Chapter

On the Stabilization of Infinite Dimensional Semilinear Systems

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Abstract

This chapter considers the question of the output stabilization for a class of infinite dimensional semilinear system evolving on a spatial domain Ω by controls depending on the output operator. First we study the case of bilinear systems, so we give sufficient conditions for exponential, strong and weak stabilization of the output of such systems. Then, we extend the obtained results for bilinear systems to the semilinear ones. Under sufficient conditions, we obtain controls that exponentially, strongly, and weakly stabilize the output of such systems. The method is based essentially on the decay of the energy and the semigroup approach. Illustrations by examples and simulations are also given.

Keywords: semilinear systems, output stabilization, feedback controls, decay estimate, semigroups

1. Introduction

We consider the following semilinear system

$$\begin{cases} \dot{z}(t) = Az(t) + v(t)Bz(t), & t \geq 0, \\ z(0) = z_0, \end{cases} \tag{1}$$

where $A : D(A) \subset H \to H$ generates a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on a Hilbert space H, endowed with norm and inner product denoted, respectively, by $\|.\|$ and $\langle ., . \rangle$, $v(.) \in V_{ad}$ (the admissible controls set) is a scalar valued control and B is a nonlinear operator from H to H with $B(0) = 0$ so that the origin be an equilibrium state of system (1). The problem of feedback stabilization of distributed system (1) was studied in many works that lead to various results. In [1], it was shown that the control

$$v(t) = -\langle z(t), Bz(t) \rangle, \tag{2}$$

weakly stabilizes system (1) provided that B be a weakly sequentially continuous operator such that, for all $\psi \in H$, we have

$$\langle BS(t)\psi, S(t)\psi \rangle = 0, \quad \forall t \geq 0 \Rightarrow \psi = 0, \tag{3}$$

and if (3) is replaced by the following assumption

$$\int_0^T |\langle BS(s)\psi, S(s)\psi\rangle| ds \geq \gamma \|\psi\|^2, \quad \forall \psi \in H \,(\text{for some } \gamma, T > 0), \tag{4}$$

then control (2) strongly stabilizes system (1) [2].

In [3], the authors show that when the resolvent of A is compact, B self-adjoint and monotone, then strong stabilization of system (1) is proved using bounded controls.

Now, let the output state space Y be a Hilbert space with inner product $\langle .,. \rangle_Y$ and the corresponding norm $\|.\|_Y$, and let $C \in \mathcal{L}(H, Y)$ be an output operator.

System (1) is augmented with the output

$$w(t) := Cz(t). \tag{5}$$

The output stabilization means that $w(t) \to 0$ as $t \to +\infty$ using suitable controls. In the case when $Y = H$ and $C = I$, one obtains the classical stabilization of the state. If Ω be the system evolution domain and $\omega \subset \Omega$, when $C = \chi_\omega$, the restriction operator to a subregion ω of Ω, one is concerned with the behaviour of the state only in a subregion of the system evolution domain. This is what we call regional stabilization.

The notion of regional stabilization has been largely developed since its closeness to real applications, and the existence of systems which are not stabilizable on the whole domain but stabilizable on some subregion ω. Moreover, stabilizing a system on a subregion is cheaper than stabilizing it on the whole domain [4–8]. In [9], the author establishes weak and strong stabilization of (5) for a class of semilinear systems using controls that do not take into account the output operator.

In this paper, we study the output stabilization of semilinear systems by controls that depend on the output operator. Firstly we consider the case of bilinear systems, then we give sufficient conditions to obtain exponential, strong and weak stabilization of the output. Secondly, we consider the case of semilinear systems, and then under sufficient conditions, we obtain controls that exponentially, strongly, and weakly stabilize the output of such systems. The method is based essentially on the decay of the energy and the semigroup approach. Illustrations by examples and simulations are also given.

This paper is organized as follows: In Section 2, we discuss sufficient conditions to achieve exponential, strong and weak stabilization of the output (5) for bilinear systems. In Section 3, we study the output stabilization for a class of semilinear systems. Section 4 is devoted to simulations.

2. Stabilization for bilinear systems

In this section, we develop sufficient conditions that allow exponential, strong and weak stabilization of the output of bilinear systems. Consider system (1) with $B : H \to H$ is a bounded linear operator and augmented with the output (5).

Definition 1.1 The output (5) is said to be:

1. weakly stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1) is global and satisfies

$$\langle Cz(t), \psi \rangle_Y \to 0, \quad \forall \psi \in Y, \quad \text{as } t \to \infty,$$

2. strongly stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1) is global and verifies

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$$\|Cz(t)\|_Y \to 0, \quad \text{as } t \to \infty,$$

and

3. exponentially stabilizable, if there exists a control $v(.) \in V_{ad}$ such that for any initial condition $z_0 \in H$, the corresponding solution $z(t)$ of system (1) is global and there exist $\alpha, \beta > 0$ such that

$$\|Cz(t)\|_Y \leq \alpha e^{-\beta t}\|z_0\|, \quad \forall t > 0.$$

Remark 1. It is clear that exponential stability of (5) $\Rightarrow$ strong stability of (5) $\Rightarrow$ weak stability of (5).

2.1 Exponential stabilization

The following result provides sufficient conditions for exponential stabilization of the output (5).

Theorem 1.2 Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and if the condition:

1. $\mathcal{R}e(\langle C^* CAy, y\rangle) \leq 0, \ \ \forall y \in D(A)$, where C^* is the adjoint operator of C,

2. $\|CS(t)y\|_Y \leq \alpha \|Cy\|_Y$ and $\|CBy\|_Y \leq \beta \|Cy\|_Y$, for some $\alpha, \beta > 0$,

3. there exist $T, \gamma > 0$ such that

$$\int_0^T |\langle C^* CBS(t)y, S(t)y\rangle| dt \geq \gamma \|Cy\|_Y^2, \forall y \in H, \tag{6}$$

hold, then there exists $\rho > 0$ for which the control

$$v(t) = -\rho \,\mathrm{sign}(\langle C^* CBz(t), z(t)\rangle)$$

exponentially stabilizes the output (5).

