Chapter

Stueckelberg-Horwitz-Piron Canonical Quantum Theory in General Relativity and Bekenstein-Sanders Gauge Fields for TeVeS

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Abstract

A consistent (off-shell) canonical classical and quantum dynamics in the framework of special relativity was formulated by Stueckelberg in 1941 and generalized to many-body theory by Horwitz and Piron in 1973 (SHP). This theory has been embedded into the framework of general relativity (GR), here denoted by SHPGR. The canonical Poisson brackets of the SHP theory remain valid (invariant under local coordinate transformations) on the manifold of GR and provide the basis for formulating a canonical quantum theory. The relation between representations based on coordinates and momenta is given by Fourier transform; a proof is given here for this functional relation on a manifold. The potential which may occur in the SHP theory emerges as a spacetime scalar mass distribution in GR. Gauge fields, both Abelian and non-Abelian on the quantum mechanical SHPGR Hilbert space in both the single particle and many-body theory, may be generated by phase transformations. Application to the construction of Bekenstein and Sanders in their solution to the lensing problem in TeVeS is discussed.

Keywords: relativistic dynamics, general relativity, quantum theory on curved space, non-Abelian gauge fields, Bekenstein-Sanders field, TeVeS

1. Introduction

The relativistic canonical Hamiltonian dynamics of Stueckelberg, Horwitz, and Piron (SHP) [1] with scalar potential and gauge field interactions for single- and many-body theories can, by local coordinate transformation, be embedded into the framework of general relativity (GR). This embedding provides a basis for the work of Horwitz et al. [2, 3] in their discussion of the origin of the field introduced by Bekenstein and Sanders [4] to explain gravitational lensing in the TeVeS formulation of modified Newtonian dynamic (MOND) theories [5–10].

The theory was originally formulated for a single particle by Stueckelberg in [11–13]. Stueckelberg envisaged the motion of a particle along a world line in spacetime that can curve and turn to flow backward in time, resulting in the phenomenon of pair annihilation in classical dynamics. The world line was then described by an invariant monotonic parameter τ. The theory was generalized by Horwitz and Piron in [14] (see also [15, 16]) to be applicable to many-body systems by assuming that the parameter τ is universal (as for Newtonian time, enabling them to solve the two-body problem classically, and later, a quantum solution was found by Arshansky and Horwitz [17–19], both for bound states and scattering theory).

Performing a coordinate transformation to general coordinates, along with the corresponding transformation of the momenta (the cotangent space of the original Minkowski manifold), one obtains [20] the SHP theory in a curved space of general coordinates and momenta with a canonical Hamilton-Lagrange (symplectic) structure. We shall refer to this generalization as SHPGR. We discuss the extension of the Abelian gauge theory described in Ref. [20] to the non-Abelian gauge discussed in [2, 3].

The invariance of the Poisson bracket under local coordinate transformations provides a basis for the canonical quantization of the theory, for which the evolution under τ is determined by the covariant form of the Stueckelberg-Schrödinger equation [1].

In this chapter, we assume a τ-independent background gravitational field; the local coordinate transformations from the flat Minkowski space to the curved space are taken to be independent of τ, consistently with an energy momentum tensor that is τ independent. In a more dynamical setting, when the energy momentum tensor depends on τ, the spacetime is evolved nontrivially [20, 21].

2. Embedding of single particle dynamics with external potential in GR

We write the SHP Hamiltonian [1, 11–13] as

$$K = \frac{1}{2M}\eta^{\mu\nu}\pi_\mu\pi_\nu + V(\xi) \tag{1}$$

where $\eta^{\mu\nu}$ is the flat Minkowski metric $(- + ++)$ and π_μ, ξ^μ are the spacetime canonical momenta and coordinates in the local tangent space of a general manifold, following Einstein's use of the equivalence principle.

The existence of a potential term (which we assume to be a Lorentz scalar), representing nongravitational forces, implies that the "free fall" condition is replaced by a local dynamics carried along by the free falling system (an additional force acting on the particle within the "elevator" according to the coordinates in the tangent space).

The canonical equations are

$$\dot{\xi}^\mu = \frac{\partial K}{\partial \pi_\mu} \quad \dot{\pi}_\mu = -\frac{\partial K}{\partial \xi^\mu} = -\frac{\partial V}{\partial \xi^\mu}, \tag{2}$$

where the dot here indicates $\frac{d}{d\tau}$, with τ the invariant universal "world time." Since then

$$\begin{aligned} \dot{\xi}^\mu &= \frac{1}{M}\eta^{\mu\nu}\pi_\nu, \\ \text{or} \qquad \pi_\nu &= \eta_{\nu\mu} M \dot{\xi}^\mu, \end{aligned} \tag{3}$$

the Hamiltonian can then be written as

$$K = \frac{M}{2}\eta_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu + V(\xi). \tag{4}$$

It is important to note that, as clear from (3), that $\dot{\xi}^0 = \frac{dt}{d\tau}$ has a sign *opposite* to π_0 which lies in the cotangent space of the manifold, as we shall see in the Poisson bracket relations below. The energy of the particle for a normal timelike particle should be positive (negative energy would correspond to an antiparticle [1, 11–13]). The *physical momenta and energy* therefore correspond to the mapping

$$\pi^\mu = \eta^{\mu\nu}\pi_\mu, \tag{5}$$

back to the tangent space. Thus, equivalently, from (2), $\dot{\xi}^\mu = (1/M)\pi^\mu$. This simple observation will be important in the discussion below of the dynamics of a particle in the framework of general relativity, for which the metric tensor is nontrivial.

