Chapter

Existence, Regularity, and Compactness Properties in the α-Norm for Some Partial Functional Integrodifferential Equations with Finite Delay

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Abstract

The objective, in this work, is to study the alpha-norm, the existence, the continuity dependence in initial data, the regularity, and the compactness of solutions of mild solution for some semi-linear partial functional integrodifferential equations in abstract Banach space. Our main tools are the fractional power of linear operator theory and the operator resolvent theory. We suppose that the linear part has a resolvent operator in the sense of Grimmer. The nonlinear part is assumed to be continuous with respect to a fractional power of the linear part in the second variable. An application is provided to illustrate our results.

Keywords: integrodifferential, mild solution, resolvent operator, fractional power operator

1. Introduction

We consider, in this manuscript, partial functional equations of retarded type with deviating arguments in terms of involving spatial partial derivatives in the following form [1]:

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + F(t, u_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_\alpha = C([-r, 0], D(A^\alpha)]), \end{cases} \tag{1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$. $B(t)$ is a closed linear operator with domain $D(B(t)) \supset D(A)$ time-independent. For $0 < \alpha < 1$, A^α is the fractional power of A which will be precise in the sequel. The domain $D(A^\alpha)$ is endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$ called $\alpha-$ norm. $\mathcal{C}_\alpha$ is the Banach space $C([-r, 0], D(A^\alpha))$ of continuous functions from $[-r, 0]$ to $D(A^\alpha)$ endowed with the following norm:

$$\|\phi\|_\alpha = \sup_{-r \leq \theta \leq 0} \|\phi(\theta)\|_\alpha \text{ for } \phi \in \mathcal{C}_\alpha.$$

$F : \mathbb{R}_+ \times \mathcal{C}_\alpha \to \mathbb{X}$ is a continuous function, and as usual, the history function $u_t \in \mathcal{C}_\alpha$ is defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \in [-r, 0].$$

As a model for this class, one may take the following Lotka-Volterra equation:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \displaystyle\int_0^t h(t-s)\dfrac{\partial^2 u(s,x)}{\partial x^2} ds \\ \qquad + \displaystyle\int_{-r}^0 g\left(t, \dfrac{\partial u(t+\theta,x)}{\partial x}\right) d\theta \text{ for } t \geq 0 \text{ and } x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0 \text{ for } t \geq 0, \\ u(\theta,x) = u_0(\theta,x) \text{ for } \theta \in [-r,0] \text{ and } x \in [0,\pi]. \end{cases} \tag{2}$$

Here $u_0 : [-r, 0] \times [0, \pi] \to \mathbb{R}, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_+ \to \mathbb{R}$ are appropriate functions.

In the particular case where $\alpha = 0$, many results are obtained in the literature under various hypotheses concerning A, B, and F (see, for instance, [2–6] and the references therein). For example, in [7], Ezzinbi et al. investigated the existence and regularity of solutions of the following equation:

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + F(t,u_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in C([-r,0];\mathbb{X}), \end{cases} \tag{3}$$

The authors obtained also the uniqueness and the representation of solutions via a variation of constant formula, and other properties of the resolvent operator were studied. In [8], Ezzinbi et al. studied a local existence and regularity of Eq. (3). To achieve their goal, the authors used the variation of constant formula, the theory of resolvent operator, and the principle contraction method. Ezzinbi et al. in [9] studied the local existence and global continuation for Eq. (3). Recall that the resolvent operator plays an important role in solving Eq. (3); in the weak and strict sense, it replaces the role of the c_0 semigroup theory. For more details in this topic, here are the papers of Chen and Grimmer [2], Hannsgen [10], Smart [11], Miller [12, 13], and Miller and Wheeler [14, 15]. In the case where the nonlinear part involves spatial derivative, the above obtained results become invalid. To overcome this difficulty, we shall restrict our problem in a Banach space $\mathbb{Y}_\alpha \subset \mathbb{X}$, to obtain our main results for Eq. (1).

Considering the case where $B = 0$, Travis and Webb in [16] obtained results on the existence, stability, regularity, and compactness of Eq. (1). To achieve their goal, the authors assumed that $-A$ is the infinitesimal generator of a compact analytic semigroup and F is only continuous with respect to a fractional power of A in the second variable. The present paper is motivated by the paper of Travis and Webb in [16].

The paper is organized as follows. In Section 2, we recall some fundamental properties of the resolvent operator and fractional powers of closed operators. The global existence, uniqueness, and continuous dependence with respect to the initial data are studied in Section 3. In Section 4, we study the local existence and bowing up phenomena. In Section 5 we prove, under some conditions, the regularity of the mild solutions. And finally, we illustrate our main results in Section 6 by examining an example.

2. Fractional power of closed operators and resolvent operator for integrodifferential equations

We shall write $\mathbb{Y}$ for $D(A)$ endowed with the graph norm $\|x\|_{\mathbb{Y}} = \|x\| + \|Ax\|$, $\mathbb{Y}_\alpha$ for $D(A^\alpha)$ and $\mathcal{L}(\mathbb{Y}_\alpha, \mathbb{X})$ will denote the space of bounded linear operators from $\mathbb{Y}_\alpha$ to $\mathbb{X}$, and for $\mathbb{Y}_0 = \mathbb{X}$, we write $\mathcal{L}(\mathbb{X})$ with norm $\|.\|_{\mathcal{L}(\mathbb{X})}$. We also frequently use the Laplace transform of f which is denoted by f^*. If we assume that $-A$ generates an analytic semigroup and, without loss of generality, that $0 \in \varrho(A)$, then one can define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $\mathbb{Y}_\alpha$ with its inverse $A^{-\alpha}$ given by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt,$$

where Γ is the gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

We have the following known results.

Theorem 2.1. [17] The following properties are true.

i. $\mathbb{Y}_\alpha = D(A^\alpha)$ is a Banach space with the norm $|x|_\alpha = \|A^\alpha x\|$ for $x \in \mathbb{Y}_\alpha$.

ii. A^α is a closed linear operator with domain $\mathbb{Y}_\alpha = \mathrm{Im}(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$.

iii. $A^{-\alpha}$ is a bounded linear operator in $\mathbb{X}$.

iv. If $0 < \alpha \le \beta$ then $D(A^\beta) \rightarrowtail D(A^\alpha)$. Moreover the injection is compact if $T(t)$ is compact for $t > 0$.

Definition 2.2. [18] A family of bounded linear operators $(R(t))_{t \ge 0}$ in $\mathbb{X}$ is called resolvent operator for the homogeneous equation of Eq. (3) if:

a. $R(0) = I$ and $\|R(t)\| \le M_1 \exp(\sigma t)$ for some $M_1 \ge 1$ and $\sigma \in \mathbb{R}$.

b. For all $x \in \mathbb{X}$, $t \to R(t)x$ is continuous for $t \ge 0$.

c. $R(t) \in \mathcal{L}(\mathbb{Y})$ for $t \ge 0$. For $x \in \mathbb{Y}$, $R(.)x \in C^1(\mathbb{R}_+, \mathbb{X}) \cap C(\mathbb{R}_+, \mathbb{Y})$, and for $t \ge 0$ we have

$$\begin{aligned} R'(t)x &= -AR(t)x + \int_0^t B(t-s)R(s)x\, ds \\ &= -R(t)Ax + \int_0^t R(t-s)B(s)x\, ds. \end{aligned} \tag{4}$$

What follows is we assume the hypothesis taken from [1] which implies the existence of an analytic resolvent operator $(R(t))_{t \ge 0}$.

(V1) $-A$ generates an analytic semigroup on $\mathbb{X}$. $(B(t))_{t \ge 0}$ is a closed operator on $\mathbb{X}$ with domain at least $D(A)$ a.e $t \ge 0$ with $B(t)x$ strongly measurable for each $x \in \mathbb{Y}$ and $\|B(t)x\| \le b(t)\|x\|_{\mathbb{Y}}$, for $b \in L^1_{loc}(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

(V2) $\rho(\lambda) = (\lambda I + A - B^*(\lambda))^{-1}$ exists as a bounded operator on $\mathbb{X}$ which is analytic for $\lambda \in \Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < \pi/2 + \delta\}$, where $0 < \delta < \pi/2$. In Λ if $|\lambda| \geq \epsilon > 0$, there exists $M = M(\epsilon) > 0$ so that $\|\rho(\lambda)\| \leq M/|\lambda|$.