Proof: System (1) has a unique mild solution $z(t)$ [10] defined on a maximal interval $[0, t_{\max}]$ by the variation of constants formula

$$z(t) = S(t)z_0 + \int_0^t v(s)S(t-s)Bz(s)ds. \tag{7}$$

From hypothesis 1, we deduce

$$\frac{d}{dt}\|Cz(t)\|_Y^2 \leq -2\rho \ |\langle C^* CBz(t), z(t)\rangle|.$$

Integrating this inequality, we get

$$\|Cz(t)\|_Y^2 - \|Cz(0)\|_Y^2 \leq -2\rho \int_0^t |\langle C^* CBz(\tau), z(\tau)\rangle| d\tau. \tag{8}$$

It follows that

$$\|Cz(t)\|_Y \leq \|Cz_0\|_Y. \tag{9}$$

For all $z_0 \in H$ and $t \geq 0$, we have

$$\begin{aligned}\langle C^* CBS(t)z_0, S(t)z_0\rangle &= \langle C^* CBz(t), z(t)\rangle - \langle C^* CBz(t), z(t) - S(t)z_0\rangle \\ &\quad + \langle C^* CB(S(t)z_0 - z(t)), S(t)z_0\rangle.\end{aligned}$$

Using hypothesis 2 and (9), we have

$$|\langle C^* CBS(t)z_0, S(t)z_0\rangle| \leq |\langle C^* CBz(t), z(t)\rangle| + 2\rho\alpha\beta\|C(z(t) - S(t)z_0)\|_Y\|Cz_0\|_Y.$$

It follows that from (7) and condition 2 that

$$|\langle C^* CBS(t)z_0, S(t)z_0\rangle| \leq |\langle C^* CBz(t), z(t)\rangle| + 2\rho\alpha^2\beta^2 T\|Cz_0\|_Y^2. \tag{10}$$

Integrating (10) over the interval $[0, T]$ and replacing z_0 by $z(t)$ and using (6), we deduce that

$$\left(\gamma - 2\rho\alpha^2\beta^2T^2\right)\|Cz(t)\|_Y^2 \leq \int_t^{t+T} |\langle C^* CBz(s), z(s)\rangle| ds. \tag{11}$$

It follows from the inequality (8) that the sequence $\|Cz(n)\|_Y$ decreases and that for all $n \in \mathbb{N}$, we have

$$\|Cz(nT)\|_Y^2 - \|Cz((n+1)T)\|_Y^2 \geq 2\rho \int_{nT}^{(n+1)T} |\langle C^* CBz(s), z(s)\rangle| ds.$$

Using (11), we deduce

$$\|Cz(nT)\|_Y^2 - \|Cz((n+1)T)\|_Y^2 \geq 2\rho\left(\gamma - 2\rho\alpha^2\beta^2T^2\right)\|Cz(nT)\|_Y^2.$$

Taking $0 < \rho < \frac{\gamma}{2\alpha^2\beta^2T^2}$, we get

$$\|Cz(nT)\|_Y^2 \geq 2\rho\left(1 + 2\rho\left(\gamma - 2\rho\alpha^2\beta^2T^2\right)\right)\|Cz((n+1)T)\|_Y^2.$$

Then

$$\|Cz(nT)\|_Y^2 \leq \frac{1}{M^n}\|Cz_0\|_Y^2.$$

where $M = \left(1 + 2\rho\left(\gamma - 2\rho\alpha^2\beta^2T^2\right)\right) > 1$.
Since $\|Cz(t)\|_Y$ decreases, we deduce that

$$\|Cz(t)\|_Y \leq \sqrt{M} e^{\frac{-\ln(M)}{2T}t}\|z_0\|, \forall t \geq 0,$$

which gives the exponential stability of the output (5).
Example 1 On $\Omega =]0, 1[$, we consider the following system

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = Az(x,t) + v(t)z(x,t) & \Omega \times]0, +\infty[\\ z(x, 0) = z_0(x) & \Omega, \end{cases} \tag{12}$$

where $H = L^2(\Omega)$ and $Az = -z$. The operator A generates a semigroup of contractions on $L^2(\Omega)$ given by $S(t)z_0 = e^{-t}z_0$. Let ω be a subregion of Ω. System (12) is augmented with the output

$$w(t) := \chi_\omega z(t), \tag{13}$$

where $\chi_\omega : L^2(\Omega) \to L^2(\omega)$, the restriction operator to ω and χ_ω^* is the adjoint operator of χ_ω. Conditions 1 and 3 of Theorem 1.2 hold, indeed: we have

$$\left\langle \chi_\omega^* \chi_\omega A y, y \right\rangle = -\|\chi_\omega y\|^2_{L^2(\omega)} \leq 0, \quad \forall y \in L^2(\Omega),$$

and for $T = 2$, we have

$$\int_0^2 \langle \chi_\omega^* \chi_\omega B e^{-t} y, e^{-t} y \rangle dt = \int_0^2 e^{-2t} dt \int_\omega |y|^2 dx = \left(\frac{1}{2} - \frac{1}{2e^4}\right) \|\chi_\omega y\|^2_{L^2(\omega)}.$$

We conclude that for all $0 < \rho < \frac{e^4 - 1}{16e^4}$, the control

$$v(t) = \begin{pmatrix} -\rho & \text{if } \|\chi_\omega z(t)\|^2_{L^2(\omega)} \neq 0, \\ 0 & \text{if } \|\chi_\omega z(t)\|^2_{L^2(\omega)} = 0, \end{pmatrix}$$

exponentially stabilizes the output (13).

2.2 Strong stabilization

The following result will be used to prove strong stabilization of the output (5).

Theorem 1.3 Let A generate a semigroup $(S(t))_{t \geq 0}$ of contractions on H and $B : H \to H$ is a bounded linear operator. If the conditions:

1. $\mathcal{R}e(\langle C^* CA\psi, \psi \rangle) \leq 0, \quad \forall \psi \in D(A)$,
2. $\mathcal{R}e(\langle C^* CB\psi, \psi \rangle \langle B\psi, \psi \rangle) \geq 0, \quad \forall \psi \in H$, hold, then control

$$v(t) = -\frac{\langle C^* CBz(t), z(t) \rangle}{1 + |\langle C^* CBz(t), z(t) \rangle|}, \tag{14}$$

allows the estimate

$$\left(\int_0^T |\langle C^* CBS(s)z(t), S(s)z(t) \rangle| ds\right)^2 = O\left(\int_t^{t+T} \frac{|\langle C^* CBz(s), z(s) \rangle|^2}{1 + |\langle C^* CBz(s), z(s) \rangle|} ds\right), \text{ as } t \to +\infty. \tag{15}$$