We now transform the local coordinates (contravariantly) according to the diffeomorphism

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} dx^\lambda \tag{6}$$

to attach small changes in ξ to the corresponding small changes in the coordinates x on the curved space, so that

$$\dot{\xi}^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda. \tag{7}$$

The Hamiltonian then becomes

$$K = \frac{M}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + V(x), \tag{8}$$

where $V(x)$ is the potential at the point ξ corresponding to the point x (a function of ξ in a small neighborhood of the point x) and

$$g_{\mu\nu} = \eta_{\lambda\sigma} \frac{\partial \xi^\lambda}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu} \tag{9}$$

Since V has significance as the source of a force in the local frame only through its derivatives, we can make this pointwise correspondence with a globally defined scalar function $V(x)$.[1]

The corresponding Lagrangian is then

$$L = \frac{M}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - V(x), \tag{10}$$

In the locally flat coordinates in the neighborhood of x^μ, the symplectic structure of Hamiltonian mechanics implies that the momentum[2] π_μ, lying in the cotangent space of the manifold $\{\xi^\mu\}$, transforms covariantly under the local transformation (5), that is, as does $\frac{\partial}{\partial \xi^\mu}$, so that we may define

[1] Since $V(x)$ has the dimension of mass, one can think of this function as a scalar mass field, reflecting forces acting in the local tangent space at each point. It may play the role of "dark energy" [2, 3]. If $V = 0$, our discussion reduces to that of the usual general relativity, but with a well-defined canonical momentum variable.

[2] We shall call the quantity π_μ in the cotangent space as *canonical momentum*, although it must be understood that its map back to the tangent space π^μ corresponds to the actual physically measureable momentum.

$$p_\mu = \frac{\partial \xi^\lambda}{\partial x^\mu} \pi_\lambda. \tag{11}$$

This definition is consistent with the transformation properties of the momentum defined by the Lagrangian (10):

$$p_\mu = \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu}, \tag{12}$$

yielding

$$p_\mu = M g_{\mu\nu} \dot{x}^\nu. \tag{13}$$

The second factor in the definition (9) of $g_{\mu\nu}$ in (13) acts on $\dot{x}^\nu$; with (7) we then have (as in (11))

$$\begin{aligned} p_\mu &= M \eta_{\lambda\sigma} \frac{\partial \xi^\lambda}{\partial x^\mu} \dot{\xi}^\sigma \\ &= \frac{\partial \xi^\lambda}{\partial x^\mu} \pi_\lambda. \end{aligned} \tag{14}$$

As we have remarked above for the locally flat space in (5), the *physical* energy and momenta are given, according to the mapping,

$$p^\mu = g^{\mu\nu} p_\nu = M \dot{x}^\nu \tag{15}$$

back to the tangent space of the manifold, which also follows directly from the local coordinate transformation of (3) and (5).

It is therefore evident from (15) that

$$\dot{p}^\mu = M \ddot{x}^\mu. \tag{16}$$

We see that $\dot{p}^\mu$, which should be interpreted as the force acting on the particle, is proportional to the *acceleration along the orbit of motion* (a covariant derivative plus a gradient of the potential), as described by the geodesic-type relation. This Newtonian-type relation in the general manifold reduces in the limit of a flat Minkowski space to the corresponding SHP dynamics and in the nonrelativistic limit, to the classical Newton law. We remark that this result does not require taking a post-Newtonian limit, the usual method of obtaining Newton's law from GR.

We now discuss the geodesic equation obtained by studying the condition

$$\ddot{\xi}^\mu = -\frac{1}{M} \dot{\pi}_\mu = -\frac{1}{M} \eta^{\mu\nu} \frac{\partial V(\xi)}{\partial \xi^\nu}. \tag{17}$$

To do this, we compute

$$\begin{aligned} \ddot{\xi}^\mu = \frac{d}{d\tau} \left(\frac{\partial \xi^\mu}{\partial x^\lambda} \dot{x}^\lambda \right) &= \frac{\partial^2 \xi^\mu}{\partial x^\lambda \partial x^\gamma} \dot{x}^\gamma \dot{x}^\lambda \\ + \frac{\partial \xi^\mu}{\partial x^\lambda} \ddot{x}^\lambda &= -\frac{1}{M} \eta^{\mu\nu} \frac{\partial x^\lambda}{\partial \xi^\nu} \frac{\partial V(x)}{\partial x^\lambda}, \end{aligned} \tag{18}$$

so that, after multiplying by $\frac{\partial x^\sigma}{\partial \xi^\mu}$ and summing over μ, we obtain

$$\ddot{x}^{\sigma} = -\frac{\partial x^{\sigma}}{\partial \xi^{\mu}} \frac{\partial^2 \xi^{\mu}}{\partial x^{\lambda} \partial x^{\gamma}} \dot{x}^{\gamma} \dot{x}^{\lambda} - \frac{1}{M} \eta^{\mu\nu} \frac{\partial x^{\lambda}}{\partial \xi^{\nu}} \frac{\partial x^{\sigma}}{\partial \xi^{\mu}} \frac{\partial V(x)}{\partial x^{\lambda}}. \tag{19}$$

Finally, with (9) and the usual definition of the connection

$$\Gamma^{\sigma}{}_{\lambda\gamma} = \frac{\partial x^{\sigma}}{\partial \xi^{\mu}} \frac{\partial^2 \xi^{\mu}}{\partial x^{\lambda} \partial x^{\gamma}} \tag{20}$$

we obtain the modified geodesic-type equation

$$\ddot{x}^{\sigma} = -\Gamma^{\sigma}{}_{\lambda\gamma} \dot{x}^{\gamma} \dot{x}^{\lambda} - \frac{1}{M} g^{\sigma\lambda} \frac{\partial V(x)}{\partial x^{\lambda}}, \tag{21}$$

from which we see that the derivative of the potential $V(\xi)$ is mapped, under this coordinate transformation into a force resulting in a modification of the acceleration along the geodesic-like curves, that is, (16) now reads

$$\dot{p}^{\mu} = M\ddot{x}^{\nu} = -M\Gamma^{\sigma}{}_{\lambda\gamma} \dot{x}^{\gamma} \dot{x}^{\lambda} - g^{\sigma\lambda} \frac{\partial V(x)}{\partial x^{\lambda}} \tag{22}$$

The procedure that we have carried out here provides a canonical dynamical structure for motion in the curvilinear coordinates. We now remark that the Poisson bracket remains valid for the coordinates $\{x, p\}$. In the local coordinates $\{\xi, \pi\}$, the τ derivative of a function $F(\xi, \pi)$ is

$$\begin{aligned} \frac{dF(\xi,\pi)}{d\tau} &= \frac{\partial F(\xi,\pi)}{\partial \xi^{\mu}} \dot{\xi}^{\mu} + \frac{\partial F(\xi,\pi)}{\partial \pi_{\nu}} \dot{\pi}_{\mu} \\ &= \frac{\partial F(\xi,\pi)}{\partial \xi^{\mu}} \frac{\partial K}{\partial \pi_{\mu}} - \frac{\partial F(\xi,\pi)}{\partial \pi_{\mu}} \frac{\partial K}{\partial \xi^{\nu}} \\ &\equiv [F, K]_{PB}(\xi, \pi). \end{aligned} \tag{23}$$