(V3) $A\rho(\lambda) \in \mathcal{L}(\mathbb{X})$ for $\lambda \in \Lambda$ and is analytic from Λ to $\mathcal{L}(\mathbb{X})$. $B^*(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ and $B^*(\lambda)\rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ for $\lambda \in \Lambda$. Given $\epsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ so that $\|A\rho(\lambda)x\| + \|B^*(\lambda)\rho(\lambda)x\| \leq (M/|\lambda|)\|x\|_{\mathbb{Y}}$ for $x \in \mathbb{Y}$ and $\lambda \in \Lambda$ with $\lambda \geq \varepsilon$ and $\|B^*(\lambda)\| \mapsto 0$ as $|\lambda| \mapsto \infty$ in Λ. In addition, $\|A\rho(\lambda)x\| \leq M|\lambda|^n \|x\|$ for some $n > 0$, $\lambda \in \Lambda$ with $\lambda \geq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in $\mathbb{Y}$ such that $A(D)$ and $B^*(\lambda)(D)$ are contained in $\mathbb{Y}$ and $\|B^*(\lambda)x\|_{\mathbb{Y}}$ is bounded for each $x \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \epsilon$.

Theorem 2.3. [1] Assume that conditions **(V1)**–**(V3)** are satisfied. Then there exists an analytic resolvent operator $(R(t))_{t \geq 0}$. Moreover, there exist positive constants N, N_α such that $\|R(t)\| \leq N \text{and} \|A^\alpha R(t)\| \leq \frac{N_\alpha}{t^\alpha}$ for $t > 0$ and $0 \leq \alpha < 1$.

We take the following hypothesis.

(H0) The semigroup $(T(t))_{t \geq 0}$ is compact for $t > 0$.

Theorem 2.4. [19] Under the conditions **(V1)**–**(V3)** and **(H0)**, the corresponding resolvent operator $(R(t))_{t \geq 0}$ is compact for $t > 0$.

3. Global existence, uniqueness, and continuous dependence with respect to the initial data

Definition 3.1. A function $u : [0, b] \to \mathbb{Y}_\alpha$ is called a strict solution of Eq. (1), if:

i. $t \to u(t)$ is continuously differentiable on $[0, b]$.

ii. $u(t) \in \mathbb{Y}$ for $t \in [0, b]$.

iii. u satisfies Eq. (1) on $[0, b]$.

Definition 3.2. A continuous function $u : [0, b] \to \mathbb{Y}_\alpha$ is called a mild solution of Eq. (1) if

$$\begin{cases} u(t) = R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)F(s, u_s)ds \text{ for } t \in [0, b], \\ u_0 = \varphi \in \mathcal{C}_\alpha. \end{cases} \tag{5}$$

Now to obtain our first result, we take the following assumption.

(H1) There exists a constant $L_F > 0$ such that

$$\|F(t, \varphi_1) - F(t, \varphi_2)\| \leq L_F \|\varphi_1 - \varphi_2\|_\alpha \text{ for } t \geq 0 \text{ and } \varphi_1, \varphi_2 \in \mathcal{C}_\alpha.$$

Theorem 3.3. Assume that **(V1)**–**(V3)** and **(H1)** hold. Then for $\varphi \in \mathcal{C}_\alpha$, Eq. (1) has a unique mild solution which is defined for all $t \geq 0$.

Proof. Let $a > 0$. For $\varphi \in \mathcal{C}_\alpha$, we define the set $\wedge$ by

$$\wedge = \{y \in C([0, a]; \mathbb{Y}_\alpha) : y(0) = \varphi(0)\}.$$

The set $\wedge$ is a closed subset of $C([0, a]; \mathbb{Y}_\alpha)$ where $C([0, a]; \mathbb{Y}_\alpha)$ is the space of continuous functions from $[0, a]$ to $\mathbb{Y}_\alpha$ equipped with the uniform norm topology

$$\|y\|_\alpha = \sup_{0 \le t \le a} \|y(t)\|_\alpha \text{ for } y \in C([0,a]; \mathbb{Y}_\alpha).$$

For $y \in \wedge$, we introduce the extension $\overline{y}$ of y on $[-r,a]$ defined by $\overline{y}(t) = y(t)$ for $t \in [0,a]$ and $\overline{y}(t) = \varphi(t)$ for $t \in [-r,0]$. We consider the operator Γ defined on $\wedge$ by

$$\Gamma y(t) = R(t)\varphi(0) + \int_0^t R(t-s)F\left(s, \overline{y}_s\right) ds \text{ for } t \in [0,a].$$

We claim that $\Gamma(\wedge) \subset \wedge$. In fact for $y \in \wedge$, we have $(\Gamma y)(0) = \varphi(0)$, and by continuity of F and $R(t)x$ for $x \in \mathbb{X}$, we deduce that $\Gamma y \in \wedge$. In order to obtain our result, we apply the strict contraction principle. In fact, let $u, v \in \wedge$ and $t \in [0,a]$. Then

$$(\Gamma(u) - \Gamma(v))(t) = \int_0^t R(t-s)(F(s, \overline{u}_s) - F(s, \overline{v}_s))\, ds.$$

Using the $\alpha-$ norm, we have

$$\begin{aligned}
\|A^\alpha(\Gamma(u) - \Gamma(v))(t)\| &\le L_F N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \|\overline{u}_s - \overline{v}_s\|_\alpha\, ds \\
&\le L_F N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \le \tau \le a} \|u(\tau) - v(\tau)\|_\alpha\, ds \\
&\le \left(L_F N_\alpha \int_0^a \frac{ds}{s^\alpha} \right) \|u - v\|_\alpha.
\end{aligned}$$

Now we choose a such that

$$L_F N_\alpha \int_0^a \frac{ds}{s^\alpha} < 1.$$

Then Γ is a strict contraction on $\wedge$, and it has a unique fixed point y which is the unique mild solution of Eq. (1) on $[0,a]$. To extend the solution of Eq. (1) in $[a, 2a]$, we show that the following equation has a unique mild solution:

$$\begin{cases} \dfrac{d}{dt} z(t) = -Az(t) + \displaystyle\int_a^t B(t-s)\, z(s)\, ds + F(t, z_t) \text{ for } t \in [a, 2a], \\ z_a = y_a \in C([-r,a], \mathbb{Y}_\alpha). \end{cases} \tag{6}$$

Notice that the solution of Eq. (6) is given by

$$z(t) = R(t-a)z(a) + \int_a^t R(t-s)F(s, z_s)\, ds \text{ for } t \in [a, 2a].$$

Let $\overline{z}$ be the function defined by $\overline{z}(t) = z(t)$ for $t \in [a, 2a]$ and $\overline{z}(t) = y(t)$ for $t \in [-r, a]$. Consider now again the set $\wedge$ defined by

$$\wedge = \{ z \in C([a, 2a]; \mathbb{Y}_\alpha) : z(a) = y(a) \},$$

provided with the induced topological norm. We define the operator Γ_a on $\wedge$ by

$$(\Gamma_a z)(t) = R(t-a)z(a) + \int_a^t R(t-s)F(s, \overline{z}_s)\, ds \text{ for } t \in [a, 2a].$$

We have $(\Gamma_a z)(a) = y(a)$ and $\Gamma_a z$ is continuous. Then it follows that $\Gamma_a \wedge \subset \wedge$. Moreover, for $u, v \in \wedge$, one has

$$\|A^\alpha(\Gamma_a(u) - \Gamma_a(v))(t)\| \leq L_F N_\alpha \int_a^t \frac{1}{(t-s)^\alpha} \|\overline{u}_s - \overline{v}_s\|_\alpha \, ds.$$

Since $\overline{u} = \overline{v} = \varphi$ in $[-r, 0]$, we deduce that

$$\|A^\alpha(\Gamma_a(u) - \Gamma_a(v))\| \leq \left(L_F N_\alpha \int_0^a \frac{ds}{s^\alpha}\right) \|u - v\|_\alpha.$$

Then we deduce that Γ_a has a unique fixed point in $\wedge$ which extends the solution y in $[a, 2a]$. Proceeding inductively, y is uniquely and continuously extended to $[na, (n+1)a]$ for all $n \geq 1$, and this ends the proof.

