi}\int_0^{\pi} f[(\alpha sin\theta cos\varphi + \beta sin\theta sin\varphi + \gamma cos\theta)]sin\theta d\theta d\varphi = 2\pi\int_0^{\pi} f(Rcosp)sinpdp = 2\pi\int_{-1}^{1} f(Rt)dt \tag{7}$$

In the formula (7), parameters α, β, and γ are some quantities. However, in Eqs. (6) and (7), parameters play the role of differential operators. The relation between quantities and operators is established by Maslov [3]. Hence, P operator maybe rewritten as follows [4]

$$\begin{aligned} P(l_0D_x; l_0D_y; l_0D_z) &= \frac{1}{2}\int_{-1}^{1} \exp\left(l_0\sqrt{\Delta}, t\right)dt = \int_0^1 \cosh\left(l_0\sqrt{\Delta}, t\right)dt \\ &= \frac{\sinh\left(l_0\sqrt{\Delta}\right)}{l_0\sqrt{\Delta}} = E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \dots \end{aligned} \tag{8}$$

In the classic continuum, we apply the impulse conservation law to any element of the medium. In this situation, we need to fill all pores over space by a force field. Instead of real stresses, which are changing very fast from one point to another, we can construct the continual image of real stresses. Namely, we use a continuous field, which is constructed by the application of the operator P to the real complicated force field. For this continuous image of real stress, $P(\sigma_{ik})$, we can apply the impulse conservation law. In the classic continuous model, this operation is made by nature itself. This model of a continuum requires some mathematical operations in order to create the continuum medium. Using operator P, we can write the equation of motion of micro-inhomogeneous body, because for an average stresses in structure, the law of impulse conservation takes the usual form, namely [4]

$$\frac{\partial}{\partial x_k}[P(\sigma_{ik})] = \rho\ddot{u}_i \tag{9}$$

In a more detailed form Eq. (9) can be rewritten as follows

$$\frac{\partial}{\partial x_k}\left[\left(E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \dots\right)\sigma_{ik}\right] = \rho\ddot{u}_i \tag{10}$$

No wonder that Eq. (9) contains derivatives of the infinite order. This circumstance is due to many degrees of freedom for structured bodies. At $l_0 \to 0$, we have the usual equations of motion for classic continuous model of space.

3. Fundamental solutions

We can pass to the image space, following Hooke's law and applying the Fourier transform along three coordinates, as [5]

$$u_i(x, y, z) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \exp\left[i\left(n_x x + n_y y + n_z z\right)\right] U_i\left(n_x, n_y, n_z\right)dn \tag{11}$$

where $n^2 = n_x^2 + n_y^2 + n_z^2$; $dn = dn_x dn_y dn_z$. The operator P leads to

$$Pu_i(x,y,z) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\sinh(l_0 n)}{l_0 n} \exp\left[i\left(n_x x + n_y y + n_z z\right)\right] U_i\left(n_x, n_y, n_z\right) dn \quad (12)$$

This allows us to calculate the Fourier transform for the fundamental solution of the system Eq. (9):

$$G_{ij} = \frac{1}{\mu n^2 - \rho\omega^2 \frac{l_0 n}{\sin(l_0 n)}} \left[\delta_{ij} - \frac{(\lambda+\mu) n_i n_j}{(\lambda + 2\mu) n^2 - \rho\omega^2 \frac{l_0 n}{\sin(l_0 n)}}\right] \frac{l_0 n}{\sin(l_0 n)} \quad (13)$$

At very small values, $l_0 n$, the sine and argument ratio approaches unity, and the Fourier transform becomes an ordinary equation for Green's tensor in an elastic continuum. The inverse Fourier transform is obtained by integration of Eq. (13) which includes simple poles corresponding to P and S waves and a set of simple poles where the sine in the denominator of Eq. (13) becomes zero. The residuals are in the simple poles, $n^2 = k_S^2 l_0 n$, where k_S^2 is the wave number of both P and S waves. At very small l_0, the ratio $\frac{l_0 n}{\sin(l_0 n)} \to 1$ and denominators in Eq. (13) become the classical equations that define the poles corresponding to compression and shear waves velocities (**Figure 2**). Assuming $n/k_s = m$ and $k_s l_0 = \varepsilon$, we obtain the equation for complex roots that describe waves from a focused source in porous and cracked solids as

$$m\sin(\varepsilon m) = \varepsilon \quad (14)$$

If $m = x + iy$ is assumed to be a complex value, for the real and imaginary parts, we have the transcendental equations

$$\begin{aligned} x\sin\varepsilon x\cosh\varepsilon y - y\sinh\varepsilon y\cos\varepsilon x &= \varepsilon^2 \\ y\sin\varepsilon x\cosh\varepsilon y + x\sinh\varepsilon y\cos\varepsilon x &= 0 \end{aligned} \quad (15)$$

We can rewrite Eq. (15) in a different form with $x^* = \varepsilon x$ and $y^* = \varepsilon y$ as new variables

$$\frac{\tan x^*}{x^*} = -\frac{\tanh y^*}{y^*};\ \sin^2 x^* + \sinh^2 y^* = \frac{\varepsilon^4}{x^{*2} + y^{*2}} \quad (16)$$