If we replace in this formula

$$\begin{aligned} \frac{\partial}{\partial \xi^{\mu}} &= \frac{\partial x^{\lambda}}{\partial \xi^{\mu}} \frac{\partial}{\partial x^{\lambda}} \\ \frac{\partial}{\partial \pi_{\mu}} &= \frac{\partial \xi^{\mu}}{\partial x^{\lambda}} \frac{\partial}{\partial p_{\lambda}}, \end{aligned} \tag{24}$$

we immediately (as assured by the invariance of the Poisson bracket under local coordinate transformations) obtain

$$\frac{dF(\xi,\pi)}{d\tau} = \frac{\partial F}{\partial x^{\mu}} \frac{\partial K}{\partial p_{\mu}} - \frac{\partial F}{\partial p_{\mu}} \frac{\partial K}{\partial x^{\nu}} \equiv [F, K]_{PB}(x, p) \tag{25}$$

In this definition of Poisson bracket, we have, as for the ξ^{μ}, π_{ν} relation,

$$\left[x^{\mu}, p_{\nu}\right]_{PB}(x, p) = \delta^{\mu}{}_{\nu}. \tag{26}$$

The Poisson bracket of x^μ with the (physical energy momentum) tangent space variable p^μ has then the tensor form

$$[x^\mu, p^\nu]_{PB}(x,p) = g^{\mu\nu}. \tag{27}$$

In the flat space limit, this relation reduces to the SHP bracket,

$$[\xi^\mu, \pi^\nu]_{PB}(\xi, \pi) = \eta^{\mu\nu}. \tag{28}$$

Continuing our analysis with p_μ (we drop the (x,p) label on the Poisson bracket henceforth),

$$\Big[p_\mu, F(x)\Big]_{PB} = -\frac{\partial F}{\partial x^\mu}, \tag{29}$$

so that p_μ acts infinitesimally as the *generator of translation* along the coordinate curves and

$$[x^\mu, F(p)]_{PB} = \frac{\partial F(p)}{\partial p_\mu}, \tag{30}$$

so that x^μ is the generator of translations in p_μ. In the classical case, if $F(p)$ is a general function of p^μ, we can write at some point x,[3]

$$[x^\mu, F(p)]_{PB} = g^{\mu\nu}(x)\frac{\partial F(p)}{\partial p^\nu}. \tag{31}$$

This structure clearly provides a phase space which could serve as the basis for the construction of a canonical quantum theory on the curved spacetime.

We now turn to a discussion of the dynamics introduced into the curved space by the procedure outlined above.

We may also write (22) in terms of the full connection form by noting that with (9),

$$\frac{\partial g_{\lambda\gamma}}{\partial x^\mu} = \eta_{\alpha\beta}\left(\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\gamma} + \frac{\partial \xi^\alpha}{\partial x^\lambda}\frac{\partial^2 \xi^\beta}{\partial x^\gamma \partial x^\mu}\right). \tag{32}$$

Multiplying by $\dot{x}^\gamma \dot{x}^\lambda$, the two terms combine to give a factor of two. We then return to the original definition of Γ in (20) in the form

$$\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} = \frac{\partial \xi^\alpha}{\partial x^\sigma}\Gamma^\sigma{}_{\lambda\mu}, \tag{33}$$

so we can write

$$\begin{aligned}\frac{\partial g_{\lambda\gamma}}{\partial x^\mu}\dot{x}^\gamma \dot{x}^\lambda &= 2\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\sigma}\frac{\partial \xi^\beta}{\partial x^\gamma}\Gamma^\sigma{}_{\lambda\mu}\dot{x}^\gamma \dot{x}^\lambda \\ &= 2g_{\sigma\gamma}\Gamma^\sigma{}_{\lambda\mu}\dot{x}^\gamma \dot{x}^\lambda.\end{aligned} \tag{34}$$

[3] In the quantized form, the factor $g^{\mu\nu}(x)$ cannot be factored out from polynomials, so, as for Dirac's quantization procedure [22–25], some care is required.

We therefore have

$$\dot{p}_\mu = -\frac{\partial V(x)}{\partial x^\mu} + M g_{\sigma\gamma} \Gamma^\sigma{}_{\lambda\mu} \dot{x}^\gamma \dot{x}^\lambda. \tag{35}$$

3. Quantum theory on the curved space

The Poisson bracket formulas (25) and (26) can be considered as a basis for defining a quantum theory with canonical commutation relations

$$[x^\mu, p_\nu] = i\hbar \delta^\mu{}_\nu, \tag{36}$$

so that

$$\left[p_\mu, F(x)\right] = -i\hbar \frac{\partial F}{\partial x^\mu}, \tag{37}$$

and

$$[x^\mu, F(p)] = i\hbar \frac{\partial F(p)}{\partial p_\mu}. \tag{38}$$

The transcription of the Stueckelberg-Schrödinger equation for a wave function $\psi_\tau(x)$ can be taken to be (see also [26–28])

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x), \tag{39}$$

where the operator valued Hamiltonian can be taken to be the Hermitian form (42), written below, on a Hilbert space defined with scalar product (with invariant measure; we write $g = -\det\{g^{\mu\nu}\}$),

$$(\psi, \chi) = \int d^4x \sqrt{g} \psi^*{}_\tau(x) \chi_\tau(x). \tag{40}$$

To construct a Hermitian Hamiltonian, we first study the properties of the canonical momentum in coordinate representation. Clearly, in coordinate representation, $-i\frac{\partial}{\partial x^\mu}$ is not Hermitian due to the presence of the factor $\sqrt{g}$ in the integrand of the scalar product. The problem is somewhat analogous to that of Newton and Wigner [29] in their treatment of the Klein-Gordon equation in momentum space. It is easily seen that the operator

$$p_\mu = -i\frac{\partial}{\partial x^\mu} - \frac{i}{2} \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\mu} \sqrt{g(x)} \tag{41}$$

is essentially self-adjoint in the scalar product (40), satisfying as well as the commutation relations (36).[4]

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[4] The physically observable momentum can be defined, as in (15), as $\frac{1}{2}\{g^{\mu\nu}, p_\nu\}$, with commutation relations of the form (27). This operator can be transformed, as for the Newton-Wigner operator [29], to the form $-i\frac{\partial}{\partial x^\mu}$ by the Foldy-Wouthuysen transformation [30] $(g(x))^{\frac{1}{4}} p_\mu (g(x))^{-\frac{1}{4}}$.