Proof: From hypothesis 1 of Theorem 1.3, we have

$$\frac{1}{2} \frac{d}{dt} \|Cz(t)\|^2_Y \leq \mathcal{R}e(v(t) \langle C^* CBz(t), z(t) \rangle).$$

In order to make the energy nonincreasing, we consider the control

$$v(t) = -\frac{\langle C^* CBz(t), z(t) \rangle}{1 + |\langle C^* CBz(t), z(t) \rangle|},$$

so that the resulting closed-loop system is

$$\dot{z}(t) = Az(t) + f(z(t)), \quad z(0) = z_0, \tag{16}$$

where

$$f(y) = -\frac{\langle C^* CBy, y\rangle}{1 + |\langle C^* CBy, y\rangle|} By, \text{ for all } y \in H$$

Since f is locally Lipschitz, then system (16) has a unique mild solution $z(t)$ [10] defined on a maximal interval $[0, t_{max}]$ by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(z(s))ds. \tag{17}$$

Because of the contractions of the semigroup, we have

$$\frac{d}{dt}\|z(t)\|^2 \leq -2\frac{\langle C^* CBz(t), z(t)\rangle\langle Bz(t), z(t)\rangle}{1 + |\langle C^* CBz(t), z(t)\rangle|}.$$

Integrating this inequality, we get

$$\|z(t)\|^2 - \|z(0)\|^2 \leq -2\int_0^t \frac{\langle C^* CBz(s), z(s)\rangle\langle Bz(s), z(s)\rangle}{1 + |\langle C^* CBz(s), z(s)\rangle|} ds.$$

It follows that

$$\|z(t)\| \leq \|z_0\|. \tag{18}$$

From hypothesis 1 of Theorem 1.3, we have

$$\frac{d}{dt}\|Cz(t)\|_Y^2 \leq -2\frac{|\langle C^* CBz(t), z(t)\rangle|^2}{1 + |\langle C^* CBz(t), z(t)\rangle|}.$$

We deduce

$$\|Cz(t)\|_Y^2 - \|Cz(0)\|_Y^2 \leq -2\int_0^t \frac{|\langle C^* CBz(s), z(s)\rangle|^2}{1 + |\langle C^* CBz(s), z(s)\rangle|} ds. \tag{19}$$

Using (17) and Schwartz inequality, we get

$$\|z(t) - S(t)z_0\| \leq \|B\|\|z_0\| \left(T\int_0^t \frac{|\langle C^* CBz(s), z(s)\rangle|^2}{1 + |\langle C^* CBz(s), z(s)\rangle|} ds\right)^{\frac{1}{2}}, \quad \forall t \in [0, T]. \tag{20}$$

Since B is bounded and C continuous, we have

$$|\langle C^* CBS(s)z_0, S(s)z_0\rangle| \leq 2K\|B\|\|z(s) - S(s)z_0\|\|z_0\| + |\langle C^* CBz(s), z(s)\rangle|, \tag{21}$$

where K is a positive constant.
Replacing z_0 by $z(t)$ in (20) and (21), we get

$$\begin{aligned}|\langle C^* CBS(s)z(t), S(s)z(t)\rangle| \leq 2K\|B\|^2\|z_0\|^2 &\left(T\int_t^{t+T} \frac{|\langle C^* CBz(s), z(s)\rangle|^2}{1+|\langle C^* CBz(s), z(s)\rangle|} ds\right)^{\frac{1}{2}} \\ &+ |\langle C^* CBz(t+s), z(t+s)\rangle|, \qquad \forall t \geq s \geq 0.\end{aligned}$$

Integrating this relation over $[0, T]$ and using Cauchy-Schwartz, we deduce

$$\int_0^T |\langle C^*CBS(s)z(t), S(s)z(t)\rangle| ds \le \left(2K\|B\|^2T^{\frac{3}{2}} + T\left(\left(1+K\|B\|\|z_0\|^2\right)\right)\right.$$

$$\left.\times \left(\int_t^{t+T} \frac{|\langle C^*CBz(s), z(s)\rangle|^2}{1+|\langle C^*CBz(s), z(s)\rangle|} ds\right)^{\frac{1}{2}}\right.,$$

which achieves the proof.

The following result gives sufficient conditions for strong stabilization of the output (5).

Theorem 1.4 Let A generate a semigroup $(S(t))_{t\ge 0}$ of contractions on H, B is a bounded linear operator. If the assumptions 1, 2 of Theorem 1.3 and

$$\int_0^T |\langle C^*CBS(t)\psi, S(t)\psi\rangle| dt \ge \gamma \|C\psi\|_Y^2, \quad \forall \psi \in H, \quad (for\ \ some\ \ T, \gamma > 0), \tag{22}$$

holds, then control (14) strongly stabilizes the output (5) with decay estimate

$$\|Cz(t)\|_Y = O\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \to +\infty. \tag{23}$$

Proof: Using (19), we deduce

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \ge 2\int_{kT}^{k(T+1)} \frac{|\langle C^*CBz(t), z(t)\rangle|^2}{1+|\langle C^*CBz(t), z(t)\rangle|} dt, \quad k \ge 0.$$

From (15) and (22), we have

$$\|Cz(kT)\|_Y^2 - \|Cz((k+1)T)\|_Y^2 \ge \beta \|Cz(kT)\|_Y^4, \tag{24}$$

where $\beta = \frac{\gamma^2}{2\left(2K\|B\|^2T^{\frac{3}{2}} + T\left(1+K\|B\|\|z_0\|^2\right)\right)^2}$.

Taking $s_k = \|Cz(kT)\|_Y^2$, the inequality (24) can be written as

$$\beta s_k^2 + s_{k+1} \le s_k, \quad \forall k \ge 0.$$

Since $s_{k+1} \le s_k$, we obtain

$$\beta s_{k+1}^2 + s_{k+1} \le s_k, \quad \forall k \ge 0.$$

Taking $p(s) = \beta s^2$ and $q(s) = s - (I+p)^{-1}(s)$ in Lemma 3.3, page 531 in [11], we deduce

$$s_k \le x(k), \quad k \ge 0,$$

where $x(t)$ is the solution of equation $x'(t) + q(x(t)) = 0, \quad x(0) = s_0$.