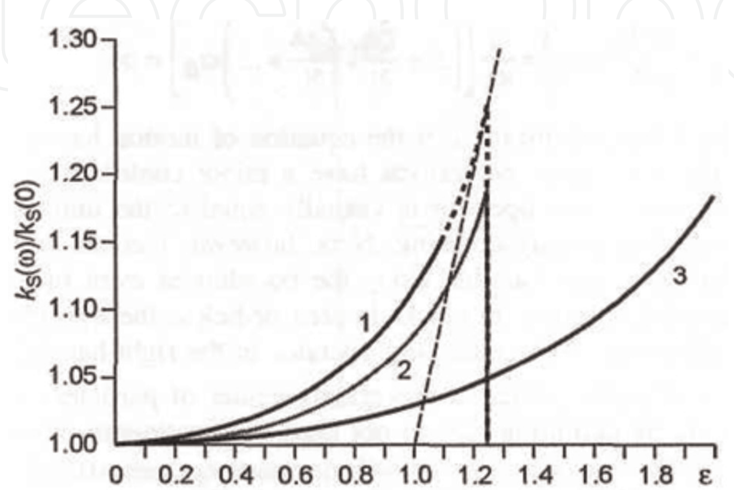


Figure 2.
Wave number ratio as a function of dimensionless ratio $\varepsilon = 2\pi l_o/\lambda_s$. Curves: 1—wave number ratio $k_s(\omega)/k_s(0)$, i.e., S-wave velocity decreasing with frequency; 2—$\gamma = V_s/V_p$ increasing with frequency; and 3—wave number ratio $k_s(\omega)/k_s(0)$ of P waves.

Equation (15) obviously has many real roots corresponding to $y = 0$. Indeed, at small ε, Eq. (15) gives the solution $m = 1$, which corresponds to the ordinary P- or S-wave velocity. At large values of m, Eq. (15) is satisfied only if εm approaches a value divisible by π, i.e., at near-zero sine that defines the characteristic anomalous velocity. The unbounded value of the wave number means that normal P and S waves coexist with arbitrarily small P and S velocity anomalies. The existence of these anomalies in a micro-heterogeneous medium has its physical explanation: energy is stored in strain (first derivatives of displacement) as well as in the curvature of higher derivatives. Therefore, there appear as velocities related to flexural and torsion waves and to numerous waves associated with oscillation of groups of particles (blocks) (**Figure 3**).

The growing of ratio $\gamma = V_S/V_P$ causes a very interesting phenomenon, namely an apparent negative Poisson value, for waves with the length not very small compared to size of a grain. The growing value of $\gamma = V_S/V_P$ means that the Poisson ratio is decreasing up to negative values [6] (**Figure 4**).

At the same time, Eqs. (14) and (15) likewise have complex roots. The first Eq. (15) shows that complex roots arise only at some values of e, which are not so small, as they satisfy the inequality $\varepsilon x > \pi/2$. **Table 1** lists complex roots corresponding to some relatively small ε. Note that the parameter ε can be expressed via the linear size-to-wavelength ratio (l_0/λ_s).

Complex roots can mean either damping or unlimited growth of wave amplitude, of course, in the presence of an energy-unbounded source. The minimum damping (growth) corresponds to $(2.0288)^{-1}$ or about a half of the normal velocity. The same process can be expected to cause both excitation and damping in porous and cracked media, depending on the phase of stationary oscillations.