Since p_μ is Hermitian in the scalar product (41), we can write the Hermitian Hamiltonian as

$$K = \frac{1}{2M} p_\mu g^{\mu\nu} p_\nu + V(x), \tag{42}$$

consistent with the local coordinate transformation of (1). The integration (40) must be considered as a total volume sum with invariant measure on the whole space, consistent with the notion of Lebesgue measure and the idea that the norm is the sum of probability measures on every subset contained. We return to this point in our discussion of the Fourier transform below.

4. Canonical quantum theory and the Fourier transform

To complete the construction of a canonical quantum theory on the curved space of GR, we discuss first the formulation of the Fourier transform $f(x) \to \tilde{f}(p)$ for a scalar function $f(x)$ (we shall use x^μ and the canonically conjugate p_μ in this discussion). Let us define ($g \equiv -\det g_{\mu\nu}$)

$$\tilde{f}(p) = \int d^4x \sqrt{g(x)} e^{ip_\mu x^\mu} f(x). \tag{43}$$

The inverse is given by

$$\int e^{-ip_\mu x^\mu} \tilde{f}(p) d^4p = \int d^4p e^{-ip_\mu (x^\mu - x\prime^\mu)} f(x') \sqrt{g(x')} d^4x' = (2\pi)^4 f(x) \sqrt{g(x)} \tag{44}$$

so that

$$\tilde{f}'(p) = \frac{1}{(2\pi)^4 \sqrt{g(x)}} \int e^{-ip_\mu x^\mu} \tilde{f}(p) d^4p. \tag{45}$$

One sees immediately that under diffeomorphisms, for which with the scalar property $f(x) = f'(x'), \tilde{f}(p) \to \tilde{f}'(p)$. The Fourier transform of $f'(x')$ is

$$\tilde{f}'(p) = \int d^4x' \sqrt{g(x')} e^{ip_\mu x\prime^\mu} f'(x'), \tag{46}$$

By change of integration variables, we have

$$\tilde{f}'(p) = \int d^4x \sqrt{g(x)} e^{ip_\mu x^\mu} f'(x), \tag{47}$$

In Dirac notation,

$$f(x) = <x|f>, \tag{48}$$

and we write as well

$$\tilde{f}(p) = <p|f>. \tag{49}$$

For

$$
\begin{aligned}
<x|p> &= \frac{1}{(2\pi)^4\sqrt{g(x)}} e^{-ip_\mu x^\mu} \\
<p|x> &= \sqrt{g(x)} e^{ip_\mu x^\mu},
\end{aligned}
\tag{50}
$$

we have, for example, the usual action of transformation functions

$$
\int <x|p><p|f> d^4p = <x|f>, \tag{51}
$$

where we have used

$$
\begin{aligned}
\int <x|p><p|x'> d^4p &= \frac{1}{(2\pi)^4\sqrt{g(x)}} \int d^4p e^{-ip_\mu x^\mu} e^{ip_\mu x'^\mu} \sqrt{g(x')} \\
&= \delta^4(x - x').
\end{aligned}
\tag{52}
$$

Note that the transformation functions $<x|p>$ and $<p|x>$ are not simple complex conjugates of each other, but require nontrivial factors of $\sqrt{g(x)}$ and its inverse to satisfy the necessary transformation laws on the manifold. Conversely (the factors $\sqrt{g(x)}$ and its inverse cancel), we should obtain

$$
\int <p'|x><x|p> d^4x = \delta^4(p' - p). \tag{53}
$$

The validity of (53) is not obvious on a curved space. We therefore provide a simple, but not trivial, proof of (53). For

$$
\int d^4p e^{ip_\mu (x^\mu - x'^\mu)} = (2\pi)^4 \frac{\delta^4(x - x')}{\sqrt{g}} \tag{54}
$$

we must have

$$
\tilde{f}(p) = \frac{1}{(2\pi)^4} \int d^4x \int d^4p' e^{i(p_\mu - p'_\mu) x^m u} \tilde{f}(p'), \tag{55}
$$

that is, exchanging the order of integrations, on the set $\{\tilde{f}(p)\}$,

$$
\Delta(p - p') = \frac{1}{(2\pi)^4} \int d^4x e^{i(p_\mu - p'_\mu) x^\mu} = \delta^4(p - p'). \tag{56}
$$

We now represent the integral as a sum over small boxes around the set of points $\{x_B\}$ that cover the space and eventually take the limit as for a Riemann integral.[5] In each small box, the coordinatization arises from an invertible transformation from the local tangent space in that neighborhood. We write

$$
x^\mu = x_B{}^\mu + \eta^\mu \quad \in \text{box B} \tag{57}
$$

where

[5] We follow here essentially the method discussed in Reed and Simon [31] in their discussion of the Lebesgue integral.

$$\eta^{\mu} = \frac{\partial x^{\mu}}{\partial \xi^{\lambda}} \xi^{\lambda} \tag{58}$$

and ξ^{λ} is in the flat local tangent space at x_B.