Now we show the continuous dependence of the mild solutions with respect to the initial data.

Theorem 3.4. Assume that **(V1)**–**(V3)** and **(H1)** hold. Then the mild solution $u(., \varphi)$ of Eq. (1) defines a continuous Lipschitz operator $U(t), t \geq 0$ in $\mathcal{C}_\alpha$ by $U(t)\varphi = u_t(., \varphi)$. That is, $U(t)\varphi$ is continuous from $[0; \infty)$ to $\mathcal{C}_\alpha$ for each fixed $\varphi \in \mathcal{C}_\alpha$. Moreover there exist a real number δ and a scalar function P such that for $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathcal{C}_\alpha$ we have

$$\|U(t)\varphi_1 - U(t)\varphi_2\| \leq P(\delta) e^{\delta t} \|\varphi_1 - \varphi_2\|_\alpha. \tag{7}$$

Proof. We use the gamma formula

$$\Gamma(1-\alpha) k^{\alpha-1} = \int_0^\infty e^{-ks} s^{-\alpha} ds,$$

where $k > 0$ (see [20], p. 265). The continuity is obvious that the map $t \to u_t(., \varphi)$ is continuous. Now, let $\varphi_1, \varphi_2 \in \mathcal{C}_\alpha$. If we pose $w(t) = u(t, \varphi_1) - u(t, \varphi_2)$, then we have

$$\|w(t)\|_\alpha \leq M_1 e^{\sigma t} \|\varphi_1 - \varphi_2\|_\alpha + L_F N_\alpha \int_0^t \frac{e^{\sigma(t-s)}}{(t-s)^\alpha} \|w_s\|_\alpha ds. \tag{8}$$

Let δ a real number be such that

$$\sigma - \delta < 0 \text{ and } M_1 \max\{e^{-\delta r}, 1\} L_F \Gamma(1-\alpha)(\delta - \sigma)^{\alpha-1} < 1.$$

We define the function P by

$$P(\delta) = M_1 M_3 M_4 \left(1 - M_1 M_4 L_F \Gamma(1-\alpha)(\delta-\sigma)^{\alpha-1}\right)^{-1}$$

where

$$M_3 = \max\{e^{\delta r}, 1\}, M_4 = \max\{e^{-\delta r}, 1\}.$$

Fix $\bar{t} > 0$ and let $E = \sup_{0 \leq s \leq \bar{t}} e^{-\delta s} \|w_s\|$. If $0 \leq \tau \leq \bar{t}$, then from Eq. (8), we have

$$\begin{aligned} e^{-\delta\tau} \|w(\tau)\|_\alpha &\leq M_1 e^{(\sigma-\delta)\tau} \|\varphi_1 - \varphi_2\|_\alpha + L_F N_\alpha \int_0^\tau \frac{e^{(\sigma-\delta)(\tau-s)}}{(\tau-s)^\alpha} e^{-\delta s} \|w_s\|_\alpha ds \\ &\leq M_1 \|\varphi_1 - \varphi_2\|_\alpha + L_F M_1 E \Gamma(1-\alpha)(\delta-\sigma)^{\alpha-1}. \end{aligned} \tag{9}$$

If $-r \le \tau \le 0$, we have

$$e^{-\delta\tau}\|w(\tau)\|_\alpha \le \|\varphi_1 - \varphi_2\|_\alpha M_3. \tag{10}$$

Therefore, Eqs. (9) and (10) imply that

$$\sup_{-r \le \tau \le \bar{t}} e^{-\delta\tau}\|w(\tau)\|_\alpha \le M_1M_3\|\varphi_1 - \varphi_2\|_{\mathcal{C}_\alpha} + L_FM_1E\Gamma(1-\alpha)(\delta-\sigma)^{\alpha-1}. \tag{11}$$

For $0 \le t \le \bar{t}$, we have

$$\begin{aligned} e^{-\delta t}\|w_t\|_\alpha &= \sup_{-r \le \theta \le 0} e^{\delta\theta}e^{-\delta(t+\theta)}\|w(t+\theta)\|_\alpha \\ &\le M_4 \sup_{-r \le \theta \le 0} e^{-\delta(t+\theta)}\|w(t+\theta)\|_\alpha \\ &\le M_4 \sup_{-r \le \tau \le \bar{t}} e^{-\delta\tau}\|w(\tau)\|_\alpha. \end{aligned} \tag{12}$$

Then from Eqs. (11) and (12), we deduce that for $0 \le t \le \bar{t}$

$$e^{-\delta t}\|w_t\|_\alpha \le M_1M_3M_4\|\varphi_1 - \varphi_2\|_\alpha + L_FM_1M_3E\Gamma(1-\alpha)(\delta-\sigma)^{\alpha-1},$$

which implies that

$$E \le M_1M_3M_4\|\varphi_1 - \varphi_2\|_\alpha + L_FM_1M_4E\Gamma(1-\alpha)(\delta-\sigma)^{\alpha-1}.$$

Then the result follows.

4. Local existence, blowing up phenomena, and the compactness of the flow

We start by generalizing a result, obtained in [19] in the case of the usual norm on $\mathbb{X}$ ($\alpha = 0$), in the case where $\alpha \ne 0$. We take the following assumption.

(H2) $B(t) \in \mathcal{L}(\mathbb{X}_\beta, \mathbb{X})$ for some $0 < \beta < 1$, a.e $t \ge 0$ and $\|B(t)x\| \le b(t)\|x\|_\beta$ for $x \in \mathbb{X}_\beta$, with $b \in \mathbf{L}^q_{loc}(0,\infty)$ where $q > 1/(1-\beta)$.

Theorem 4.1. Assume that **(V1)**–**(V3)** and **(H2)** hold. Then for any $a > 0$, there exists a positive constant $M = M(a)$ such that for $x \in \mathbb{X}$ we have

$$\|A^\alpha(R(t+h)x - R(h)R(t)x)\| \le M\int_0^h \frac{ds}{s^\alpha}\|x\| \text{ for } 0 \le h < t \le a.$$

Proof. Let $a > 0$ and $x \in \mathbb{X}$. Then

$$\begin{aligned} \frac{d}{dt}R(t+h)x &= -AR(t+h)x + \int_0^{t+h} B(t+h-s)R(s)x\,ds \\ &= -AR(t+h)x + \int_0^t B(t-s)R(s+h)x\,ds \\ &\quad + \int_t^{t+h} B(t+h-s)R(s)x\,ds. \end{aligned}$$

We deduce that $R(t+h)x$ satisfies the equation of the form

$$\frac{d}{dt}y(t) = -Ay(t) + \int_0^t B(t-s)y(s)\,ds + f(t).$$

Then by the variation o constante formula, it follows that

$$\begin{aligned} R(t+h)x &= R(t)R(h)x + \int_0^t R(t-s)\int_s^{s+h} B(s+h-u)R(u)x\,du\,ds \\ &= R(h)R(t)x + \int_0^h R(h-s)\int_0^t B(u)R(s+t-u)x\,du\,ds. \end{aligned}$$

Which yields that

$$R(t+h)x - R(h)R(t)x = \int_0^h R(h-s)\int_0^t B(u)R(s+t-u)x\,du\,ds.$$

Taking the $\alpha-$norm, we obtain that

$$\begin{aligned} \|A^\alpha(R(t+h)x - R(h)R(t)x)\| &\le N_\alpha \int_0^h \frac{1}{(h-s)^\alpha}\left\|\int_0^t B(u)R(s+t-u)x\,du\right\|ds \\ &\le N_\alpha \int_0^h \frac{1}{(h-s)^\alpha}\int_0^t b(u)\|A^\beta R(t+s-u)x\|du\,ds \\ &\le N_\alpha N_\beta \int_0^h \frac{ds}{(h-s)^\alpha}\int_0^t \frac{b(u)}{(t-u)^\beta}\|x\|du. \end{aligned}$$

Let p be such that $1/q + 1/p = 1$, so $p < 1/\beta$. Then it follows that

$$\|A^\alpha(R(t+h)x - R(h)R(t)x)\| \le N_\alpha N_\beta \|b\|_{\mathbf{L}^q(0,a)}\left\|u^{-\beta}\right\|_{\mathbf{L}^p(0,a)}\int_0^h \frac{ds}{s^\alpha}\|x\|.$$

And the proof is complete.