Since $x(k) \ge s_k$ and $x(t)$ decreases give $x(t) \ge 0$, $\forall t \ge 0$. Furthermore, it is easy to see that $q(s)$ is an increasing function such that

$$0 \le q(s) \le p(s), \forall s \ge 0.$$

We obtain $-\beta x(t)^2 \le x'(t) \le 0$, which implies that

$$x(t) = O\left(t^{-1}\right), \quad as \quad t \to +\infty.$$

Finally the inequality $s_k \leq x(k)$, together with the fact that $\|Cz(t)\|_Y$ decreases, we deduce the estimate (23).

Example 2 Let us consider a system defined on $\Omega =]0, 1[$ by

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = Az(x,t) + v(t)a(x)z(x,t) & \Omega \times]0, +\infty[\\ z(x,0) = z_0(x) & \Omega \\ z(0,t) = z(1,t) = 0 & t > 0, \end{cases} \tag{25}$$

where $H = L^2(\Omega)$, $Az = -z$, and $a \in L^\infty(]0, 1[)$ such that $a(x) \geq 0$ a.e on $]0, 1[$ and $a(x) \geq c > 0$ on subregion ω of Ω and $v(.) \in L^\infty(0, +\infty)$ the control function. System (25) is augmented with the output

$$w(t) = \chi_\omega z(t). \tag{26}$$

The operator A generates a semigroup of contractions on $L^2(\Omega)$ given by $S(t)z_0 = e^{-t}z_0$. For $z_0 \in L^2(\Omega)$ and $T = 2$, we obtain

$$\int_0^2 \langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle dt = \int_0^2 e^{-2t} dt \int_\omega a(x)|z_0|^2 dx \geq \beta \|\chi_\omega z_0\|^2_{L^2(\omega)},$$

with $\beta = c\int_0^2 e^{-2t} dt > 0$.

Applying Theorem 1.4, we conclude that the control

$$v(t) = -\frac{\int_\omega a(x)|z(x,t)|^2 dx}{1 + \int_\omega a(x)|z(x,t)|^2 dx}$$

strongly stabilizes the output (26) with decay estimate

$$\|\chi_\omega z(t)\|_{L^2(\omega)} = O\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \to +\infty.$$

2.3 Weak stabilization

The following result provides sufficient conditions for weak stabilization of the output (5).

Theorem 1.5 Let A generate a semigroup $(S(t))_{t \geq 0}$ of contractions on H and B is a compact operator. If the conditions:

1. $\mathcal{R}e(\langle C^* CA\psi, \psi \rangle) \leq 0, \quad \forall \psi \in D(A)$,
2. $\mathcal{R}e(\langle C^* CB\psi, \psi \rangle \langle B\psi, \psi \rangle) \geq 0, \quad \forall \psi \in H$,
3. $\langle C^* CBS(t)\psi, S(t)\psi \rangle = 0, \quad \forall t \geq 0 \Rightarrow C\psi = 0$ hold, then control (14) weakly stabilizes the output (5).

Proof: Let us consider the nonlinear semigroup $\Gamma(t)z_0 := z(t)$ and let (t_n) be a sequence of real numbers such that $t_n \to +\infty$ as $n \to +\infty$.

From (18), $\Gamma(t_n)z_0$ is bounded in H, then there exists a subsequence $(t_{\phi(n)})$ of (t_n) such that

$$\Gamma\left(t_{\phi(n)}\right)z_0 \rightharpoonup \psi, \quad as\ n \to \infty.$$

Since B is compact and C continuous, we have

$$\lim_{n\to+\infty}\left\langle C^*CBS(t)\Gamma\left(t_{\phi(n)}\right)z_0, S(t)\Gamma\left(t_{\phi(n)}\right)z_0\right\rangle = \langle C^*CBS(t)\psi, S(t)\psi\rangle.$$

For all $n \geq$, we set

$$\Lambda_n(t) := \int_{\phi(n)}^{\phi(n)+t} \frac{|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|^2}{1+|\langle C^*CB\Gamma(s)z_0, \Gamma(s)z_0\rangle|} ds.$$

It follows that $\forall t \geq 0$, $\Lambda_n(t) \to 0$ as $n \to +\infty$.
Using (15), we get

$$\lim_{n\to+\infty}\int_0^t \left|\left\langle C^*CBS(s)\Gamma\left(t_{\phi(n)}\right)z_0, S(s)\Gamma\left(t_{\phi(n)}\right)z_0\right\rangle\right| ds = 0.$$

Hence, by the dominated convergence Theorem, we have

$$\int_0^t |\langle C^*CBS(s)\psi, S(s)\psi\rangle| ds = 0.$$

We conclude that

$$\langle C^*CBS(s)\psi, S(s)\psi\rangle = 0, \quad \forall s \in [0, t].$$

Using condition 3 of Theorem 1.5, we deduce that

$$C\Gamma\left(t_{\phi(n)}\right)z_0 \rightharpoonup 0, \quad \text{as} \quad n \to +\infty. \tag{27}$$

On the other hand, it is clear that (27) holds for each subsequence $\left(t_{\phi(n)}\right)$ of (t_n) such that $C\Gamma\left(t_{\phi(n)}\right)z_0$ weakly converges in Y. This implies that $\forall \varphi \in Y$, we have $\langle C\Gamma(t_n)z_0, \varphi\rangle \to 0$ as $n \to +\infty$ and hence

$$C\Gamma(t)z_0 \rightharpoonup 0, \quad \text{as} \quad t \to +\infty.$$

Example 3 Consider a system defined in $\Omega =]0, +\infty[$, and described by

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = -\dfrac{\partial z(x,t)}{\partial x} + v(t)Bz(x,t) & x \in \Omega, \quad t > 0 \\ z(x,0) = z_0(x) & x \in \Omega \\ z(0,t) = z(\infty,t) = 0 & t > 0, \end{cases} \tag{28}$$

where $Az = -\frac{\partial z}{\partial x}$ with domain
$D(A) = \left\{z \in H^1(\Omega) \mid z(0) = 0,\ z(x) \to 0\ \ as\ \ x \to +\infty\right\}$ and $Bz(.) = \int_0^1 z(x)dx(.)$ is the control operator. The operator A generates a semigroup of contractions

$$(S(t)z_0)(x) = \begin{cases} z_0(x-t) & \text{if } x \geq t \\ 0 & \text{if } x < t. \end{cases}$$

Let $\omega =]0, 1[$ be a subregion of Ω and system (28) is augmented with the output

$$w(t) = \chi_\omega z(t). \qquad (29)$$

We have

$$\langle \chi_\omega^* \chi_\omega Az, z \rangle = -\int_0^1 z'(x) z(x) dx = -\frac{z^2(1)}{2} \leq 0,$$

so, the assumption 1 of Theorem 1.5 holds. The operator B is compact and verifies

$$\langle \chi_\omega^* \chi_\omega BS(t) z_0, S(t) z_0 \rangle = \left(\int_0^{1-t} z_0(x) dx \right)^2, \quad 0 \leq t \leq 1.$$

Thus

$$\langle \chi_\omega^* \chi_\omega BS(t) z_0, S(t) z_0 \rangle = 0, \quad \forall t \geq 0 \quad \Rightarrow z_0(x) = 0, \ a.e \text{ on } \omega.$$

Then, the control

$$v(t) = -\frac{\left(\int_0^1 z(x,t) dx \right)^2}{1 + \left(\int_0^1 z(x,t) dx \right)^2}, \qquad (30)$$

weakly stabilizes the output (29).