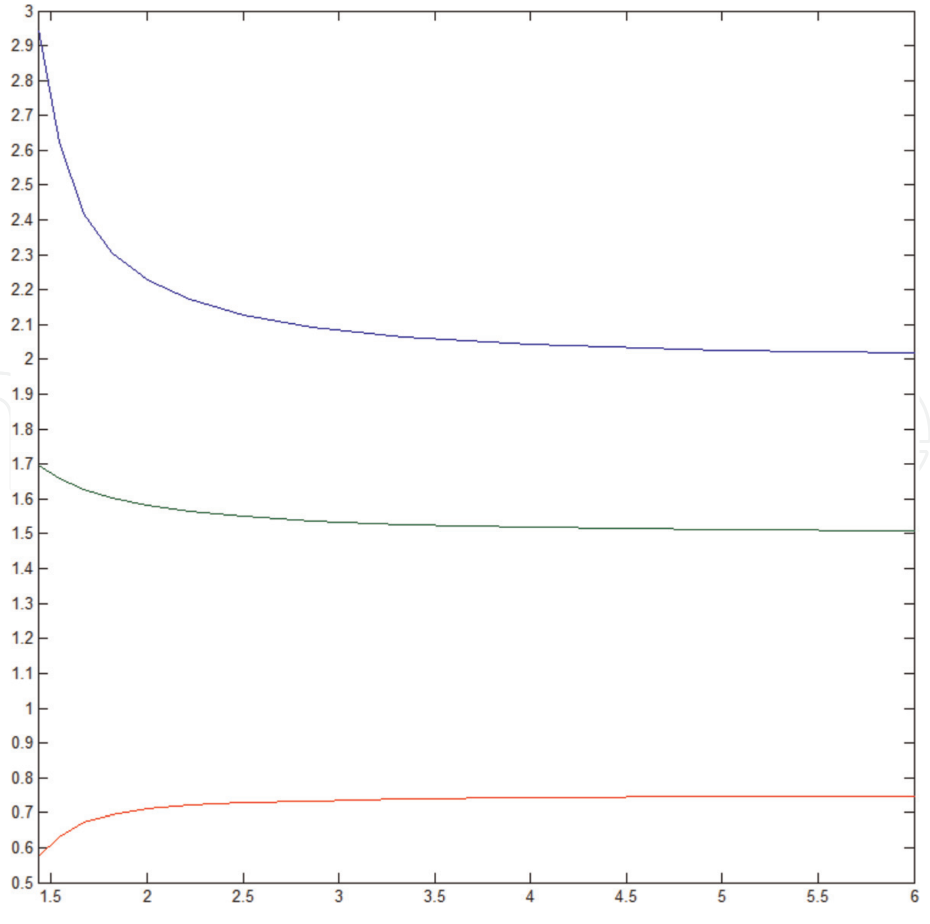


Figure 3.
The decreasing P-wave velocity (the upper line) and S-wave velocity (the middle curve) and the growth of their ratio $\gamma = V_S/V_P$ (the lower line) due to increasing size of microstructure. The ratio $\gamma = V_S/V_P$ more than 0.705 corresponds to the negative Poisson ratio [6]. The vertical scale is the wave velocities (km/s) and a horizontal scale is the ratio between the size of the microstructure and wavelength.

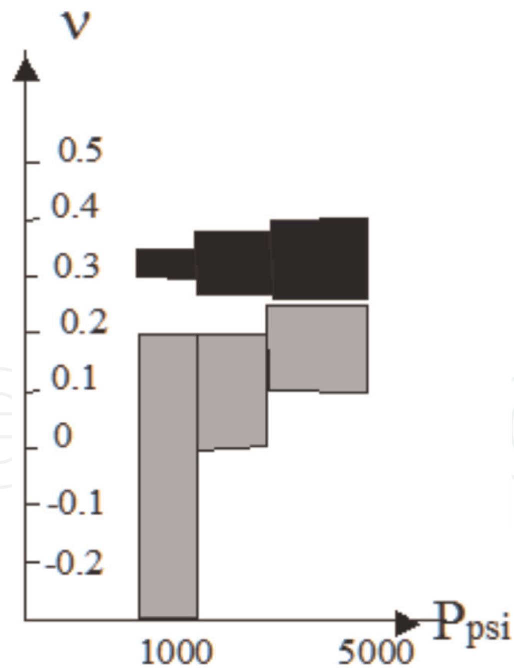


Figure 4.
Gregory experimental data. Poison ratio (the vertical axis) versus pressure. Black color corresponds to water saturated porous shales and gray color corresponds to dry shales with the same porosity. In this case, negative Poisson ratios are possible.

ε	x	y
0.2147	2.0288	0.0548
0.2507	2.0645	0.5838
0.2771	2.1064	0.8880
0.3253	2.1560	1.1838
0.3918	2.2157	1.5122

Table 1.
The value epsilon means dimensionless product of structure size into wavenumber of usual S waves in continuous medium. Value x means the real value of product of structural wavenumber into structure size. Value y is the imaginary part of it.

4. One-dimensional case: plane wave and instabilities

In one-dimensional case, the Eq. (10) takes more simple expression

$$u''\left(E+\frac{l_0^2}{3!}\Delta+\frac{l_0^4}{5!}\Delta\Delta+\dots\right)+k_S^2u=0 \tag{17}$$

This equation by substitution $u=\exp(ikx)$ gives us the dispersion equation for an unknown wave number k, or for unknown wave velocity, which depends on the size of structure l_0 or specific surface of sample σ_0:

$$\frac{\sin(kl_0)}{kl_0}=\frac{k_S^2}{k^2} \tag{18}$$

It is evident that by $l_0\to 0$, the wave number $k\to k_S$, i.e., the wave velocity is equal to V_P or V_S, elastic wave velocity. However, if l_0 is not a very small value, the wave velocity decreases up to zero by $kl_0\to m\pi$, if m is the integer number. Hence, this model describes along with usual seismic waves many waves of very small velocities, which are not bound below.

This effect is more for P waves than for S ones. Eq. (14) shows that if the Poisson ratio is measured on samples by velocities V_P and V_S, their ratio V_S/V_P grows by growing l_0, and this effect can produce abnormally small Poisson's ratio, up to negative volume of it.

It is evident that at $kl = m\pi$, m is the integer number and the value $k \to \infty$. It means that there are waves with arbitrary small velocities not bounded below. Beside it, Eq. (15) has complex roots too, because $\sin(kl_0)$ may be negative, while the second term in Eq. (18) contains $(k_S/k)^2$. Eq. (18) means that the complex roots do not by small values of x, because the right-hand expression is a negative value. In order to be complex roots, an evident condition is necessary, i.e., $tanx > \pi/2$. The physical sense of it means that the complex roots are possible, if the wavelength is four times (or more than four times) more than the size of the structure. These complex roots mean that amplitude of oscillations may be increasing or decreasing up to infinity or, may be, to zero. These roots are responsible for catastrophe's behavior of structured bodies.