The local existence result is given by the following Theorem.

Theorem 4.2. Suppose that **(V1)**–**(V3)**, **(H0)**, and **(H2)** hold. Moreover, assume that F defined from $J \times \Omega$ into $\mathbb{X}$ is continuous where $J \times \Omega$ is an open set in $\mathbb{R}_+ \times \mathcal{C}_\alpha$. Then for each $\varphi \in \Omega$, Eq. (1) has at least one mild solution which is defined on some interval $[0, b]$.

Proof. Let $\varphi \in \Omega$. For any real $\zeta \in J$ and $p > 0$, we define the following sets:

$$I_\zeta = \{t : 0 \le t \le \zeta\} \quad \text{and} \quad H_p = \left\{\phi \in \mathcal{C}_\alpha : \|\phi\|_\alpha \le p\right\}.$$

For $\phi \in H_p$, we choose ζ and p such that $(t, \phi + \varphi) \in I_\zeta \times H_p$ and $H_p \subseteq \Omega$. By continuity of F, there exists $N_1 \ge 0$ such that $\|F(t, \phi + \varphi)\| \le N_1$ for (t, ϕ) in $I_\zeta \times H_p$. We consider $\overline{\varphi} \in C([-r, \zeta]; \mathbb{Y}_\alpha)$ as the function defined by $\overline{\varphi}(t) = R(t)\varphi(0)$ for $t \in I_\zeta$ and $\overline{\varphi}_0 = \varphi$. Suppose that $\overline{p} < p$ and choose $0 < b < \zeta$ such that

$$N_\alpha N_1 \int_0^b \frac{ds}{s^\alpha} < \overline{p} \quad \text{and} \quad \|\overline{\varphi}_t - \varphi\|_\alpha \le p - \overline{p} \text{ for } t \in I_b.$$

Let $K_0 = \left\{\eta \in C([-r, b]; \mathbb{Y}_\alpha) : \eta_0 = 0 \text{ and } \|\eta_t\|_\alpha \le \overline{p} \text{ for } 0 \le t \le b\right\}$. Then we have $\|F(t, \overline{\varphi}_t + \eta_t)\| \le N_1$ for $0 \le t \le b$ and $\eta \in K_0$, since $\|\eta_t + \overline{\varphi}_t - \varphi\|_\alpha \le p$. Consider the mapping S from K_0 to $C([-r, b]; \mathbb{Y}_\alpha)$ defined by $(S\eta)(0) = 0$

$$(S\eta)(t) = \int_0^t R(t-s)F(s, \eta_s + \overline{\varphi}_s)\,ds \text{ for } 0 \le t \le b. \tag{13}$$

Notice that finding a fixed point of S in K_0 is equivalent to finding a mild solution of Eq. (1) in K_0. Furthermore, S is a mapping from K_0 to K_0, since if $\eta \in K_0$ we have $(S\eta)_0 = 0$ and

$$\|(S\eta)(t)\|_\alpha \le \int_0^t \|A^\alpha R(t-s)F(s, \eta_s + \overline{\varphi}_s)\|ds.$$

Then

$$\begin{aligned}\|(S\eta)(t)\|_\alpha &\le N_\alpha N_1 \int_0^t \frac{ds}{(t-s)^\alpha} \\ &\le N_\alpha N_1 \int_0^b \frac{ds}{s^\alpha} < \overline{p}\end{aligned}$$

which implies that $S(K_0) \subset K_0$. We claim that $\{(S\eta)(t)) : \eta \in K_0\}$ is compact in $\mathbb{Y}_\alpha$ for fixed $t \in [-r, b]$. In fact, let β be such that $0 < \alpha \le \beta < 1$. The above estimate show that $\{A^\beta (S\eta)(t) : \eta \in K_0\}$ is bounded in $\mathbb{X}$. Since $A^{\alpha-\beta}$ is compact operator, we infer that $\{A^{\alpha-\beta}A^\beta (S\eta)(t) : \eta \in K_0\}$ is compact in $\mathbb{X}$, hence $\{(S\eta)(t) : \eta \in K_0\}$ is compact in $\mathbb{Y}_\alpha$. Next, we show that $\{(S\eta)(t) : \eta \in K_0\}$ is equicontinuous. The equicontinuity of $\{(S\eta)(t) : \eta \in K_0\}$ at $t = 0$ follows from the above estimation of $(S\eta)(t)$. Now let $0 < t_0 < t \le b$ with t_0 be fixed. Then we have

$$\begin{aligned}\|A^\alpha((S\eta)(t) - (S\eta)(t_0))\| \le &\int_0^{t_0} \|A^\alpha(R(t-s) - R(t-t_0)R(t_0-s))F(s, \eta_s + \overline{\varphi}_s)\|\,ds \\ &+ \left\|A^\alpha(R(t-t_0) - I)\int_0^{t_0} R(t_0-s)F(s, \eta_s + \overline{\varphi}_s)\,ds\right\| \\ &+ \int_{t_0}^t \|A^\alpha R(t-s)F(s, \eta_s + \overline{\varphi}_s)\|\,ds.\end{aligned} \tag{14}$$

Using Theorem 4.1, it follows that

$$\begin{aligned}&\|A^\alpha((S\eta)(t) - (S\eta)(t_0))\| \\ &\le t_0 N_1 M \int_0^{t-t_0} \frac{ds}{s^\alpha} + \left\|(R(t-t_0) - I)A^\alpha \int_0^{t_0} R(t_0-s)F(s, \eta_s + \overline{\varphi}_s)ds\right\| \\ &+ N_\alpha N_1 \int_0^{t-t_0} \frac{1}{s^\alpha}ds.\end{aligned}$$

As the set $\{(S\eta)(t_0) : \eta \in K_0\}$ is compact in $\mathbb{Y}_\alpha$, we have

$$\lim_{t \to t_0^+} \|(S\eta)(t) - (S\eta)(t_0)\|_\alpha = 0 \;\text{ uniformly }\; \text{in } \eta \in K_0.$$

We obtain the same results by taking t_0 be fixed with $0 < t < t_0 \le b$. Then we claim that $\lim_{t \to t_0} \|(S\eta)(t) - (S\eta)(t_0)\|_\alpha = 0$ uniformly in $\eta \in K_0$ which means that $\{(S\eta)(t) : \eta \in K_0\}$ is equicontinuous. Then by Ascoli-Arzela theorem, $\{S\eta : \eta \in K_0\}$

is relatively compact in K_0. Finally, we prove that S is continuous. Since F is continuous, given $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\sup_{0 \leq s \leq b} \|\eta(s) - \hat{\eta}(s)\|_\alpha < \delta \text{ implies that } \|F(s, \eta_s + \overline{\varphi}_s) - F(s, \hat{\eta}(s) + \overline{\varphi}_s)\| < \varepsilon.$$

Then for $0 \leq t \leq b$, we have

$$\begin{aligned} \|(S\eta)(t) - (S\hat{\eta})(t)\|_\alpha &\leq N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \|F(s, \eta_s + \overline{\varphi}_s) - F(s, \hat{\eta}(s) + \overline{\varphi}_s)\| ds \\ &\leq N_\alpha \varepsilon \int_0^t \frac{ds}{s^\alpha}. \end{aligned}$$

This yields the continuity of S, and using Schauder's fixed point theorem, we deduce that S has a fixed point. Then the proof of the theorem is complete.

The following result gives the blowing up phenomena of the mild solution in finite times.