3. Stabilization for semilinear systems

In this section, we give sufficient conditions for exponential, strong and weak stabilization of the output (5). Consider the semilinear system (1) augmented with the output (5).

4. Exponential stabilization

In this section, we develop sufficient conditions for exponential stabilization of the output (5).

The following result concerns the exponential stabilization of (5).

Theorem 1.6 Let A generate a semigroup $(S(t))_{t \geq 0}$ of contractions on H and B be locally Lipschitz. If the conditions:

1. $\mathcal{R}e(\langle C^* CAy, y \rangle) \leq 0, \quad \forall y \in D(A)$,

2. $\mathcal{R}e(\langle C^* CBy, y \rangle \langle By, y \rangle) \geq 0, \quad \forall y \in H$,

3. there exist $T, \gamma > 0$, such that

$$\int_0^T |\langle C^* CBS(t) y, S(t) y \rangle| dt \geq \gamma \|Cy\|_Y^2, \quad \forall y \in H, \qquad (31)$$

hold, then the control

$$v(t) = \begin{cases} -\dfrac{\langle C^* CBz(t), z(t)\rangle}{\|z(t)\|^2}, & \text{if} \quad z(t) \neq 0, \\ 0, & \text{if} \quad z(t) = 0, \end{cases} \tag{32}$$

exponentially stabilizes the output (5).

Proof: Since $(S(t))_{t\geq 0}$ is a semigroup of contractions, we have

$$\frac{d}{dt}\|z(t)\|^2 \leq 2\mathcal{R}e(v(t)\langle Bz(t), z(t)\rangle).$$

Integrating this inequality, and using hypothesis 2 of Theorem 1.6, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{33}$$

For all $z_0 \in H$ and $t \geq 0$, we have

$$\begin{aligned}\langle C^* CBS(t)z_0, S(t)z_0\rangle &= \langle C^* CBz(t), z(t)\rangle - \langle C^* CBz(t), z(t) - S(t)z_0\rangle \\ &\quad + \langle C^* CBS(t)z_0 - C^* CBz(t), S(t)z_0\rangle.\end{aligned}$$

Since B is locally Lipschitz, there exists a constant positive L that depends on $\|z_0\|$ such that

$$|\langle C^* CBS(t)z_0, S(t)z_0\rangle| \leq |\langle C^* CBz(t), z(t)\rangle| + 2\alpha L\|z(t) - S(t)z_0\|\|z_0\|, \tag{34}$$

where α is a positive constant.

Using (33), we deduce

$$|\langle C^* CBz(t), z(t)\rangle| \leq |v(z(t))|\|z(t)\|\|z_0\|, \quad \forall t \in [0, T]. \tag{35}$$

While from the variation of constants formula and using Schwartz's inequality, we obtain

$$\|z(t) - S(t)z_0\| \leq L\left(T\int_0^T |v(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}}. \tag{36}$$

Integrating (34) over the interval $[0, T]$ and taking into account (35) and (36), we get

$$\begin{aligned}\int_0^T |\langle C^* CBS(t)z_0, S(t)z_0\rangle| dt &\leq 2\alpha T^{\frac{3}{2}}L^2\|z_0\|\left(\int_0^T |v(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\quad + T^{\frac{1}{2}}\|z_0\|\left(\int_0^T |v(z(t))|^2\|z(t)\|^2 dt\right)^{\frac{1}{2}}.\end{aligned}$$

Now, let us consider the nonlinear semigroup $U(t)z_0 := z(t)$ [1].

Replacing z_0 by $U(t)z_0$ in (37), and using the superposition properties of the semigroup $(U(t))_{t\geq 0}$, we deduce that

$$\begin{aligned}\int_0^T |\langle C^* CBS(s)U(t)z_0, S(s)U(t)z_0\rangle| ds &\leq 2\alpha T^{\frac{3}{2}}L^2\|U(t)z_0\| \\ &\quad \times \left(\int_t^{t+T} |v(U(s)z_0)|^2\|U(s)z_0\|^2 ds\right)^{\frac{1}{2}} \\ &\quad + T^{\frac{1}{2}}\|U(t)z_0\|\left(\int_t^{t+T} |v(U(s)z_0)|^2\|U(s)z_0\|^2 ds\right)^{\frac{1}{2}}\end{aligned} \tag{37}$$

Thus, by using (31) and (37), it follows that

$$\gamma \|CU(t)z_0\|_Y \leq M\left(\int_t^{t+T} |v(U(s)z_0)|^2 \|U(s)z_0\|^2 ds\right)^{\frac{1}{2}}, \tag{38}$$

where $M = \left(2\alpha TL^2 + 1\right)T^{\frac{1}{2}}$ is a non-negative constant depending on $\|z_0\|$ and T.
From hypothesis 1 of Theorem 1.6, we have

$$\frac{d}{dt}\|CU(t)z_0\|_Y^2 \leq -2|v(U(t)z_0)|^2 \|U(t)z_0\|^2. \tag{39}$$

Integrating (39) from nT and $(n+1)T$, $(n \in \mathbb{N})$, we obtain

$$\|CU(nT)z_0\|_Y^2 - \|CU((n+1)T)z_0\|_Y^2 \geq 2\int_{nT}^{(n+1)T} |v(U(s)z_0)|^2 \|U(s)z_0\|^2 ds.$$

Using (38), (39) and the fact that $\|CU(t)z_0\|_Y$ decreases, it follows

$$\left(1 + 2\left(\frac{\gamma}{M}\right)^2\right)\|CU((n+1)T)z_0\|_Y^2 \leq \|CU(nT)z_0\|_Y^2.$$

Then

$$\|CU((n+1)T)z_0\|_Y \leq \beta \|CU(nT)z_0\|_Y,$$

where $\beta = \frac{1}{\left(1+2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}}$.
By recurrence, we show that $\|CU(nT)z_0\|_Y \leq \beta^n \|Cz_0\|_Y$.
Taking $n = E\left(\frac{t}{T}\right)$ the integer part of $\frac{t}{T}$, we deduce that

$$\|CU(t)z_0\|_Y \leq \mathrm{R}e^{-\sigma t}\|z_0\|,$$

where $R = \alpha\left(1 + 2\left(\frac{\gamma}{M}\right)^2\right)^{\frac{1}{2}}$, with $\alpha > 0$ and $\sigma = \frac{\ln\left(1+2\left(\frac{\gamma}{M}\right)^2\right)}{2T} > 0$, which achieves the proof.