Hence, if there is a source of sufficient energy, even some small oscillations can produce catastrophes. It is interesting that nonlinear deforming of samples decreases this effect, because a wave velocity for rocks is decreasing, by growing amplitude of wave. It means that the wave number is growing by the same frequency in the pure elastic process. In **Figure 6**, the real roots of dispersion, Eq. (18) are shown. The vertical axis shows a dimensionless frequency, namely ε, while horizontal axis shows us the real and imaginary parts of wave numbers. In **Figures 5**–**7** [7], complex roots as a function of dimensionless frequency ε are shown. Every point is a position of some root, namely a real part, an imaginary one, and a dimensionless frequency. The more is the spreading of ε values, the greater is the number of complex roots.

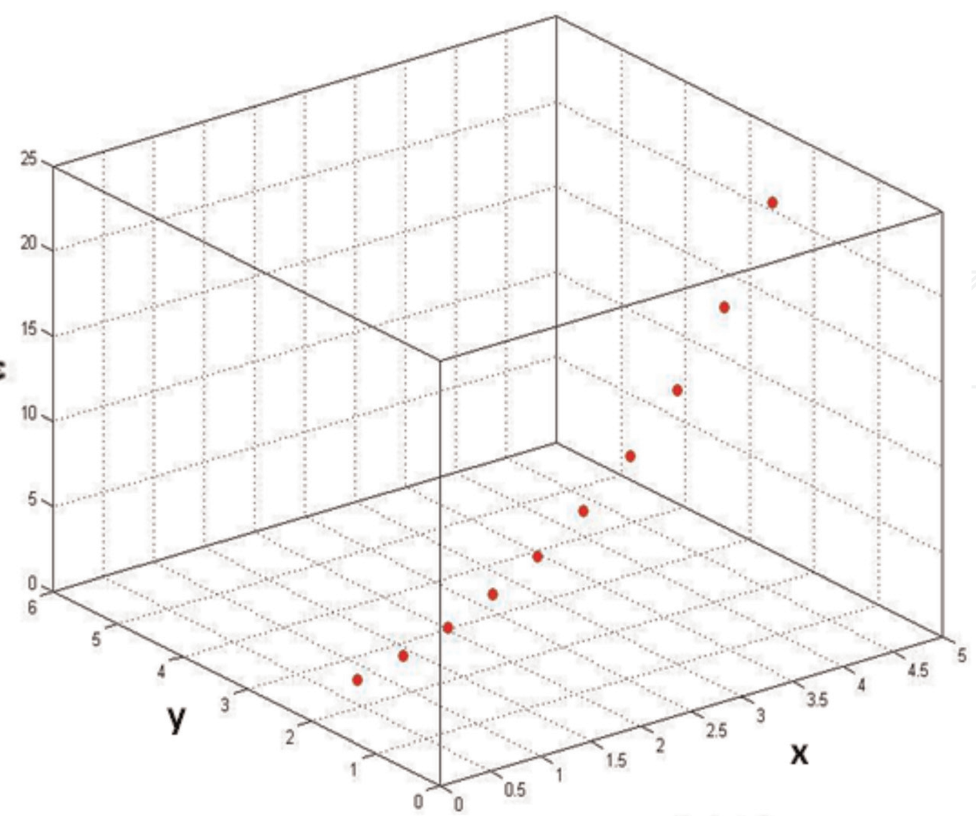


Figure 5.
The position of complex roots depends on the value ε. The more the value ε, the more numbers of roots. The first value of ε corresponds to values of first row from ***Table 1***.