Theorem 4.3. Assume that **(V1)**–**(V3)**, **(H0)**, and **(H2)** hold and F is a continuous and bounded mapping. Then for each $\varphi \in \mathcal{C}_\alpha$, Eq. (1) has a mild solution $u(.,\varphi)$ on a maximal interval of existence $[-r, b_\varphi)$. Moreover if $b_\varphi < \infty$, then $\overline{\lim}_{t \to b_\varphi^-} \|u(t,\varphi)\|_\alpha = +\infty$.

Proof. Let $u(.,\varphi)$ be the mild solution of Eq. (1) defined on $[0, b]$. Similar arguments used in the local existence result can be used for the existence of $b_1 > b$ and a function $u(., u_b)$ defined from $[b, b_1]$ to $\mathbb{Y}_\alpha$ satisfying

$$u(t, u_b(.,\varphi)) = R(t)u(b,\varphi) + \int_b^t R(t-s)F(s, u_s)ds \text{ for } t \in [b, b_1].$$

By a similar proceeding, we show that the mild solution $u(.,\varphi)$ can be extended to a maximal interval of existence $[-r, b_\varphi)$. Assume that $b_\varphi < +\infty$ and $\overline{\lim}_{t \to b_\varphi^-} \|u(t,\varphi)\|_\alpha < +\infty$. There exists $N_2 > 0$ such that $\|F(s, u_s)\| \leq N_2$, for $s \in [0, b_\varphi)$. We claim that $u(.,\varphi)$ is uniformly continuous. In fact, let $0 < h \leq t \leq t + h < b_\varphi$. Then

$$\begin{aligned} u(t+h) - u(t) &= (R(t+h) - R(t))\varphi(0) + \int_0^t (R(t+h-s) - R(t-s))F(s, u_s)\, ds \\ &\quad + \int_t^{t+h} R(t+h-s)F(s, u_s)\, ds. \end{aligned}$$

By continuity of $A^\alpha R(t)$, we claim that $A^\alpha (R(t+h) - R(t))\varphi(0)$ is uniformly continuous on each compact set. Moreover, Theorem 4.1 implies that $A^\alpha (R(t+h-s) - R(t-s))F(s, u_s) \to 0$ uniformly in t when $h \to 0$. In fact we have

$$\begin{aligned} &\int_0^t \|(R(t+h-s) - R(t-s))F(s, u_s)\|_\alpha ds \\ &\qquad \leq \int_0^t \|(R(t+h-s) - R(h)R(t-s))F(s, u_s)\|_\alpha ds \\ &\qquad + \left\|(R(h) - I)A^\alpha \textstyle\int_0^t R(t-s)F(s, u_s)\, ds\right\| \end{aligned}$$

Then using Theorem 4.1, we obtain that

$$\int_0^t \|(R(t+h-s)-R(t-s))F(s,u_s)\|_\alpha ds$$
$$\leq b_\varphi N_2 M \int_0^h \frac{ds}{s^\alpha} + \left\|(R(h)-I)A^\alpha \int_0^t R(t-s)F(s,u_s)\,ds\right\|.$$

We claim that the set $\left\{A^\alpha \int_0^t R(t-s)F(s,u_s)\,ds : t \in \left[0, b_\varphi\right)\right\}$ is relatively compact. In fact, let $(t_n)_{n\geq 0}$ be a sequence of $\left[0, b_\varphi\right)$. Then there exist a subsequence $\left(t_{n_k}\right)_k$ and a real number t_0 such that $t_{n_k} \to t_0$. Using the dominated convergence theorem, we deduce that

$$\int_0^{t_{nk}} A^\alpha R\left(t_{n_k}-s\right)F(s,u_s)\,ds \to \int_0^{t_0} A^\alpha R(t_0-s)F(s,u_s)\,ds.$$

This implies that $\left\{A^\alpha \int_0^t R(t-s)F(s,u_s)\,ds : t \in \left[0, b_\varphi\right)\right\}$ is relatively compact. Now using Banach-Steinhaus' theorem, we deduce that

$$(R(h)-I)A^\alpha \int_0^t R(t-s)F(s,u_s)\,ds \to 0$$

uniformly when $h \to 0$ with respect to $t \in \left[0, b_\varphi\right)$. Moreover we have

$$\| \int_t^{t+h} R(t+h-s)F(s,u_s)\,ds\|_\alpha \leq N_2 N_\alpha \int_0^h \frac{ds}{s^\alpha}.$$

Consequently $\|u(t+h)-u(t)\|_\alpha \to 0 \ \ as \ \ h \to 0$ uniformly in $t \in \left[0, b_\varphi\right)$. If $h \leq 0$, that is, for $t \leq t_0$, we have

$$u(t)-u(t_0) = (R(t)-R(t_0))\varphi(0) - \int_0^t (R(t_0-s)-R(t_0-t)R(t-s))F(s,u_s)\,ds$$
$$-(R(t_0-t)-I)\int_0^t R(t-s)F(s,u_s)\,ds - \int_t^{t_0} R(t_0-s)F(s,u_s)\,ds,$$

one can show similar results by using the same reasoning. This implies that $u(.,\varphi)$ is uniformly continuous. Therefore $\lim_{t\to b_\varphi^-} u(t,\varphi)$ *exists in* $\mathbb{Y}_\alpha$. And consequently, $u(.,\varphi)$ can be extended to b_φ which contradicts the maximality of $\left[0, b_\varphi\right)$.

The next result gives the global existence of the mild solutions under weak conditions of F. To achieve our goal, we introduce a following necessary result which is a consequence of Lemma 7.1.1 given in ([21], p. 197, Exo 4).

Lemma 4.4. [21] Let $\alpha, a, b \geq 0$, $\beta < 1$ and $0 < d < \infty$. Also assume that v is nonnegative and locally integrable on $[0,d)$ with

$$v(t) \leq \frac{a}{t^\alpha} + b\int_0^t \frac{v(s)}{(t-s)^\beta} ds \text{ for } t \in (0,d).$$

Then there exists a constant $M_2 = M_2(a,b,\alpha,\beta,d) < \infty$ such that $v(t) \leq M_2/t^\alpha$ on $(0,d)$.

Theorem 4.5. Assume that **(V1)**–**(V3)**, **(H0)**, and **(H2)** hold and F is a completely continuous function on $\mathbb{R}_+ \times \mathcal{C}_\alpha$. Moreover suppose that there exist continuous nonnegative functions f_1 and f_2 such that $\|F(t,\varphi)\| \leq f_1(t)\|\varphi\|_\alpha + f_2(t)$ for $\varphi \in \mathcal{C}_\alpha$ and $t \geq 0$. Then Eq. (1) has a mild solution which is defined for $t \geq 0$.

Proof. Let $[0, b_\varphi)$ be the maximal interval of existence of a mild solution $u(., \varphi)$. Assume that $b_\varphi < +\infty$. By Theorem 4.3 we have $\overline{\lim}_{t \to t_\varphi^-} \|u(t, \varphi)\|_\alpha = +\infty$. Recall that the solution of Eq. (1) is given by $u_0 = \varphi$ and

$$u(t, \varphi) = R(t)\varphi(0) + \int_0^t R(t-s)F(s, u_s(., \varphi))\, ds \text{ for } t \in [0, b_\varphi).$$

Then taking the $\alpha-$norm, we obtain

$$\|u(t, \varphi)\|_\alpha \le \|R(t)\| \|\varphi(0)\|_\alpha + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^\alpha}\, ds + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \|u_s(., \varphi)\|_\alpha ds,$$

where $k_1 = \max_{0 \le t \le b_\varphi} |f_1(t)|$ and $k_2 = \max_{0 \le t \le b_\varphi} |f_2(t)|$. Then we deduce that

$$\|u(t, \varphi)\|_\alpha \le N \|\varphi(0)\|_\alpha + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \le \tau \le s} \|u(\tau, \varphi)\|_\alpha\, ds + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^\alpha}. \quad (15)$$