We now write the integral (56) as

$$\begin{aligned}\Delta(p - p') &= \frac{1}{(2\pi)^4} \Sigma_B \int_B d^4\eta e^{i\left(p_\mu - p'_\mu\right)\left(x_B{}^\mu + \eta^\mu\right)} \\ &= \frac{1}{(2\pi)^4} \Sigma_B e^{i\left(p_\mu - p'_\mu\right)x_B{}^\mu} \int_B d^4\eta e^{i\left(p_\mu - p'_\mu\right)\eta^\mu}.\end{aligned} \tag{59}$$

Let us call

$$I_B = \int_B d^4\eta e^{i\left(p_\mu - p'_\mu\right)\eta^\mu}. \tag{60}$$

In this neighborhood, call

$$\frac{\partial x^{\mu}}{\partial \xi^{\lambda}} = a^{\mu}{}_{\lambda}(B), \tag{61}$$

which we assume a constant matrix (Lorentz transformation) in each box. In (60), we then have

$$I_B = \int_B \frac{d^4\xi}{\sqrt{\det a}} e^{i\left(p_\mu - p'_\mu\right)a^\mu{}_\lambda(B)\xi^\lambda}. \tag{62}$$

However, we can make a change of variables; we are left with

$$I_{B'} = \int_B d^4\xi e^{i\left(p_\mu - p'_\mu\right)\xi^\mu}. \tag{63}$$

in each box.

However, the transformations $a^{\mu}{}_{\lambda}(B)$ in the neighborhood of each point B may be different, and therefore the set of transformed boxes may not cover (boundary deficits) the full domain of spacetime coordinates (one can easily estimate that the deficit from an arbitrarily selected set can be infinite in the limit).

We may avoid this problem by assuming geodesic completeness of the manifold and taking the covering set of boxes, constructed of parallel transported edges, along geodesic curves. Parallel transport of the tangent space boxes then fills the space in the neighborhood of the geodesic curve we are following, and each infinitesimal box may carry an invariant volume (Liouville-type flow) transported along a geodesic curve. For successive boxes along the geodesic curve, since the boundaries are determined by parallel transport (rectilinear shift in the succession of local tangent spaces), there is no volume deficit between adjacent boxes.

We may furthermore translate a geodesic curve to an adjacent geodesic by the mechanism discussed in [32], so that boxes associated with adjacent geodesics are also related by parallel transport. In this way, we may fill the entire geodesically accessible spacetime volume.

Let us assign a measure to each point B:

$$\Delta\mu(B, p - p') \equiv I_B. \tag{64}$$

We may then write (59) as

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\Sigma_B e^{i(p_\mu - p'_\mu)x_B{}^\mu}\Delta\mu(B, p-p'), \tag{65}$$

Our construction has so far been based on elements constructed in the tangent space in the neighborhood of each point B. Relating all points along a geodesic to the corresponding tangent spaces and putting each patch in correspondence by continuity, we may consider the set $\{x_B\}$ to be in correspondence with an extended flat space $\{\xi\}$, for which $x_B \sim \xi_B$ to obtain[6]

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\Sigma_B e^{i(p_\mu - p'_\mu)\xi_B{}^\mu}\Delta\mu(\xi_B, p-p'), \tag{66}$$

In the limit of vanishing spacetime box volume, this approaches the Lebesgue-type integral on a flat space:

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\int e^{i(p_\mu - p'_\mu)\xi^\mu} d\mu(\xi, p-p'). \tag{67}$$

If the measure is differentiable, we could write

$$d\mu(\xi, p-p') = m(\xi, p-p')d^4\xi. \tag{68}$$

Since the kernel $\Delta(p-p')$ is to act on elements of a Hilbert space $\left\{\tilde{f}(p)\right\}$, the support for $p' \to \infty$ vanishes, so that $p-p'$ is essentially bounded, as we discuss below. In the small box, say, size ϵ,

$$\begin{aligned}\int_{-\epsilon/2}^{\epsilon/2} d\xi^0 d\xi^1 d\xi^2 d\xi^3 e^{i(p_\mu - p'_\mu)\xi^\mu} &= (2i)^4 \Pi_{j=0}^{j=3}\frac{\sin\left(p_j - p'_j\right)\frac{\epsilon}{2}}{\left(p_j - p'_j\right)} \\ &\to \epsilon^4 \sim d^4\xi,\end{aligned} \tag{69}$$

so that $m(\xi, p-p') = 1$, and we have

$$\Delta(p-p') = \frac{1}{(2\pi)^4}\int e^{i(p_\mu - p'_\mu)\xi^\mu} d^4\xi, \tag{70}$$

or[7]

$$\Delta(p-p') = \delta^4(p-p'). \tag{71}$$

It is clear that the assertion (69) requires some discussion. For $\epsilon \to 0$ we must be sure that p' does not become too large, so that our local measure is equivalent to $d^4\xi$. In one of the dimensions, what we want to find are conditions for which

$$\frac{sinp\epsilon}{p} \to \epsilon \tag{72}$$

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[6] Similar to the method followed in the simpler case of constant curvature by Georgiev [33].

[7] Note that Abraham et al. [34] apply the formal Fourier transform on a manifold in three dimensions without proof.

for $\epsilon \to 0$, where we have written p for $p - p'$. As a distribution, on functions $g(p)$, the left member of (72) acts as

$$G(\epsilon) \equiv \int_{-\infty}^{\infty} \frac{sinp\epsilon}{p} g(p). \tag{73}$$

The function $G(\epsilon)$ is analytic in the neighborhood of $\epsilon = 0$ if $p^n g(p)$ has a Fourier transform for all n and the series is convergent in this neighborhood, since $G(0)$ is identically zero and successive derivatives correspond to the Fourier transforms of $p^n g(p)$ (differentiating under the integral). This implies that if the (usual) Fourier transform of $g(p)$ is a C^∞ function (as a simple sufficient condition) in the local tangent space $\{\xi\}$ and we have appropriate convergence properties, we can reliably use the first order term in the Taylor expansion;

$$\frac{d}{d\epsilon} G(\epsilon)\bigg|_{\epsilon=0} = \int \cos\epsilon p \;\; g(p)\bigg|_{\epsilon=0} \tag{74}$$

so that, for $\epsilon \to 0$,

$$G(\epsilon) \to \epsilon\tilde{g}(0), \tag{75}$$

where $\tilde{g}(\xi)$ is the Fourier transform of $g(p)$. As a distribution on such functions $g(p)$, the assertion (3.39) then follows.

5. Application to the Bekenstein-Sanders fields

We have discussed the construction of a canonical quantum theory in terms of an embedding of the SHP relativistic classical and quantum theory into general relativity. We show in this section that this systematic embedding provides a framework for the method developed by Bekenstein and Milgrom for understanding the MOND [5–10] that appeared necessary to explain the galactic rotation curves [35].