4.1 Strong stabilization

The following result provides sufficient conditions for strong stabilization of the output (5).

Theorem 1.7 Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H and B be locally Lipschitz. If the conditions:

1. $\mathcal{R}e(\langle C^*CAy, y\rangle) \leq 0, \quad \forall y \in D(A)$,
2. $\mathcal{R}e(\langle C^*CBy, y\rangle\langle By, y\rangle) \geq 0, \quad \forall y \in H$,
3. there exist $T, \gamma > 0$, such that

$$\int_0^T |\langle C^*CBS(t)y, S(t)y\rangle| dt \geq \gamma \|Cy\|_Y^2, \quad \forall y \in H, \tag{40}$$

hold, then the control

$$v(t) = -\langle C^* CBz(t), z(t)\rangle, \tag{41}$$

strongly stabilizes the output (5).

Proof: From hypothesis 1 of Theorem 1.7, we obtain

$$\frac{d}{dt}\|Cz(t)\|_Y^2 \leq -2|\langle C^* CBz(t), z(t)\rangle|^2. \tag{42}$$

Integrating this inequality, gives

$$2\int_0^t |\langle C^* CBz(s), z(s)\rangle|^2 ds \leq \|Cz(0)\|_Y^2.$$

Thus

$$\int_0^{+\infty} |\langle C^* CBz(s), z(s)\rangle|^2 ds < +\infty, \tag{43}$$

From the variation of constants formula and using Schwartz's inequality, we deduce

$$\|z(t) - S(t)z_0\| \leq LT^{\frac{1}{2}}\left(\int_0^T |\langle C^* CBz(s), z(s)\rangle|^2 ds\right)^{\frac{1}{2}}. \tag{44}$$

Integrating (34) over the interval $[0, T]$ and taking into account (44), we obtain

$$\int_0^T |\langle C^* CBS(s)z_0, S(s)z_0\rangle| ds \leq 2\alpha L^2 T^{\frac{3}{2}}\|z_0\|^2\left(\int_0^T |\langle C^* CBz(s), z(s)\rangle|^2 ds\right)^{\frac{1}{2}} + T^{\frac{1}{2}}\left(\int_0^T |\langle C^* CBz(s), z(s)\rangle|^2 ds\right)^{\frac{1}{2}}.$$

Replacing z_0 by $z(t)$ and using the superposition property of the solution, we get

$$\int_0^T |\langle C^* CBS(s)z(t), S(s)z(t)\rangle| ds \leq 2\alpha L^2 T^{\frac{3}{2}}\|z_0\|^2\left(\int_t^{t+T} |\langle C^* CBz(s), z(s)\rangle|^2 ds\right)^{\frac{1}{2}} + T^{\frac{1}{2}}\left(\int_t^{t+T} |\langle C^* CBz(s), z(s)\rangle|^2 ds\right)^{\frac{1}{2}}. \tag{45}$$

By (43), we get

$$\int_t^{t+T} |\langle C^* CBS(s)z(t), S(s)z(t)\rangle| ds \to 0, \text{ as } t \to +\infty. \tag{46}$$

From (40) and (46), we deduce that $\|Cz(t)\|_Y \to 0$, as $t \to +\infty$, which completes the proof.

Proposition 1.8 Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H, B be locally Lipschitz and the assumptions 1, 2 and 3 of Theorem 1.7 hold, then the control (41) strongly stabilizes the output (5) with decay estimate

$$\|Cz(t)\|_Y = O\left(t^{-\frac{1}{2}}\right), \ as\ t \to +\infty. \tag{47}$$

Proof: Using (45), we get

$$\int_0^T |\langle C^*CBS(s)U(t)z_0, S(s)U(t)z_0\rangle| ds \le \theta\sqrt{\xi(t)}, \tag{48}$$

where $\theta = \left(2\alpha TL^2\|z_0\|^2 + 1\right)T^{\frac{1}{2}}$ and $\xi(t) = \left(\int_t^{t+T} |\langle C^*CBU(s)z_0, U(s)z_0\rangle|^2 ds\right)$.
From (40) and (48), we deduce that

$$\varrho\sqrt{\xi(nT)} \ge \|CU(nT)z_0\|_Y^2, \quad \forall n \ge 0, \tag{49}$$

where $\varrho = \frac{1}{\gamma}\theta$.
Integrating the above inequality gives

$$\frac{d}{dt}\|CU(t)z_0\|_Y^2 \le -2|\langle C^*CBU(t)z_0, U(t)z_0\rangle|^2,$$

from nT to $(n+1)T$, $(n \in \mathbb{N})$ and using (49), we obtain

$$\|CU(nT)z_0\|_Y^2 - \|CU(nT+T)z_0\|_Y^2 \ge 2\xi(nT), \quad \forall n \ge 0.$$

We obtain

$$\varrho^2\|CU(nT+T)z_0\|_Y^2 - \varrho^2\|CU(nT)z_0\|_Y^2 \le -2\|CU(nT)z_0\|_Y^4, \quad \forall n \ge 0. \tag{50}$$

Let us introduce the sequence $r_n = \|CU(nT)z_0\|_Y^2, \quad \forall n \ge 0.$
Using (50), we deduce that

$$\frac{r_n - r_{n+1}}{r_n^2} \ge \frac{2}{\varrho^2}, \quad \forall n \ge 0.$$

Since the sequence (r_n) decreases, we get

$$\frac{r_n - r_{n+1}}{r_n . r_{n+1}} \ge \frac{2}{\varrho^2}, \quad \forall n \ge 0,$$

and also

$$\frac{1}{r_{n+1}} - \frac{1}{r_n} \ge \frac{2}{\varrho^2}, \quad \forall n \ge 0.$$

We deduce that

$$r_n \le \frac{r_0}{\frac{2r_0}{\varrho^2}n + 1}, \quad \forall n \ge 0.$$

Finally, introducing the integer part $n = E\left(\frac{t}{T}\right)$ and from (42), the function $t \to \|CU(t)z_0\|_Y$ decreases. We deduce the estimate

$$\|Cz(t)\|_Y = O\left(t^{-1/2}\right), \quad as\ t \to +\infty.$$

4.2 Weak stabilization

The following result discusses the weak stabilization of the output (5).