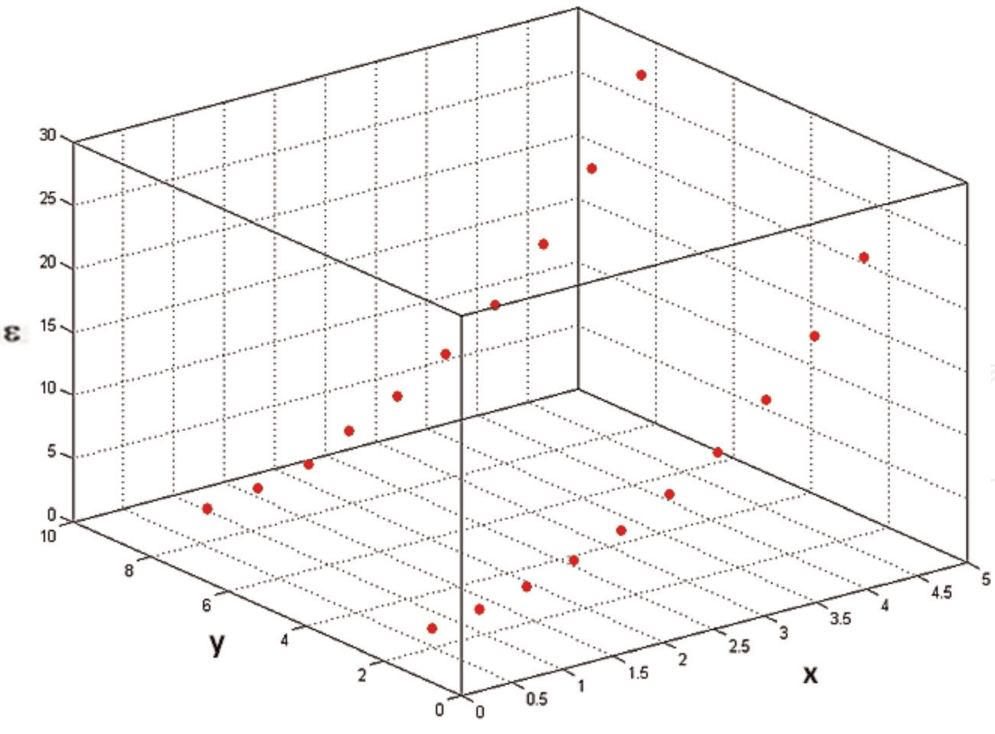


Figure 6.
The position of complex roots depends on the value ε. The more the value ε, the more numbers of roots. The second value of ε corresponds to values of second row from ***Table 1****.*

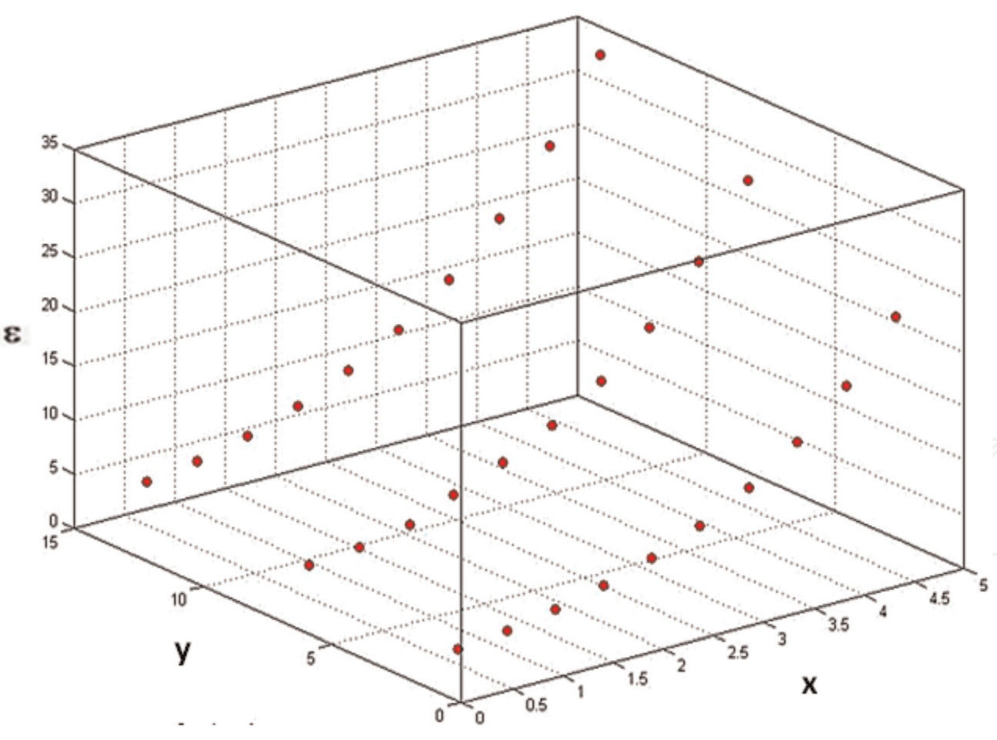


Figure 7.
The position of complex roots depends on the value ε. The more the value ε, the more numbers of roots. The third value of ε corresponds to values of third row from ***Table 1****.*

5. Pointing vector and equation of equilibrium for blocked media

The equation of equilibrium for micro-structured media can be written from Eq. (9) as

$$\frac{\partial}{\partial x_k}[P(\sigma_{ik})] = P\left[\frac{\partial \sigma_{ik}}{\partial x_k}\right] = 0; \frac{\partial \sigma_{ik}}{\partial x_k} = P^{-1}(0) \tag{19}$$

The inverse operator $P^{-1}(0)$ contains zero, but not zero only. It contains some periodic functions and the average value equal to zero. For example, such construction satisfies to Eq. (19)

$$P\left\{\sum \mathrm{Im}\left\{\exp\left[in\frac{\pi}{l_0}(k_1x + k_2y + k_3z)\right]\right\}\right\} = 0 \tag{20}$$

If n is integer number, $k_1 = sin\theta cos\varphi; k_2 = sin\theta cos\varphi; k_3 = cos\theta$. Physical sense of it means that volume forces are equal to zero in average sense, not at any point. Using mentioned inverse operator, we can write the equilibrium equation for blocked media in the form

$$\frac{\partial \sigma_{ik}}{\partial x_k} = \phi\sigma_0^2\sigma_{ik}^0u_k^0\sum_{n=1}^{\infty}\mathrm{Im}\left\{\exp\left[in\frac{\pi}{l_0}(k_1x + k_2y + k_3z)\right]\right\} \tag{21}$$

In Eq. (21) σ_0^2 is a quadrat of specific surface; $\sigma_{ik}^0u_k^0 = A_i^0$ is the pointing vector of usual continuous body, and ϕ is a dimensionless constant, which must be obtained. These values we can put as constants in small structure volume. The integration with respect to spherical angles gives us a result that the imaginary part of exponent is zero in average sense, namely