The remarkable development of observational equipment and power of computation has resulted in the discovery that Newtonian gravitational physics leads to a prediction for the dynamics of stars in galaxies that is not consistent with observation. It was proposed that there should be a matter present which does not radiate light which would resolve this difficulty, but so far no firm evidence of the existence of such matter has emerged. Milgrom [5–10] proposed a modification of Newton's law (MOND) which could resolve the problem. However, since Newton's law of gravitation emerges in the "post-Newtonian approximation" to the geodesic motion in Einstein's theory of gravity [35], the modification of Newton's law must involve a modification of Einstein's theory.[8] Such a modification was proposed by Bekenstein and Milgrom [5–10] in terms of a conformal factor multiplying the usual Einstein metric.

The origin of such a conformal factor can be found in the potential term of the special relativistic SHP theory. The embedding of this theory in GR [20] brings this potential term as a world scalar. The Hamiltonian for the general relativistic case then has the form (8). It was shown by Horwitz et al. [37] that a very sensitive test

[8] Yahalom [36] has proposed an alternative view involving the retardation effects associated with gravitational waves, presently being tested and developed. We do not discuss this approach further here.

by geodesic deviation can be formulated by to study stability by transforming a standard nonrelativistic Hamiltonian of the form

$$H = \frac{\mathbf{p}^2}{2M} + V(\mathbf{r}) \tag{76}$$

to the form

$$H = \frac{1}{2M} p_i g^{ij}(\mathbf{r}) p_j, \tag{77}$$

with

$$g^{ij}(\mathbf{r}) = \phi(\mathbf{r}) \frac{E}{E - V} \delta^{ij}, \tag{78}$$

that is, a conformal factor on the original metric. Applying the same idea to the Hamiltonian (8), with the $g^{\mu\nu}(x)$ of Einstein replaced by the conformal form

$$\tilde{g}\mu\nu(x) = \phi(x) g^{\mu\nu}(x) \tag{79}$$

where

$$\phi(x) = \frac{k}{k - V(x)}, \tag{80}$$

with k a point in the spectrum of K, so that

$$H = \frac{1}{2M} p_\mu \tilde{g}\mu\nu(x) p_\nu. \tag{81}$$

We see that we can in this way achieve the structure proposed by Bekenstein and Milgrom [5–10] systematically. Moreover, in addition to providing a mechanism for achieving a realization of the MOND theory, in the original form (8), the world scalar term $V(x)$ could represent the so-called dark energy [2, 3], establishing a relation between the MOND picture and the anomalous expansion of the universe, a question presently under study.

The theory proposed by Bekenstein and Milgrom [5–10] did not, however, account for the lensing of light observed when light passes a galaxy. To solve this problem, Bekenstein and Sanders [4] proposed the introduction of a *vector field* $n^\mu(x)$, satisfying the normalization constraint

$$n^\mu n_\mu = -1, \tag{82}$$

so that the vector is timelike.

This vector field can then be used to construct a modified meric of the form

$$\tilde{g}\mu\nu_T = \phi(g^{\mu\nu}(x) + n^\mu(x) n^\nu(x)) + \phi^{-1} n^\mu(x) n^\nu(x). \tag{83}$$

With this modification, Bekenstein and Sanders [4] could explain the lensing effect. In the following, we show that this new field may arise from a *non-Abelian* gauge transformation [38, 39] on the quantum theory that we have discussed in Section 3. Although Contaldi et al. [40] point out that a gauge field in this context can have caustic singularities due to the presence of a massive system, Horwitz et al. [2, 3] show that in the limit in which the gauge field approaches the Abelian limit, as required by Bekenstein and Sanders [4], there is a residual term that can cancel the caustic singularities.

To preserve the normalization condition (83), it is clear that we have the possibility of moving the n field on a hyperbola with a Lorentz transformation, which we can perform by a gauge transformation.

A Lorentz transformation on n^μ is noncommutative, and therefore the gauge field is non-Abelian [21].

An analogy can be drawn to the usual Yang-Mills gauge on $SU(2)$, where there is a two-valued index for the wave function $\psi_\alpha(x)$. The gauge transformation is a two-by-two matrix function of x and acts only on the indices α. The condition of invariant absolute square (probability) is

$$\sum_\alpha \left| \sum_\beta U_{\alpha\beta}\psi_\beta \right|^2 = \sum |\psi_\alpha|^2 \tag{84}$$

Generalizing this structure, one can take the indices α to be infinite dimensional, and even continuous, so that (84) becomes (in the spectral representation for n^μ)

$$\int (dn) \left| \int (dn')U(n,n')\psi(n',x) \right|^2 = \int (dn)|\psi(n,x)|^2, \tag{85}$$

implying that $U(n,n')$ (at each point x) is a unitary operator on a Hilbert space $L^2(dn)$. Since we are assuming that n^μ lies on a hyperbola determined by (83), the measure is

$$(dn) = \frac{d^3n}{n^0}, \tag{86}$$

that is, a three-dimensional Lorentz invariant integration measure.

We now examine the gauge condition:

$$(p^\mu - \epsilon n'^\mu)U\psi = U(p^\mu - \epsilon n^\mu)\psi \tag{87}$$

Since the Hermitian operator p_μ acts as a derivative under commutation relations, we obtain

$$n'_\mu = Un_\mu U^{-1} - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}U^{-1}, \tag{88}$$

in the same form as the Yang-Mills theory [38, 39]. It is evident in the Yang-Mills theory, due to the operator nature of the second term, the field will be algebra-valued, and thus we have the usual structure of the Yang-Mills non-Abelian gauge theory. Here, if the transformation U is a Lorentz transformation, the numerical-valued field n_μ would be carried, at least in the first term, to a new value on a hyperbola. However, the second term is operator valued on $L^2(dn)$, and thus, as in the Yang-Mills theory, n'_μ would become operator valued. Therefore, in general, the gauge field n^μ is operator valued.