Theorem 1.9 Let A generate a semigroup $(S(t))_{t\geq 0}$ of contractions on H, B be locally Lipschitz and weakly sequentially continuous. If assumptions 1, 2 of Theorem 1.7 and

$$\langle C^* CBS(t)y, S(t)y\rangle = 0, \quad \forall t\geq 0 \Rightarrow Cy = 0, \tag{51}$$

hold, then the control

$$v(t) = -\langle C^* CBz(t), z(t)\rangle, \tag{52}$$

weakly stabilizes the output (5).

Proof: Let us consider $\psi \in Y$ and $(t_n)\geq 0$ be a sequence of real numbers such that $t_n \to +\infty$, as $n \to +\infty$.

Using (42), we deduce that the sequence $h_n = \langle Cz(t_n), \psi\rangle_Y$ is bounded.

Let $h_{\gamma(n)}$ be an arbitrary convergent subsequence of h_n.

From (33), the subsequence $z\left(t_{\gamma(n)}\right)$ is bounded in H, so we can extract a subsequence still denoted by $z\left(t_{\gamma(n)}\right)$ such that $z\left(t_{\gamma(n)}\right) \rightharpoonup \varphi \in H$, as $n \to +\infty$.

Since C is continuous, B is weakly sequentially continuous and $S(t)$ is continuous $\forall t\geq 0$, we get

$$\lim_{n\to+\infty} \left\langle C^* CBS(t)z\left(t_{\gamma(n)}\right), S(t)z\left(t_{\gamma(n)}\right)\right\rangle = \langle C^* CBS(t)\varphi, S(t)\varphi\rangle.$$

From (46), we have

$$\int_0^T \left\langle C^* CBS(s)z\left(t_{\gamma(n)}\right), S(s)z\left(t_{\gamma(n)}\right)\right\rangle ds \to 0, \quad \text{as} \quad n \to +\infty.$$

Using the dominated convergence Theorem, we deduce that

$$\langle C^* CBS(t)\varphi, S(t)\varphi\rangle = 0, \quad \text{for all} \quad t\geq 0,$$

which implies, according to (51), that $C\varphi = 0$, and hence $h_n \to 0$, as $t \to +\infty$.

We deduce that $\langle Cz(t), \psi\rangle_Y \to 0$, as $t \to +\infty$. In other words $Cz(t) \rightharpoonup 0$, as $t \to +\infty$, which achieves the proof.

Example 4 Let us consider the system defined in $\Omega =]0, +\infty[$ by

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = -\dfrac{\partial z(x,t)}{\partial x} + v(t)Bz(x,t), & x\in\Omega, \quad t>0, \\ z(x,0) = z_0(x), & x\in\Omega, \end{cases} \tag{53}$$

where $H = L^2(\Omega)$, $Az = -\frac{\partial z}{\partial x}$ with domain $D(A) = \left\{z\in H^1(\Omega) \mid z(0) = 0, \ z(x) \to 0, \ \text{as} \ x \to +\infty\right\}$, $Bz = |\int_0^1 z(x)dx|$ the control operator and $v(.)\in L^2(0,+\infty)$. The operator A generates a semigroup of contractions

$$(S(t)z_0)(x) = \begin{cases} z_0(x-t), & \text{if} \quad x\geq t, \\ 0, & \text{if} \quad x<t. \end{cases}$$

Let $\omega =]0, 1[$ be a subregion of Ω and system (53) is augmented with the output

$$w(t) = \chi_\omega z(t). \tag{54}$$

The operator B is sequentially continuous and verifies

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = | \int_0^{1-t} z_0(x)dx| \int_0^{1-t} z_0(x)dx, \quad 0 \leq t \leq 1.$$

Thus

$$\langle \chi_\omega^* \chi_\omega BS(t)z_0, S(t)z_0 \rangle = 0, \quad \forall t \geq 0 \quad \Rightarrow z_0(x) = 0 \text{ a.e } x \in]0, 1[, \quad \text{i.e} \quad \chi]0, 1[z_0 = 0.$$

Then, the control

$$v(t) = -| \int_0^1 z(x,t)dx| \int_0^1 z(x,t)dx, \tag{55}$$

weakly stabilizes the output (54).

5. Simulations

Consider system (53) with $z(x, 0) = \sin(\pi x)$, and augmented with the output (54).

For $\omega =]0, 2[$, we have

Figure 1 shows that the output (54) is weakly stabilized on ω with error equals 6.8×10^{-4} and the evolution of control is given by **Figure 2**.

For $\omega =]0, 3[$, we have

Figure 3 shows that the output (54) is weakly stabilized on ω with error equals 9.88×10^{-4} and the evolution of control is given by **Figure 4**.

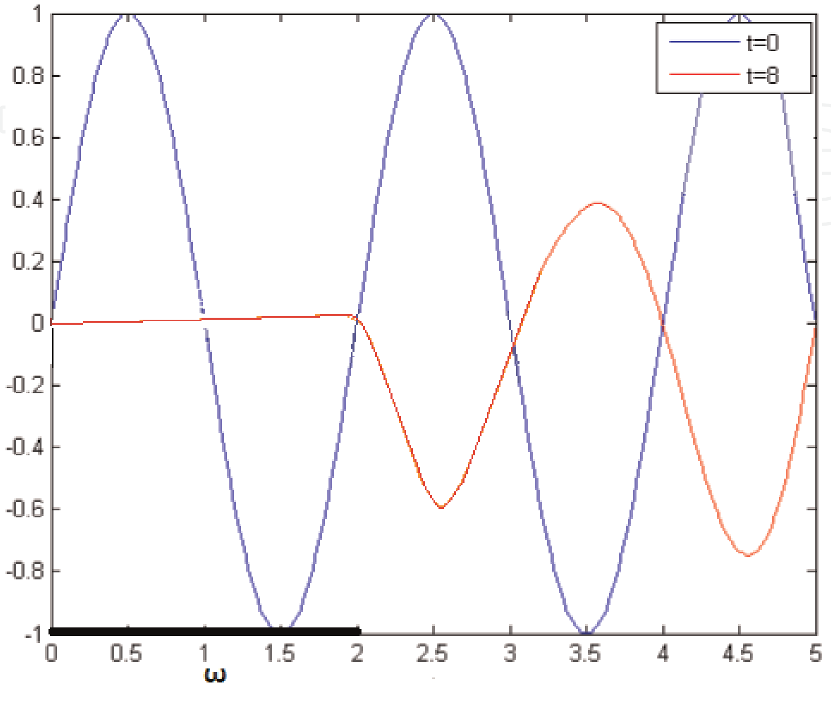

Figure 1.
The stabilization on $\omega =]0, 2[$.