Now we claim that the function

$$t \to \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \le \tau \le s} \|u(\tau, \varphi)\|_\alpha ds,$$

is nondecreasing. In fact, let $0 \le t_1 \le t_2$. Then

$$\begin{aligned}
\int_0^{t_1} \frac{1}{(t_1-s)^\alpha} \sup_{-r \le \tau \le s} \|u(\tau, \varphi)\|_\alpha ds &= \int_0^{t_1} \frac{1}{s^\alpha} \sup_{-r \le \tau \le t_1 - s} \|u(\tau, \varphi)\|_\alpha ds \\
&\le \int_0^{t_2} \frac{1}{s^\alpha} \sup_{-r \le \tau \le t_2 - s} \|u(\tau, \varphi)\|_\alpha ds \\
&= \int_0^{t_2} \frac{1}{(t_2-s)^\alpha} \sup_{-r \le \tau \le s} \|u(\tau, \varphi)\|_\alpha ds
\end{aligned}$$

which yields the result. Then it follows from Eq. (15) that

$$\sup_{-r \le s \le t} \|u(s, \varphi)\|_\alpha \le N \|\varphi(0)\|_\alpha + k_2 N_\alpha \int_0^{b_\varphi} \frac{ds}{s^\alpha}\, ds + k_1 N_\alpha \int_0^t \frac{1}{(t-s)^\alpha} \sup_{-r \le \tau \le s} \|u(\tau, \varphi)\|_\alpha ds.$$

Then using Lemma 4.4, we deduce that $u(., \varphi)$ is bounded in $[0, b_\varphi)$. Then we obtain that $\overline{\lim}_{t \to b_\varphi^-} \|u(t, \varphi)\|_\alpha < \infty$, which contradicts our hypothesis. Then the mild solution is global.

We focus now to the compactness of the flow defined by the mild solutions.

Theorem 4.6. Assume that (**V1**)–(**V3**) and (**H0**)–(**H2**) hold. Then the flow $U(t)$ defined from $\mathcal{C}_\alpha$ to $\mathcal{C}_\alpha$ by $U(t)\varphi = u_t(., \varphi)$ is compact for $t > r$, where $u_t(., \varphi)$ denotes the mild solution starting from φ.

Proof. We use Ascoli-Arzela's theorem. Let $E = \{\varphi_\gamma : \gamma \in \Gamma\}$ be a bounded subset of $\mathcal{C}_\alpha$ and let $t > r$ be fixed, but arbitrary. We will prove that $\overline{U(t)E}$ is compact. It follows from (**H1**) and inequality Eq. (7) that there exists N_5 such that

$$\|F(t, u_t(\varphi_\gamma))\| \le N_2 \|u_t(\varphi_\gamma))\| + \|F(t, 0)\| = N_5 \text{ for } \gamma \in \Gamma.$$

For each $\gamma \in \Gamma$, we define $f_\gamma \in \mathcal{C}_\alpha$ by $f_\gamma = u_t(.,\varphi_\gamma)$. We show now that for fixed $\theta \in [-r, 0]$, the set $\{f_\gamma(\theta) : \gamma \in \Gamma\}$ is precompact in $\mathbb{Y}_\alpha$. For any $\gamma \in \Gamma$, we have

$$f_\gamma(\theta) = R(t+\theta)\varphi_\gamma(0) + \int_0^{t+\theta} R(t+\theta-s)F(s, u_s(.,\varphi))\,ds.$$

As $R(t)$ is compact for $t > 0$, we need only to prove that the set

$$\left\{\int_0^{t+\theta} R(t+\theta-s)F\left(s, u_s\left(.,\varphi_\gamma\right)\right)ds : \gamma \in \Gamma\right\}$$

is compact. Also we have

$$\mu\left(\left\{R(\varepsilon)\int_0^{t+\theta-\varepsilon} R(t+\theta-\varepsilon-s)F\left(s, u_s\left(.,\varphi_\gamma\right)\right)ds : \gamma \in \Gamma\right\}\right) = 0,$$

where μ is the measure of non-compactness. Moreover, using Theorem 4.1, we have

$$\begin{aligned}
&\left\|A^\alpha\left(\int_0^{t+\theta-\varepsilon} R(t+\theta-\varepsilon-s) - R(\varepsilon)R(t+\theta-\varepsilon-s)F(s, u_s(.,\varphi_\gamma))\,ds\right)\right\| \\
&\qquad \le \int_0^{t+\theta-\varepsilon} \|(R(t+\theta-s) - R(\varepsilon)R(t+\theta-\varepsilon-s))F(s, u_s(.,\varphi))\|_\alpha\,ds \\
&\qquad \le N_5 M\int_0^\varepsilon \frac{ds}{s^\alpha} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

We deduce that

$$\mu\left(\left\{\int_0^{t+\theta-\varepsilon} R(t+\theta-s)F\left(s, u_s\left(.,\varphi_\gamma\right)\right)ds : \gamma \in \Gamma\right\}\right) = 0.$$

On the other hand, for $0 < \alpha \le \beta < 1$, we have

$$\begin{aligned}
\left\|A^\beta\int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)F(s, u_s(.,\varphi_\gamma))ds\right\| &\le \int_{t+\theta-\varepsilon}^{t+\theta} \left\|R(t+\theta-s)F(s, u_s(.,\varphi_\gamma))\right\|_\beta ds \\
&\le N_\beta N_5\int_{t+\theta-\varepsilon}^{t+\theta} \frac{ds}{(t+\theta-s)^\beta} \\
&= N_\beta N_5\int_0^\varepsilon \frac{ds}{s^\beta} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Thus $\left\{A^\beta\int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)F\left(s, u_s\left(.,\varphi_\gamma\right)\right)ds : \gamma \in \Gamma\right\}$ is a bounded subset of $\mathbb{X}$. The precompactness in $\mathbb{Y}_\alpha$ now follows from the compactness of $A^{-\beta} : \mathbb{X} \to \mathbb{Y}_\alpha$. Then the set $\{(U(t)E)(\theta) : -r \le \theta \le 0\}$ is precompacted in $\mathbb{Y}_\alpha$. We prove that the family $\{f_\gamma : \gamma \in \Gamma\}$ is equicontinuous. Let γ in Γ, $0 < \varepsilon < t - r$, and $-r \le \hat{\theta} \le \theta \le 0$ with $\hat{\theta}$ be fixed and $h = \theta - \hat{\theta}$. Then

$$\begin{aligned}\left\|A^\alpha\left(f_\gamma(h+\hat\theta)-f_\gamma(\hat\theta)\right)\right\| &\leq \|R(t+\hat\theta+h)-R(t+\hat\theta)\varphi_\gamma(0)\|_\alpha\\ &+\int_0^{t+\hat\theta}\|A^\alpha(R(t+\hat\theta+h-s)-R(h)R(t+\hat\theta-s))F(s,u_s(.,\varphi_\gamma)\|ds\\ &+\left\|(R(h)-I)A^\alpha\int_0^{t+\hat\theta}R(t+\hat\theta-s)F(s,u_s(.,\varphi_\gamma))\,ds\right\|\\ &+\int_{t+\hat\theta}^{t+\hat\theta+h}\|A^\alpha R(t+\hat\theta+h-s)F(s,u_s(.,\varphi_\gamma))\|\,ds.\end{aligned}$$

Then it follows that

$$\begin{aligned}\left\|A^\alpha\left(f_\gamma(h+\hat\theta)-f_\gamma(\hat\theta)\right)\right\| &\leq \|(R(t+\hat\theta+h)-R(t+\hat\theta))A^\alpha\varphi_\gamma(0)\|+MN_5(t+\hat\theta)\int_0^h\frac{ds}{s^\alpha}\\ &+\left\|(R(h)-I)A^\alpha\int_0^{t+\hat\theta}R(t+\hat\theta-s)F(s,u_s(.,\varphi_\gamma))\,ds\right\|\\ &+N_5N_\alpha\int_0^h\frac{ds}{s^\alpha}.\end{aligned}$$

Using the compactness of the set $\left\{A^\alpha\int_0^{t+\theta}R(t+\theta-s)F\left(s,u_s\left(.,\varphi_\gamma\right)\right)ds:\gamma\in\Gamma\right\}$ and the continuity of $t\to R(t)x$ for $x\in\mathbb{X}$, the right side of the above inequality can be made sufficiently small for $h>0$ small enough. Then we conclude that $\left\{f_\gamma:\gamma\in\Gamma\right\}$ is equicontinuous. Consequently, by Ascoli-Arzela's theorem, we conclude that the set $\{U(t)\varphi:\varphi\in E\}$ is compact, which means that the operator $U(t)$ is compact for $t>r$.

