Chapter

Non-Gaussian Entanglement and Wigner Function

Mustapha Ziane and Morad El Baz

Abstract

A measure of non-Gaussian entanglement in continuous variable (CV) systems based on the volume of the negative part of the Wigner function is proposed. We analyze comparatively this quantity with a numerical evaluation of the negativity of the partial transpose (NPT) considering a system of Bell states formed in the coherent state basis (quasi-Bell states).

Keywords: Wigner function, negativity, non-Gaussian state, nonclassicality, non-Gaussianity, quasi-Bell states, coherent states

1. Introduction

Continuous variable (CV) quantum optical systems are well-established tools for both theoretical and experimental investigations of quantum information processing (QIP) [1, 2]. Entangled states represent key resources, both for quantum computers and for many communication schemes [1, 3], an can be realized with Gaussian two-mode states; these states are relatively easy to work with theoretically and are also commonly produced in a laboratory. It has been successfully applied to implement various important protocols, such as quantum teleportation [4–6], quantum dense coding [7–9], and entanglement swapping [10]. This advancement comes from the development of Gaussian optical operations, such as beam splitting, phase shifting, squeezing, displacement, and homodyne detection. Recently, it became evident that the understanding of entanglement behavior beyond Gaussian systems is a necessity [11–13]. Furthermore, recent theoretical investigations have shown some limits to the Gaussian operations. For example, the no-go theorem relating to the distillation of entanglement shard by distant parties using only Gaussian local operations and classical communications (LOCC) [14, 15]. Moreover, on the theoretical level, the study of entanglement in many-body systems has been limited to Gaussian states [16–19] where the quantification of quantum correlations (QC) reduces to the study of the covariance matrix, but the non-Gaussian entanglement doesn't have such a simplified approach.

The problem of quantifying entanglement in non-Gaussian systems, in a way that is independent of particular external parameters, hasn't solved yet; it is our main objective in this paper. An entanglement measure E of the state ρ should satisfy some criteria [20] to be an entanglement monotone. Many quantities have been proposed as a quantifier of entanglement in discrete variables (DV) and CV Gaussian states. Recently, however, two entanglement measures that are much

more amenable to evaluation have been proposed, the negativity of the partial transpose (NPT) and its logarithmic extension [21].

In this chapter, we are interested in establishing a direct measure of entanglement in non-Gaussian systems. This measure is based on the Wigner representation in the phase space of the non-Gaussian states. That is, they are defined in terms of the quantification of the degree of the negativity of Wigner function (NWF) [22, 23]. The most distinctive feature of this entanglement measure is the ease of calculated with a numerical integration.

2. Two-mode quasi-Bell state: an entangled non-Gaussian state

The simplest example of a non-Gaussian state is the single-photon state. There are also other examples that can be generated by excitations of Gaussian states [24, 25]. Here we are going to use quasi-classical state that has been extensively studied for its nonclassical proprieties and violation of Bell inequalities; it is the superposition of two-mode standard coherent states (SCS). Let us consider two modes of electromagnetic fields A and B with corresponding annihilation operators $\hat{a}$ and $\hat{b}$. Two-mode coherent states are defined by $|\alpha, \beta\rangle = D_a(\alpha)D_b(\beta)|0, 0\rangle$, where $|0, 0\rangle$ is the two-mode vacuum state and $D_i(\alpha)$ is the displacement operator of the mode $i(i = A, B)$. The state $|\alpha, \beta\rangle$ can be expressed into the form

$$|\alpha, \beta\rangle = e^{-\left(|\alpha|^2+|\beta|^2\right)/2} \sum_{n,\, m}^{\infty} \frac{\alpha^n \beta^n}{\sqrt{n!m!}} |n, m\rangle, \tag{1}$$

where $|n_1, n_2\rangle$ are the two-mode Fock states. The quasi-Bell coherent states (QBS) are defined by the following superpositions of two-mode coherent states:

$$|\psi_{\pm}\rangle = N_{\pm}(|\alpha, \beta\rangle \pm |-\alpha, -\beta\rangle), \tag{2}$$

$$|\phi_{\pm}\rangle = N_{\pm}(|\alpha, -\beta\rangle \pm |-\alpha, \beta\rangle), \tag{3}$$

where $N_{\pm} = 1\sqrt{2\exp\left(-2|\alpha|^2 - 2|\beta|^2\right) + 2}$ is the normalization factor.