It follows from (87) that the "field strengths"

$$f^{\mu\nu} = \frac{\partial n^\mu}{\partial x_\nu} - \frac{\partial n^\nu}{\partial x_\mu} + i\epsilon[n^\mu, n^\nu]. \tag{89}$$

Under a gauge transformation $n^\mu \to n'^\mu$, the new fields create field strengths in the transformed form

$$f'^{\mu\nu} = \frac{\partial n'^\mu}{\partial x_\nu} - \frac{\partial n'^\nu}{\partial x_\mu} + i\epsilon[n'^\mu, n'^\nu] \tag{90}$$

according to

$$f'^{\mu\nu}(x) = Uf^{\mu\nu}(x)U^{-1}, \tag{91}$$

just as in the finite-dimensional Yang-Mills theories.
For

$$U \cong 1 + iG, \tag{92}$$

where G is infinitesimal, (87) becomes

$$n'^\mu = n^\mu + i[G, n^\mu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x_\mu} + O\left(G^2\right). \tag{93}$$

Then,

$$\begin{aligned} n'^\mu n'_\mu &\cong n^\mu n_\mu + i\left(n^\mu\left[G, n_\mu\right] + [G, n^\mu]\right)n_\mu \\ &+ \frac{1}{\epsilon}\left(\frac{\partial G}{\partial x_\mu}n_\mu + n^\mu\frac{\partial G}{\partial x^\mu}\right). \end{aligned} \tag{94}$$

Let us take

$$\begin{aligned} G &= -\frac{i\epsilon}{2}\sum\left\{\omega_{\lambda\gamma}(n,x), \left(n^\lambda\frac{\partial}{\partial n_\gamma} - n^\gamma\frac{\partial}{\partial n_\lambda}\right)\right\} \\ &\equiv \frac{\epsilon}{2}\sum\left\{\omega_{\lambda\gamma}(n,x), N^{\lambda\gamma}\right\} \end{aligned} \tag{95}$$

where symmetrization is required since $\omega_{\lambda\gamma}$ is a function of n as well as x and

$$N^{\lambda\gamma} = -i\left(n^\lambda\frac{\partial}{\partial n_\gamma} - n^\gamma\frac{\partial}{\partial n^\lambda}\right). \tag{96}$$

Our investigation in the following will be concerned with a study of the infinitesimal gauge neighborhood of the Abelian limit, where the components of n^μ do not commute and therefore still constitute a Yang-Mills-type field. We shall show in the limit that the corresponding field equations acquire nonlinear terms and may therefore nullify the difficulty found by Contaldi et al. [40] demonstrating a dynamical instability for an Abelian vector-type TeVeS gauge field. They found that nonlinear terms associated with a non-Maxwellian-type action, such as $(div\mathbf{n})^2$, could nullify this caustic singularity, so that the nonlinear terms we find as a residue of the Yang-Mills structure induced by our gauge transformation might achieve this effect in a natural way.

Now, the second term of (94), which is the commutator of G with $n^\mu n_\mu$, vanishes, since this product is Lorentz invariant (the symmetrization in G does not affect this result).

We now consider the third term in (94).

$$\begin{aligned}\frac{1}{\epsilon}\frac{\partial G}{\partial x_\mu}n_\mu + n^\mu\frac{\partial G}{\partial x^\mu} &= \frac{1}{2}\left\{\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}, N^{\lambda\gamma}\right\}n_\mu + n_\mu\left\{\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}, N^{\lambda\gamma}\right\} \\ &= \frac{1}{2}N^{\lambda\gamma}\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}n_\mu + \frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}N^{\lambda\gamma}n_\mu + n_\mu N^{\lambda\gamma}\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu} + n_\mu\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}N^{\lambda\gamma}\end{aligned} \tag{97}$$

There are two terms proportional to

$$\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}n_\mu.$$

If we take

$$\omega_{\lambda\gamma}(n,x) = \omega_{\lambda\gamma}(k^\nu x_\nu), \tag{98}$$

where $k^\nu n_\nu = 0$, then

$$\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}n_\mu = k^\mu n_\mu \omega'_{\lambda\gamma} = 0. \tag{99}$$

For the remaining two terms,

$$\begin{aligned} &n_\mu N^{\lambda\gamma}\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu} + \frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}N^{\lambda\gamma}n_\mu \\ &= N^{\lambda\gamma}n^\mu\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu} \\ &+ \left[n_\mu, N^{\lambda\gamma}\right]\frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu} + \frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}n_\mu N^{\lambda\gamma} \\ &+ \frac{\partial\omega_{\lambda\gamma}}{\partial x_\mu}\left[N^{\lambda\gamma}, n_\mu\right].\end{aligned} \tag{100}$$

The commutators contain only terms linear in n^μ and they cancel; the remaining terms are zero, and therefore the condition $n^\mu n_\mu = -1$ is invariant under this gauge transformation. It involves the coefficient $\omega_{\lambda\gamma}$ which is a function of the projection of x^μ onto a hyperplane orthogonal the n^μ. The vector k^μ of course depends on n^μ. We take, for definiteness, $k^\mu = n^\mu(n\cdot b) + b^\mu$, for some $b^\mu \neq 0$.

We now consider the derivation of field equations from a Lagrangian constructed with the ψ_s and $f^{\mu\nu}f_{\mu\nu}$. We take the Lagrangian to be of the form

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_m, \tag{101}$$

where

$$\mathcal{L}_f = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu} \tag{102}$$

and

$$\mathcal{L}_m = \psi^*\left(i\frac{\partial}{\partial\tau} - \frac{1}{2M}(p^\mu - \epsilon n^\mu)\left(p_\mu - \epsilon n_\mu\right) - \Phi\right)\psi + \text{c.c.} \tag{103}$$

In carrying out the variation of $\mathcal{L}_m$, the contributions of varying the ψs with respect to n vanish due to the field equations (Stueckelberg-Schrödinger equation) obtained by varying ψ^* (or ψ), and therefore in the variation with respect to n, only the explicit presence of n in (103) need be taken into account.