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\exp\left[\frac{\pi n}{l_0}i(xsin\theta cos\varphi + ysin\theta sin\varphi + zcos\theta)\right]sin\theta d\theta d\varphi$$

$$\int_0^{\pi}\exp\left(ir\frac{n\pi}{l_0}cosp\right)sinpdp = \frac{1}{2}\int_{-1}^{1}\exp\left(ir\frac{n\pi}{l_0}t\right)dt = \frac{l_0}{n\pi r}\sin\left(\frac{rn\pi}{l_0}\right) + i0 \tag{22}$$

Partial solution of Eq. (22) is a convolution of Green tensor with right hand of Eq. (21), that is,

$$u_i^1(x) = \phi\frac{1}{\mu}\sigma_0^2\sigma_{mk}^0u_k^0(x)\mathrm{Im}\iiint\Gamma_{mi}(x,y)\exp\left[ik_m(x_m - y_m)\right]dV_y \tag{23}$$

Taking into account that the sizes of area much more, than sizes of structure, the area of integration is the infinite large one. In this case, integral Eq. (13) practically is the Fourier transform of fundamental solution of usual elastic equilibrium equations

$$u_i^{1n}(x) = \phi\frac{1}{\mu}\sigma^2\sigma_{mk}^0u_k^0(x)\left(\frac{l_0}{n\pi}\right)^2\left[\delta_{mi} - (1-\gamma^2)k_mk_i\right]\exp\left[\frac{in\pi(k_lx_l))}{l_0}\right] \tag{24}$$

In Eq. (24) the imaginary part of the exponent is used. Hence, the additional value in average sense is equal to zero. Using relation Eq. (1) $\sigma_0l_0 = 4(1-f)$, we get a partial solution, which depends on porosity only

$$u_i^{1n}(x) = \phi\frac{1}{\mu}\sigma_{mk}^0u_k^0(x)\left(\frac{4(1-f)}{n\pi}\right)^2\left[\delta_{mi} - (1-\gamma^2)k_mk_i\right]\exp\left[i\frac{n\pi(k_lx_l)}{l_0}\right] \tag{25}$$

If these indexes coincide, $i = m$, we get

$$< u_i^{1n}(x) > = \frac{\phi}{\mu} U_{i0} \left(\frac{4(1-f)}{n\pi} \right)^2 \left[1 - \frac{1-\gamma^2}{3} \right] \exp \left[\frac{in\pi(k_l x_l))}{l_0} \right] \tag{26}$$

Take into account that the average value of a quadrat of cosine is $< k_m k_i > = \frac{\delta_{km}}{3}$. There is a summation with respect to n, and U_{i0} is a Pointing vector for usual continuous model of the medium. This value is a small one of the second order compared to usual displacement, because a Pointing vector, divided on the shear module is order to strain, multiplied to size of structure l_0.

Strains. By differentiating of an integral Eq. (23) take into account that the main part of the field contains in fast changing exponent, not in Green tensor itself, i.e.,

$$u_{i,j}^1(x) \approx \frac{\phi}{\mu} \sigma^2 \sigma_{mk}^0 u_k^0(x) i k_j \mathrm{Im} \iiint \Gamma_{mi}(x,y) \exp \left[i k_l (x_l - y_l) \right] dV_y \tag{27}$$

$$\begin{aligned} e_{ij}^{1n} &= \frac{i}{2} \left(u_{i,j}^1 + u_{j,i}^1 \right) = \frac{\phi}{2\mu} \sigma_{mk}^0 \frac{u_k^0(x)}{l_0} \frac{4}{\pi} 4(1-f)^2 \left\{ \left[\delta_{mi} - (1-\gamma^2) k_m k_i \right] k_j \right. \\ &\quad \left. + \left[\delta_{mj} - (1-\gamma^2) k_m k_j \right] k_i \right\} \frac{1}{n} \exp \left[i \frac{n\pi(k_1 x + k_2 y + k_3 z)}{l_0} \right] \\ &= \phi \frac{8\gamma^2}{3\pi\mu} \frac{1}{l_0} \left[U_{0i} k_j + U_{0j} k_i \right] (1-f)^2 \frac{1}{n} \exp \left[i \frac{n\pi(k_l x_l)}{l_0} \right] \end{aligned} \tag{28}$$

According to Eq. (9) the additional dilatation is

$$\theta^{(n)} = \phi \frac{16\gamma^2}{3\pi\mu} \frac{1}{l_0} [U_{0n}] (1-f)^2 \frac{1}{n} \exp \left[i \frac{n\pi(k_l x_l)}{l_0} \right] \tag{29}$$

Let us integrate the normal component of the Pointing vector on the small sphere with radius r. This integral must be equal to density of potential energy E (divergence of Pointing vector) namely,

$$< U_{0n} > = \frac{E}{\sigma_0} = 4(1-f) E l_0 \tag{30}$$

The average value of fast-changing exponent in Eqs. (28) and (29) on spherical angles is

$$< \exp \frac{1}{n} \left[i \frac{n\pi(k_l x_l)}{l_0} \right] > = \frac{4(1-f)}{\pi n^2} \int_0^{n\pi} \frac{sinx}{x} dx = \frac{4(1-f)}{\pi n^2} Si(n\pi) \tag{31}$$

The additional dilatation due to randomly oriented volume forces (an average value of these forces is zero) may be written as

$$\theta = \phi \frac{16\gamma^2}{3\pi\mu} \frac{4(1-f)^3 E}{\pi} \sum_{n=1}^{\infty} \frac{Si(n\pi)}{n^2} \tag{32}$$

In Eq. (32) the symbol $Si(n\pi)$ means an integral sine of argument $(n\pi)$. The left hand in Eq. (32) is an additional expansion or compression, so called as dilatancy. It depends on the potential energy of the continuous body E, which may contain shear energy only, but it produces additional expansion or compression. It is a quadrat effect too.