The Wigner function $\mathcal{W}(\hat{R}, \alpha, \beta)$ of the state (1) is given by

$$\mathcal{W}(\hat{R}, \alpha, \beta) = \frac{1}{\pi} \exp\left(f(\alpha, q_1, p_1) + f(\beta, q_2, p_2)\right), \tag{4}$$

where $\hat{R} = \left(q_1, p_1, q_2, p_2\right)^T$ is the quadrature operators vector and $f(x, y, z) = -2|x|^2 + \sqrt{2}(x^* + x)y + i\sqrt{2}(x - x^*)z - y^2 - z^2$. For the quasi-Bell entangled coherent states Eq. (2), the Wigner function is given by [26, 27].

$$\begin{aligned} \mathcal{W}^{\pm}_{QCS}(\hat{R}, \alpha, \beta) = N^2_{\alpha, \beta, \pm}\big[&\mathcal{W}(\hat{R}_1, \alpha, \alpha)\mathcal{W}(\hat{R}_2, \beta, \beta) \\ &\pm \mathcal{W}(\hat{R}_1, \alpha, -\alpha)\mathcal{W}(\hat{R}_2, \beta, -\beta) \\ &\pm \mathcal{W}(\hat{R}_1, -\alpha, \alpha)\mathcal{W}(\hat{R}_2, -\beta, \beta) \\ &+\mathcal{W}(\hat{R}_1, -\alpha, -\alpha)\mathcal{W}(\hat{R}_2, -\beta, -\beta)\big], \end{aligned} \tag{5}$$

where $\hat{R}_1$ and $\hat{R}_2$ are the quadrature operators vectors of the first and second modes and $\mathcal{W}(\hat{R}_i, x, y)$ is the Wigner function of one-mode coherent state with $i = 1, 2$; $\{x, y\} = \{\pm\alpha, \pm\beta\}$ satisfies the normalization condition

$\int\int\int\int_{\hat{R}} \mathcal{W}(\hat{R}, \alpha, \beta,) d\hat{R} = 1$. Hence the doubled volume of the integrated negative part of the Wigner function of the state (2) may be written as

$$\delta(\psi_{\pm}) = \iiiint_{\hat{R}} \left| \mathcal{W}^{\pm}_{QCS}(\hat{R}, \alpha, \beta) \right| d\hat{R} - 1. \tag{6}$$

By definition, the quantity δ is equal to 0 for coherent and squeezed vacuum states, for which $\mathcal{W}$ is nonnegative. In this work we shall treat δ as a parameter characterizing the properties of the state under consideration.

It is clear from expression (5) and the plot in **Figure 1** that the Wigner function of the quasi-Bell state (2) is non-Gaussian. In order to characterize this non-Gaussianity, several measures of the degree of non-Gaussianity were proposed [28, 29]. According to [29], the degree of non-Gaussianity of state ρ is defined by

$$\delta_{NG}(\rho) = S(\rho \| \tau). \tag{7}$$

where $S(\rho_1 \| \rho_2)$ is the quantum relative entropy between states ρ_1 and ρ_2. Here τ is the reference Gaussian state with the same first and second moments of ρ. This property of reference state τ leads to $\mathrm{Tr}[\rho \ln \tau] = \mathrm{Tr}[\tau \ln \tau]$, so that

$$\delta_{NG}(\rho) = S(\tau) - S(\rho), \tag{8}$$

where $S(\rho)$ is the Von Neumann entropy of the state ρ. Also $S(\tau) = h(d_+) + h(d_-)$, where

$$h(x) = \left(x + \frac{1}{2}\right) \ln \left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) \ln \left(x - \frac{1}{2}\right) \tag{9}$$