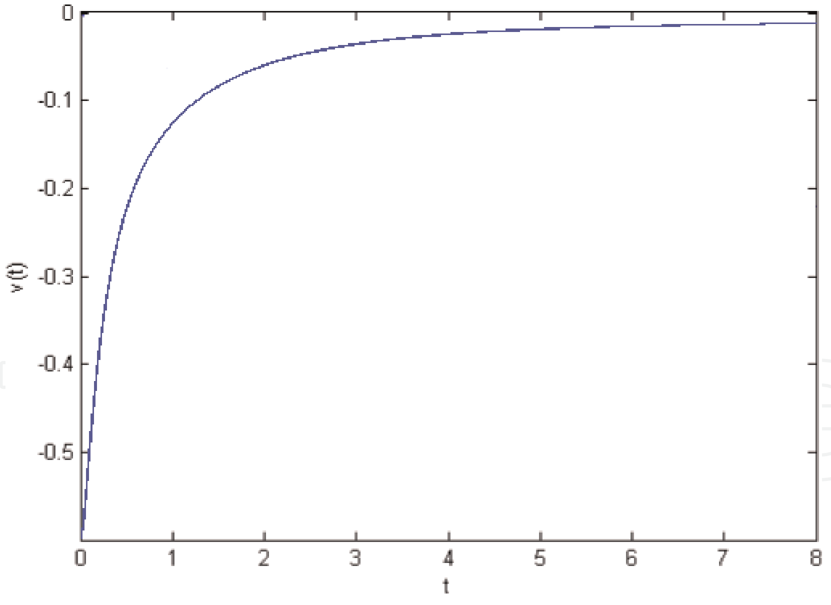

Figure 2.
The evolution control in the interval $]0, 8]$.

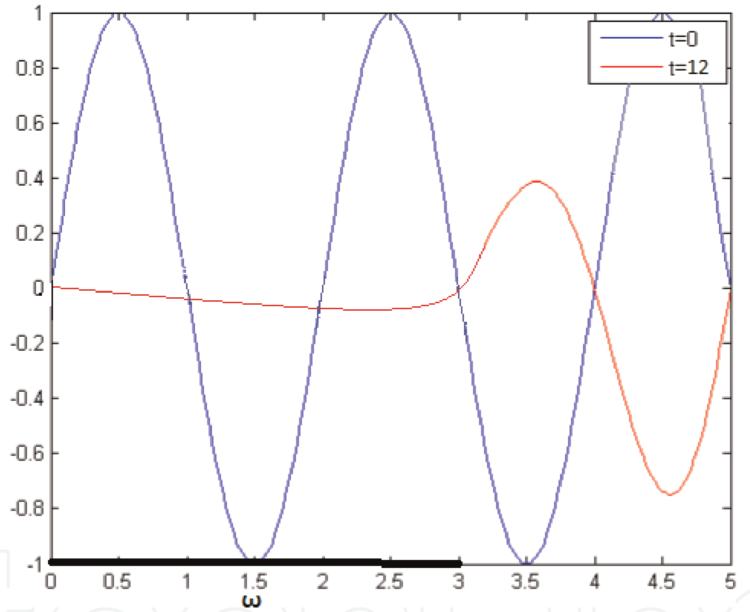

Figure 3.
The stabilization on $\omega =]0, 3[$.

Remark 2. It is clear that the control (55) stabilizes the state on ω, but do not take into account the residual part $\Omega\ \omega$.

6. Conclusions

In this work, we discuss the question of output stabilization for a class of semilinear systems. Under sufficient conditions, we obtain controls depending on the output operator that strongly and weakly stabilizes the output of such systems. This work gives an opening to others questions; this is the case of output stabilization for hyperbolic semilinear systems. This will be the purpose of a future research paper.

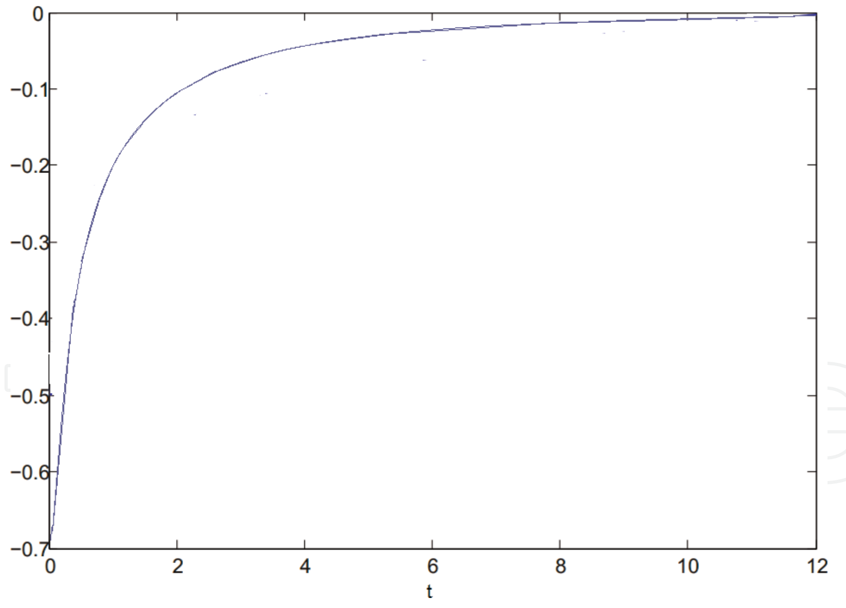

Figure 4.
The evolution control in the interval $]0, 12]$.

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