More strong effect is related with product of high-changing volume force (equal to zero in average) into displacement. This product in not equal to zero in average, because it contains a quadrat of high-changing sine, which is equal to number one third in three dimension space.

$$E_n = \phi \frac{1}{\mu} \sigma^0_{mk} u^0_k(x) \left(\frac{4(1-f)}{n\pi} \right)^2 \left[\delta_{mi} - \left(1-\gamma^2\right) k_m k_i \right] \frac{1}{3} \ \phi \sigma_0^2 U_{0i} \tag{33}$$

If indexes coincide, $m = i$, we get the additional potential energy, due to fluctuations

$$E_n = \frac{\phi^2}{3(\lambda+2\mu)} \left[U_{01}^2 + U_{02}^2 + U_{03}^2 \right] \sigma_0^2 \left(\frac{4(1-f)}{n\pi} \right)^2 \tag{34}$$

The summation with respect to index n from unit up to infinity gives

$$E = \frac{8\phi^2}{9(\lambda+2\mu)} \sigma_0^2 \left[U_{01}^2 + U_{02}^2 + U_{03}^2 \right] (1-f)^2 \tag{35}$$

In spite of a fact that the Pointing vector is the small value of more high order, than stresses, the high value σ_0^2 (quadrat of specific surface) in Eq. (35) can produce not small common effect. The indefinite factor ϕ depends on the real structure of pore space and macro-stress-strain state. However, in some simple situations, it can calculate elementary. For example, at rigid pressing of globe by spherical force (radial displacements are constants), the stress-strain state is a hydrostatic state in average, but not such state at any point. The compressional energy is proportional to compress module of skeleton and its volume plus the incompressibility of fluid and its volume, namely [8–11]

$$E = \left(\lambda + \frac{2\mu}{3} \right) \frac{\theta_1^2}{2} (1-f) + \rho c^2 \frac{\theta_0^2}{2} f \tag{36}$$

Indexes unit and zero in Eq. (36) mean solid and liquid parameters. The dilatation of two-phase body gives by the formula

$$\theta = (1-f)\theta_1 + f\theta_0; \theta_0 = \theta_1 \tag{37}$$

If we have uniform random distribution of phases, the average energy is

$$E = E_1(1-f) + E_0 f \tag{38}$$

In Eq. (38) f is the porosity and E_1 and E_0 are the energies of solid and liquid. The dispersion of random value relates with random volume forces, i.e.,

$$[E_1 - (E_1(1-f) + E_0 f)]^2 (1-f) + f[E_0 - (E_1(1-f) + E_0 f)]^2 = (E_1 - E_0)^2 f(1-f) \tag{39}$$