5. Regularity of the mild solutions

We define the set $\mathcal{C}_\alpha^1$ by $\mathcal{C}_\alpha^1=C^1([-r,0];\mathbb{Y}_\alpha)$ as the set of continuously differentiable functions from $[-r,0]$ to $\mathbb{Y}_\alpha$. We assume the following hypothesis.

(**H3**) F is continuously differentiable, and the partial derivatives D_tF and $D_\varphi F$ are locally Lipschitz in the classical sense with respect to the second argument.

Theorem 5.1. Assume that (**V1**)–(**V3**), (**H1**), and (**H3**) hold. Let φ in $\mathcal{C}_\alpha^1$ be such that $\varphi(0)\in\mathbb{Y}$ and $\dot\varphi(0)=-A\varphi(0)+F(0,\varphi)$. Then the corresponding mild solution u becomes a strict solution of Eq. (1).

Proof. Let $a>0$. Take $\varphi\in\mathcal{C}_\alpha^1$ such that $\varphi(0)\in\mathbb{Y}$ and $\dot\varphi(0)=-A\varphi(0)+F(0,\varphi)$, and let u be the mild solution of Eq. (1) which is defined on $[0,+\infty[$. Consider the following equation:

$$\begin{cases}v(t)=R(t)\dot\varphi(0)+\displaystyle\int_0^tR(t-s)\left[D_tF(s,u_s)+D_\varphi F(s,u_s)v_s\right]ds\\ \qquad+\displaystyle\int_0^tR(t-s)B(s)\varphi(0)\,ds\text{ for }t\geq0,\\ v_0=\dot\varphi.\end{cases}\tag{16}$$

Using the strict contraction principle, we can show that there exists a unique continuous function v solution in $[0,a]$ of Eq. (16). We introduce the function w defined by

$$w(t)=\begin{cases}\varphi(0)+\int_0^t v(s)\,ds & \text{if } t\geq 0,\\ \varphi(t) & \text{if } -r\leq t\leq 0.\end{cases}$$

Then it follows

$$w_t=\varphi+\int_0^t v_s\,ds \text{ for } t\in[0,a].$$

Consequently, the maps $t\to w_t$ and $t\to\int_0^t R(t-s)F(s,w_s)ds$ are continuously differentiable, and the following formula holds

$$\begin{aligned}\frac{d}{dt}\int_0^t R(t-s)F(s,w_s)\,ds&=R(t)F(0,w_0)+\int_0^t R(t-s)\left[D_tF(s,w_s)+D_\varphi F(s,w_s)v_s\right]ds\\&=R(t)F(0,\varphi)+\int_0^t R(t-s)\left[D_tF(s,w_s)+D_\varphi F(s,w_s)v_s\right]ds.\end{aligned}$$

This implies that

$$\int_0^t R(s)F(0,\varphi)\,ds=\int_0^t R(t-s)F(s,w_s)\,ds-\int_0^t\int_0^s R(s-\tau)\left[D_tF(\tau,w_\tau)+D_\varphi F(\tau,w_\tau)v_\tau\right]d\tau ds.$$

On the other hand, from equality Eq. (4), we have

$$-\int_0^t R(s)A\varphi(0)\,ds=R(t)\varphi(0)-\varphi(0)-\int_0^t\int_0^s R(s-\tau)B(\tau)\varphi(0)\,d\tau ds.$$

We rewrite w as follows:

$$\begin{aligned}w(t)=\varphi(0)&-\int_0^t R(s)A\varphi(0)\,ds+\int_0^t R(s)F(0,\varphi)\,ds\\&+\int_0^t\int_0^s R(s-\tau)\left[D_tF(\tau,u_\tau)+D_\varphi F(\tau,u_\tau)v_\tau\right]d\tau ds\\&+\int_0^t\int_0^s R(s-\tau)B(\tau)\varphi(0)\,d\tau ds.\end{aligned}$$

Then it follows that

$$\begin{aligned}w(t)=R(t)\varphi(0)&+\int_0^t R(t-s)F(s,w_s)\,ds\\&+\int_0^t\int_0^s R(s-\tau)[(D_\tau F(\tau,u_\tau)-D_\tau F(\tau,w_\tau))]\,d\tau ds\\&+\int_0^t\int_0^s \left(D_\varphi F(\tau,u_\tau)v_\tau-D_\varphi F(\tau,w_\tau)v_\tau\right)d\tau ds.\end{aligned}$$

We deduce, for $t\in[0,a]$, that

$$\begin{aligned}\|u(t)-w(t)\|_\alpha\leq&\int_0^t\|A^\alpha R(t-s)(F(s,u_s)-F(s,w_s))\|\,ds\\&+\int_0^t\int_0^s\|A^\alpha R(s-\tau)(D_\tau F(\tau,u_\tau)-D_\tau F(\tau,w_\tau))\|d\tau ds\\&+\int_0^t\int_0^s\left\|A^\alpha R(s-\tau)(D_\varphi F(\tau,u_\tau)-D_\varphi F(\tau,w_\tau))v_\tau\right\|d\tau ds.\end{aligned}\tag{17}$$

The set $H = \{u_s, w_s : s \in [0, a]\}$ is compact in $\mathcal{C}_\alpha$. Since the partial derivatives of F are locally Lipschitz with respect to the second argument, it is well-known that they are globally Lipschitz on H. Then we deduce that

$$\begin{aligned}\|u(t) - w(t)\|_\alpha &\leq N_\alpha h(a) \int_0^t \frac{1}{(t-s)^\alpha} \|u_s - w_s\|_\alpha \, ds \\ &\leq N_\alpha h(a) \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \leq \tau \leq a} \|u(\tau) - w(\tau)\|_\alpha ds,\end{aligned}$$

where $h(a) = L_F N_\alpha + a N_\alpha \mathrm{Lip}(D_t F) + a N_\alpha \mathrm{Lip}(D_\varphi F)$, with $\mathrm{Lip}(D_\varphi F)$ and $\mathrm{Lip}(D_t F)$ the Lipschitz constant of $D_\varphi F$ and $D_t F$, respectively, which implies that

$$\|u - w\|_\alpha \leq \left(N_\alpha h(a) \int_0^a \frac{ds}{s^\alpha} \right) \|u - w\|_\alpha.$$

If we choose a such that

$$N_\alpha h(a) \int_0^a \frac{ds}{s^\alpha} < 1,$$

then $u = w$ in $[0, a]$. Now we will prove that $u = w$ in $[0, +\infty)$. Assume that there exists $t_0 > 0$ such that $u(t_0) \neq w(t_0)$. Let $t_1 = \inf\{t > 0 : \|u(t) - w(t)\| > 0\}$. By continuity, one has $u(t) = w(t)$ for $t \leq t_1$, and there exists $\varepsilon > 0$ such that $\|u(t) - w(t)\| > 0$ for $t \in (t_1, t_1 + \varepsilon)$. Then it follows that for $t \in (t_1, t_1 + \varepsilon)$,

$$\|u(t) - w(t)\|_\alpha \leq N_\alpha h(\varepsilon) \int_0^\varepsilon \frac{ds}{s^\alpha} \sup_{\varepsilon \leq \tau \leq t_1 + \varepsilon} \|u(\tau) - w(\tau)\|_\alpha.$$

Now choosing ε such that

$$N_\alpha h(\varepsilon) \int_0^\varepsilon \frac{ds}{s^\alpha} < 1,$$

then $u = w$ in $[t_1, t_1 + \varepsilon]$ which gives a contradiction. Consequently, $u(t) = w(t)$ for $t \geq 0$. We conclude that $t \to u_t$ from $[0, +\infty)$ to $\mathbb{Y}_\alpha$ and $t \to F(t, u_t)$ from $[0, +\infty) \times \mathcal{C}_\alpha$ to $\mathbb{X}$ are continuously differentiable. Thus, we claim that u is a strict solution of Eq. (1) on $[0, +\infty)$ [22–31].