and $d^2_{\pm} = \frac{1}{2}\left(\Delta(\delta) \pm \sqrt{\Delta(\delta)^2 - 4I_4}\right)$ are the symplectic eigenvalues of the covariance matrix σ of the reference Gaussian state τ. Here $\Delta(\delta) = I_1 + I_2 + 2I_3$, where $I_1 = \det(\mathbf{A})$, $I_2 = \det(\mathbf{B})$, $I_3 = \det(\mathbf{C})$, and $I_4 = \det(\sigma)$ are the four local symplectic invariants of the covariance matrix:

$$\sigma = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^{\dagger} & \mathbf{B} \end{pmatrix}, \tag{10}$$

where

$$\sigma_{ij} = \frac{1}{2}\left\langle \left\{ R_i, R_j \right\} \right\rangle - \left\langle R_i \right\rangle \left\langle R_j \right\rangle. \tag{11}$$

For the considered states (2), we suppose that the two fields have the same mode ($\alpha = \beta$); we find

$$\sigma_{\psi_+} = \begin{pmatrix} u_+ & 0 & r_+ & 0 \\ 0 & v_+ & 0 & s_+ \\ r_+ & 0 & u_+ & 0 \\ 0 & s_+ & 0 & v_+ \end{pmatrix}, \tag{12}$$

$$\sigma_{\psi_-} = \begin{pmatrix} u_- & 0 & r_- & 0 \\ 0 & v_- & 0 & s_- \\ r_- & 0 & u_- & 0 \\ 0 & s_- & 0 & v_- \end{pmatrix}, \tag{13}$$