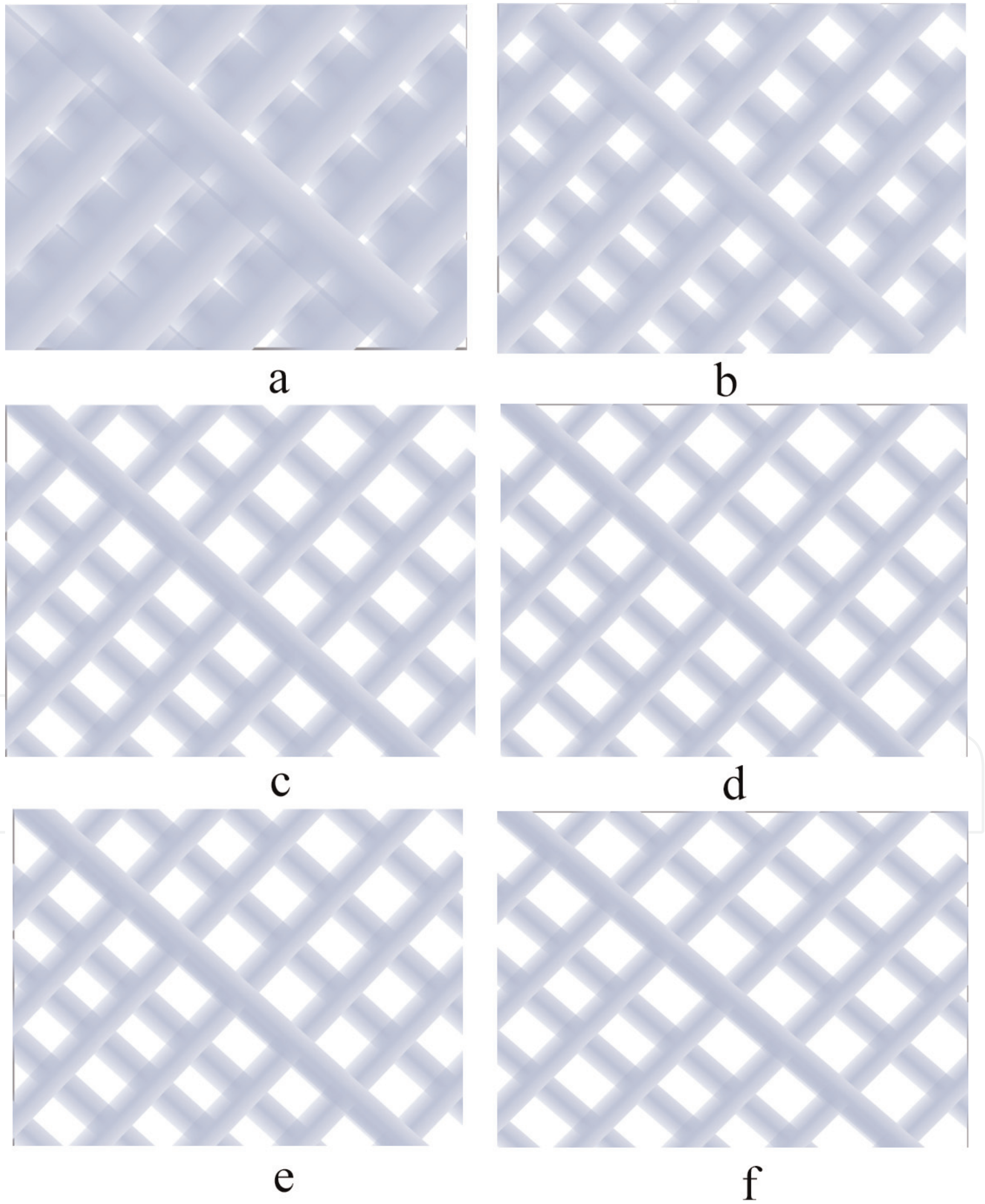
Equation (39) gives the additional energy for very simple macro-hydrostatic state in average. This is the additional of interphase acting. It is equal to additional energy, which is given by Eq. (15). It is reasonable that at unit or zero porosity, an additional energy is equal to zero. The second result is, if the phase energy is equal, the mentioned additional one is equal to zero too. Hence, the indefinite factor ϕ^2 given by the simple equation is

$$\frac{1}{\mu}|E_1 - E_0|\sqrt{f(1-f)} = \frac{8\phi^2}{9(\lambda + 2\mu)}\sigma_0^2\left[U_{01}^2 + U_{02}^2 + U_{03}^2\right](1-f)^2 \tag{40}$$

6. The arriving of plasticity

In spite of that, the additional average strains is small, does not means, that these strains are small in the any point of the volume. Equations (28) and (29) show that on the planes $k_1x \pm k_2y \pm k_3z = 2l_0q$ (q is an integer number), the exponent is not a highly changed value, because it is equal to1 or -1.

In plane situation, the role of these planes plays orthogonal lines $k_1x \pm k_2y = 2l_0q$.



Figures 8.
Successive process of strains localization due to decreasing strains inside of quadrats, making orthogonal lines and increasing them near lines itself.

The series Eqs. (28) and (29) with respect to n in vicinity of mentioned planes are divergent (harmonic) series. It means that the field is decreasing inside of quadrats, making planes, and concentrating in vicinity of planes. Mentioned planes are analogs of slipping lines (lines of Luders) [12] in classic plasticity of the compressible medium. In practical, the number n in Eqs. (28) and (29) is bounded by the elastic limit of the second strain invariant. The field of strains is growing into planes (lines) and decreasing inside of them. This process is called as localization of strains. This localization begins in elasticity, with contrary of classic plasticity and elasticity. The other specific feature of this process is the finite distance between planes (lines). This distance is equal to l_0 (the inverse value of specific surface of sample), while in classic plasticity, this distance is infinitely small. The geological sense of it is interesting. In order to transform the matter from elasticity to plasticity, there is no necessary to have the plastic state at any point of the medium. Plasticity may concentrated near planes, and the other volume can be in elastic state. Rock may flow comparatively light, if they have pores and cracks. On the **Figure 8** shown successive process of localization of strains due to decreasing field inside of quadrats, making by orthogonal lines and increasing them near lines itself.

7. Conclusions

1. The model of the structured continuum with specific surface of the blocked medium or average size of structure, gives us the differential equations of motion of the infinite order. This model includes collective properties of pore space like the porosity and specific surface and predicts besides usual elastic waves many unusual waves with very small velocities.

2. This model predicts the decreasing of the Poisson ratio (up to negative values) due to finite size of microstructure. The reason for this is the decreasing of wave velocity with finite specific surface of the rock.

3. The localization of stresses and strains in structured media begins in elastic state of deforming.

4. The small areas of a stress-strain concentration looks like usual orthogonal sliding lines in classic plasticity. However, they have a finite effective thickness, which depends on the average size of the structure and the elastic strain limit. Besides, there is a finite distance between analogs of sliding lines, which is equal to the average distance from one pore to another one, or between cracks.

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