Note that for the general case of n generally operator valued, we can write

$$\psi^*(p^\mu - \epsilon n^\mu)\left(p_\mu - \epsilon n_\mu\right)\psi = ((p^\mu - \epsilon n^\mu)\psi)^*\left(p_\mu - \epsilon n_\mu\right)\psi, \tag{104}$$

since the Lagrangian density (108) contains an integration over $(dn')(dn'')$ (in spectral representation, considered in lowest order, as well as an integration over $(dx)\sqrt{g(x)}$ in the action). In the limit in which n is evaluated in the spectral representation, and noting that p_μ is represented by an imaginary differential operator, we can write this as

$$\psi^*(p^\mu - \epsilon n^\mu)\left(p_\mu - \epsilon n_\mu\right)\psi = -(p^\mu + \epsilon n^\mu))\psi^*\left(p_\mu - \epsilon n_\mu\right)\psi, \tag{105}$$

that is, replacing explicitly p_μ by $-i(\partial/\partial x^\mu) \equiv -i\partial_\mu$ (since it acts by commutator with the fields); we have

$$\delta_n \mathcal{L}_m = \frac{-i\epsilon}{2M}\{\psi^*(\partial_\mu - i\epsilon n_\mu)\psi - ((\partial_\mu + i\epsilon n_\mu)\psi^*)\psi\}\delta n^\mu, \tag{106}$$

or

$$\delta_n \mathcal{L}_m = j_\mu(n,x)\delta n^\mu, \tag{107}$$

where j_μ has the usual form of a gauge invariant current.

For the calculation of the variation of $\mathcal{L}_f$, we note that the commutator term in (89) is, in lowest order, a c-number function.

Calling

$$\omega'{}_\lambda{}^\mu n^\lambda \equiv v^\mu, \tag{108}$$

we compute the variation of

$$[n'^\mu, n'^\nu] = 2i(k^\nu v^\mu - k^\mu v^\nu) \tag{109}$$

Then, for

$$\delta_n[n'^\mu, n'^\nu] = \delta_{n^\gamma}\frac{\partial}{\partial n^\gamma}[n'^\mu, n'^\nu], \tag{110}$$

we compute

$$\frac{\partial}{\partial n^\gamma}[n'^\mu, n'^\nu] = 2i\left(\frac{\partial k^\nu}{\partial n^\gamma}v^\mu + k^\nu\frac{\partial v^\mu}{\partial n^\gamma} - (\mu \leftrightarrow \nu)\right). \tag{111}$$

With our choice of $k^\nu = n^\nu(n\cdot b) + b^\nu$,

$$\frac{\partial k^\nu}{\partial n^\gamma} = \delta^\nu{}_\gamma(n\cdot b) + n^\nu b_\gamma, \tag{112}$$

so that

$$\frac{\partial}{\partial n^\gamma}[n'^\mu, n'^\nu] = 2i\big(\big(\delta^\nu{}_\gamma(n \cdot b) + n^\nu b_\gamma\big)v^\mu + k^\nu \frac{\partial v^\mu}{\partial n^\gamma} - (\mu \leftrightarrow \nu)\big). \tag{113}$$

Here,

$$\frac{\partial v^\mu}{\partial n^\gamma} = \omega\prime^\mu{}_\gamma + \omega''{}_\lambda{}^\mu n^\lambda \frac{\partial k_\sigma}{\partial n^\gamma} x^\sigma,$$

so we see that

$$\frac{\partial}{\partial n^\gamma}[n'^\mu, n'^\nu] \equiv \mathcal{O}_\gamma{}^{\mu\nu}, \tag{114}$$

where the quantity $\mathcal{O}_\gamma{}^{\mu\nu}\delta n^\gamma$ depends on the first and second derivatives of ω^μ_λ, in general, nonlinear in n^μ. We therefore have

$$\delta_n[n'^\mu, n'^\nu] = \mathcal{O}_\gamma{}^{\mu\nu}\delta n^\gamma \tag{115}$$

In the limit that $\omega \to 0$, its derivative and higher derivatives which appear in $\mathcal{O}_\gamma{}^{\mu\nu}$ may not vanish (somewhat analogous to the case in gravitational theory when the connection form vanishes, but the curvature does not), so that this term can contribute in the limit of the an Abelian gauge.

Returning to the variation of $\mathcal{L}_f$, we see that

$$\begin{aligned}\delta\mathcal{L}_f &= -\frac{1}{4}\big(\big(\partial^\mu \delta n^\nu - \partial^\nu \delta n_\mu + i\epsilon\delta[n^\mu, n^\mu]\big)f_{\mu\nu} \\ &\quad + f^{\mu\nu}\big(\partial_\mu \delta n_\nu - \partial_\nu \delta n_\mu + i\epsilon\delta\big[n_\mu, n_\mu\big]\big)\big) \\ &= -\partial^\nu f_{\mu\nu}\delta n^\mu + 2if_{\mu\nu}\delta[n^\mu, n^\nu],\end{aligned} \tag{116}$$

where we have taken into account the fact that $\big[n_\mu, n_\mu\big]$ is a c-number function and integrated by parts the derivatives of δn. We then obtain

$$\delta\mathcal{L}_f = -\partial^\nu f_{\mu\nu}\delta n^\mu + 2i\epsilon f_{\lambda\sigma}\mathcal{O}^{\lambda\sigma}{}_\mu \delta n^\mu \tag{117}$$

Since the coefficient of δn^μ must vanish, we obtain the Yang-Mills equations for the fields given the source currents:

$$\partial^\nu f_{\mu\nu} = j_\mu - 2i\epsilon f_{\lambda\sigma}\mathcal{O}^{\lambda\sigma}{}_\mu, \tag{118}$$

which is nonlinear in the fields n^μ, as we have seen, even in the Abelian limit, where, from (106),

$$j_\mu = -i\frac{\epsilon}{2M}\big\{\psi^*\big(\partial_\mu - i\epsilon n_\mu\big)\psi - \big(\big(\partial_\mu + i\epsilon n_\mu\big)\psi^*\big)\psi\big\}. \tag{119}$$

6. Summary

In this chapter, we have shown that the formulation of MOND theory by Bekenstein and Milgrom [5–10] can have a systematic origin within the framework of the embedding of the SHP [1] theory into general relativity [20]. The SHP formalism admits a scalar potential term that appears both in the conformal factor giving rise to the MOND functions in the galaxy and, in the original form of the Hamiltonian, to a possible candidate for "dark energy." The solution of the lensing problem by Bekenstein and Sanders [4] by introduction of a local vector field was also shown to arise in a natural way in terms of a non-Abelian gauge field, for which, in the Abelian limit, there is a residual term that can cancel the caustic singularity found by Contaldi et al. [40] which can arise in a purely Abelian gauge theory.

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