6. Application

For illustration, we propose to study the model Eq. (2) given in the Introduction. We recall that this is defined by

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \dfrac{\partial^2}{\partial x^2} w(t,x) + \displaystyle\int_0^t h(t-s) \frac{\partial^2}{\partial x^2} w(s,x) ds \\ \qquad\qquad + \displaystyle\int_{-r}^0 g\left(t, \frac{\partial}{\partial x} w(t+\theta, x)\right) d\theta \text{ for } t \geq 0 \text{ and } x \in [0, \pi], \\ w(t,0) = w(t,\pi) = 0 \text{ for } t \geq 0, \\ w(\theta, x) = w_0(\theta, x) \text{ for } \theta \in [-r, 0] \text{ and } x \in [0, \pi], \end{cases} \tag{18}$$

where $w_0 : [-r, 0] \times [0, \pi] \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_+ \to \mathbb{R}+$ are appropriate functions. To study this equation, we choose $\mathbb{X} = L^2([0, \pi])$, with its usual norm $\|.\|$. We define the operator $A : \mathbb{Y} = D(A) \subset \mathbb{X} \to \mathbb{X}$ by

$$Aw = -w'' \text{ with domain } D(A) = H^2(0, \pi) \cap H_0^1(0, \pi),$$

and $B(t)x = h(t)Ax \in \mathbb{X}$, *for* $\geq 0, x \in \mathbb{Y}$. For $\alpha = 1/2$, we define $\mathbb{Y}_{1/2} = \left(D\left(A^{1/2}\right), |\cdot|_{1/2}\right)$ where $|x|_{1/2} = \|A^{1/2}x\|$ for each $x \in \mathbb{Y}_{1/2}$. We define $\mathcal{C}_{1/2} = C([-r, 0], \mathbb{Y}_{1/2})$ equipped with norm $|\cdot|_\infty$ and the functions u and φ and F by $u(t) = w(t, x)$, $\varphi(\theta)(x) = w_0(\theta, x)$ for a.e $x \in [0, \pi]$ and $\theta \in [-r, 0]$, $t \geq 0$, and finally

$$F(t, \varphi)(x) = \int_{-r}^{0} g\left(t, \frac{\partial}{\partial x}\varphi(\theta)(x)\right) d\theta \text{ for a.e } x \in [0, \pi] \text{ and } \varphi \in \mathcal{C}_{1/2}.$$

Then Eq. (18) takes the abstract form

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) + \displaystyle\int_0^t B(t-s)u(s)\,ds + F(t, u_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_{1/2} = C\left([-r, 0], D\left(A^{1/2}\right)]\right), \end{cases} \tag{19}$$

The $-A$ is a closed operator and generates an analytic compact semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$. Thus, there exists δ in $(0, \pi/2)$ and $M \geq 0$ such that $\Lambda = \left\{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$ is contained in $\rho(-A)$, the resolvent set of $-A$, and $\|R(\lambda, -A)\| < M/|\lambda|$ for $\lambda \in \Lambda$. The operator $B(t)$ is closed and for $x \in \mathbb{Y}$, $\|B(t)x\| \leq h(t)\|x\|_{\mathbb{Y}}$. The operator A has a discrete spectrum, the eigenvalues are n^2, and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \cdots$. Moreover the following formula holds:

i. $Au = \sum_{n=1}^{\infty} n^2 \langle u, e_n \rangle e_n \; u \in D(A)$.

ii. $A^{-1/2}u = \sum_{n=1}^{\infty} \frac{1}{n} \langle u, e_n \rangle e_n$ for $u \in \mathbb{X}$.

iii. $A^{1/2}u = \sum_{n=1}^{\infty} n \langle u, e_n \rangle e_n$ for $u \in D\left(A^{1/2}\right) = \left\{u \in \mathbb{X} : \sum_{n=1}^{\infty} \frac{1}{n} \langle u, e_n \rangle e_n \in \mathbb{X}\right\}$.

One also has the following result.
Lemma 6.1 [16] *Let* $\varphi \in \mathbb{Y}_{1/2}$. *Then* φ *is absolutely continuous,* $\varphi' \in \mathbb{X}$ *and*

$$\|\varphi'\| = \|A^{\frac{1}{2}}\varphi\|.$$

We assume the following assumptions.
(**H4**) The scalar function $h(.) \in L^1(0, \infty)$ and satisfies $g_1(\lambda) = 1 + h^*(\lambda) \neq 0$ (h^* the Laplace transform of h) and $\lambda g_1^{-1}(\lambda) \in \Lambda$ for $\lambda \in \Lambda$. Further, $h^*(\lambda) \to 0$ as $|\lambda| \to \infty$, for $\lambda \in \Lambda$ and $(h^*(\lambda))^{-1} = \circ(|\lambda|^n)$.
(**H5**) The function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz with respect to the second variable.
By assumption (**H4**), the operator $\rho(\lambda) = (\lambda I + g_1(\lambda)A)^{-1} = g_1^{-1}(\lambda)\left(\lambda g_1^{-1}(\lambda)I + A\right)^{-1}$ exists as a bounded operator on $\mathbb{X}$,

which is analytic in Λ and satisfies $\|\rho(\lambda)\| < M/|\lambda|$. On the other hand, for $x \in \mathbb{X}$, we have

$$\begin{aligned} A\rho(\lambda)x &= A\left(\lambda I + g_1(\lambda)A\right)^{-1}x \\ &= \left(A + \lambda g_1^{-1}(\lambda)I - \lambda g_1^{-1}(\lambda)I\right)\left(\lambda I + g_1(\lambda)A\right)^{-1}x \\ &= g_1^{-1}(\lambda)\left[I - \lambda g_1^{-1}(\lambda)\left(\lambda g_1^{-1}(\lambda)I + A\right)^{-1}\right]x. \end{aligned}$$

Since $\lambda g_1^{-1}(\lambda)\left(\lambda g_1^{-1}(\lambda)I + A\right)^{-1}$ is bounded because $g_1^{-1}(\lambda) \in \Lambda$, then $\|A\rho(\lambda)x\|$ has the growth properties of $g_1^{-1}(\lambda)$ which tends to 1 if $|\lambda|$ goes to infinity. Then we deduce that $A\rho(\lambda) \in \mathcal{L}(\mathbb{X})$. Moreover, it is analytic from Λ to $\mathcal{L}(\mathbb{X})$. Now, for $x \in \mathbb{Y}$, one has

$$A\rho(\lambda)x = g_1^{-1}(\lambda)\left(\lambda g_1^{-1}(\lambda)I + A\right)^{-1}Ax \text{ and } B^*(\lambda)\rho(\lambda)x = h^*(\lambda)\rho(\lambda)Ax.$$

Then it follows that

$$\|A\rho(\lambda)x\| \leq M/|\lambda|\|x\|_{\mathbb{Y}} \text{ and } \|B^*(\lambda)\rho(\lambda)\| \leq M/|\lambda|\|x\|_{\mathbb{Y}}.$$

We deduce that $A\rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, $B^*(\lambda) = h^*(\lambda)A \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, and $B^*(\lambda)\rho(\lambda) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. Considering $D = C_0^\infty([0, \pi])$, we see that the conditions **(V1)**–**(V3)** and **(H0)** are verified. Hence the homogeneous linear equation of Eq. (18) has an analytic compact resolvent operator $(R(t))_{t \geq 0}$. The function F is continuous in the first variable from the fact that g is continuous in the first variable. Moreover from Lemma 6.1 and the continuity of g, we deduce that F is continuous with respect to the second argument. This yields the continuity of F in $\mathbb{R}_+ \times \mathcal{C}_{1/2}$. In addition, by assumption **(H5)** we deduce that

$$\|F(t, \varphi_1) - F(t, \varphi_2)\| \leq rL_f\|\varphi_1 - \varphi_2\|_{\mathcal{C}_{1/2}}.$$

Then F is a continuous globally Lipschitz function with respect to the second argument. We obtain the following important result.

Proposition 6.2. Suppose that the assumptions **(H4)**–**(H5)** hold. Then Eq. (19) has a mild solution which is defined for $t \geq 0$.

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