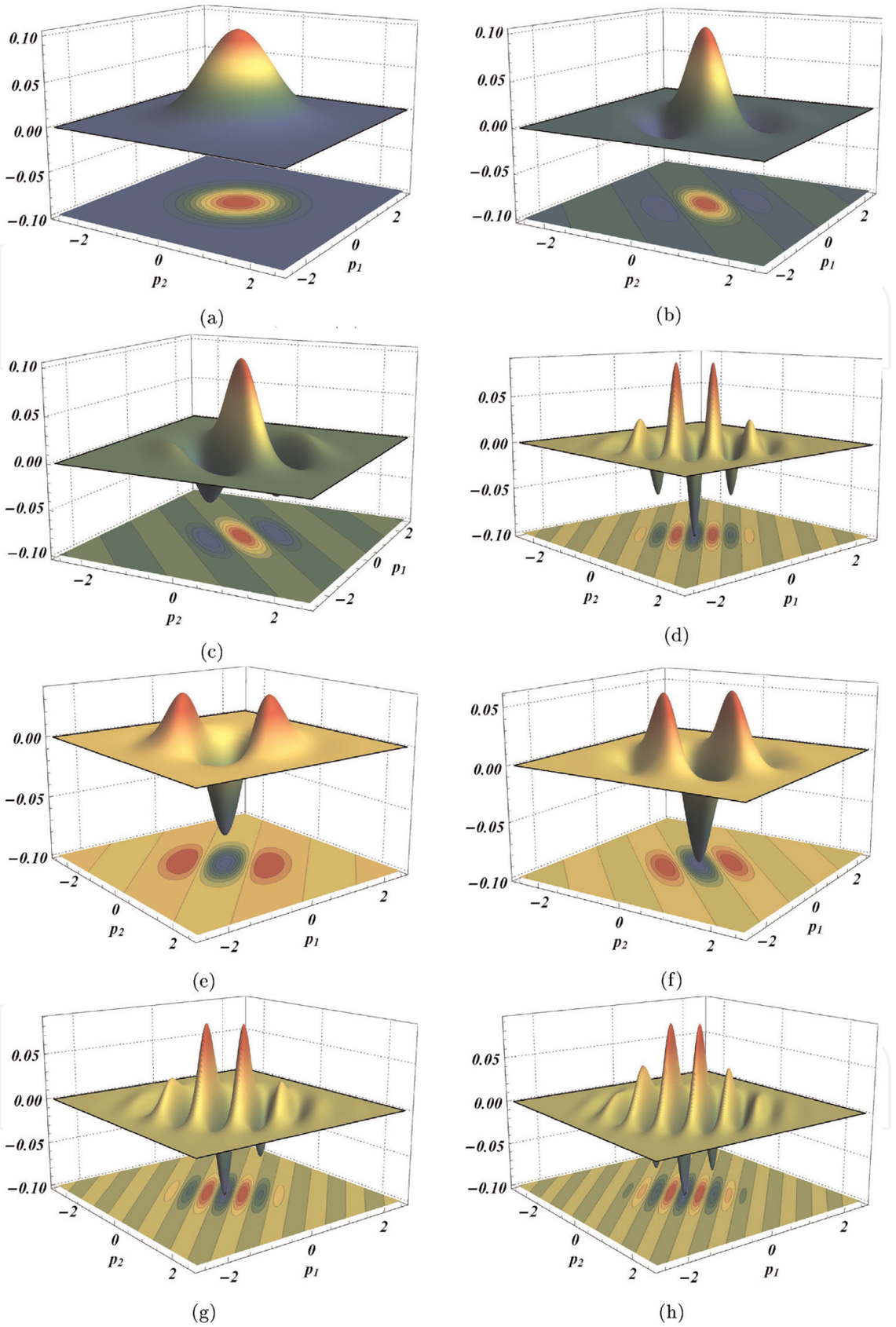

Figure 1.
Wigner function of quasi-Bell states (2). (a) Wigner function of ψ_+ for $|\alpha| = 0$. (b) Wigner function of ψ_+ for $|\alpha| = 2$. (c) Wigner function of ψ_+ for $|\alpha| = 1$. (d) Wigner function of ψ_+ for $|\alpha| = 2$. (e) Wigner function of ψ_- for $|\alpha| = 0.5$. (f) Wigner function of ψ_- for $|\alpha| = 1$. (g) Wigner function of ψ_- for $|\alpha| = 1.5$. (h) Wigner function of ψ_- for $|\alpha| = 2$.

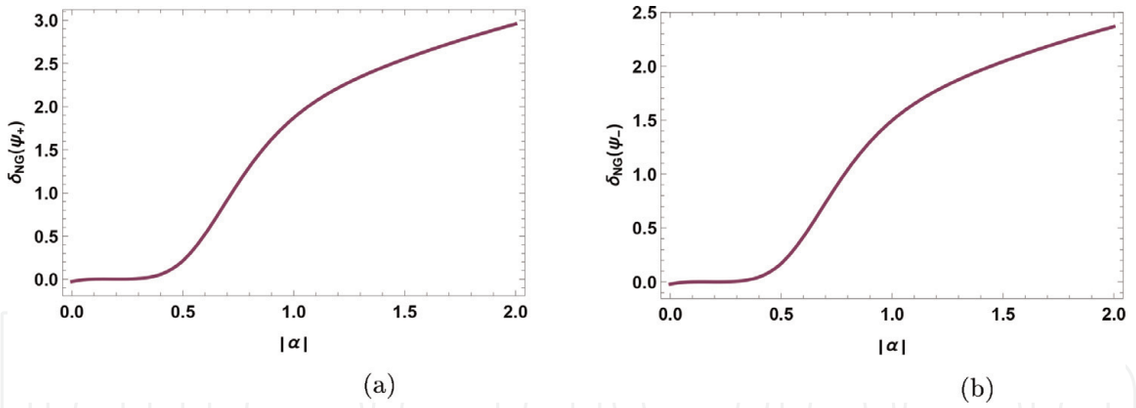

Figure 2.
Non-Gaussianity versus $|\alpha|$. (a) for states ψ_+ and (b) for state ψ_-.

where we have defined

$$u_\pm = N^2_{\alpha,\beta,\pm}\left(4\alpha^2 \pm \Gamma^2 + 1\right), \qquad r_\pm = 4\alpha^2 N^2_{\alpha,\beta,\pm}, \tag{14}$$

$$v_\pm = N^2_{\alpha,\beta,\pm}\left(\mp 4\alpha^2\Gamma^2 \mp \Gamma^2 + 1\right), \qquad s_\pm = \mp 4\Gamma^2\alpha^2 N^2_{\alpha,\beta,\pm}, \tag{15}$$

with $\Gamma = \langle\alpha| - \alpha\rangle = Exp\left(-\frac{|\alpha|^2}{2}\right)$. **Figure 2** shows the behavior of non-Gaussianity of states (2) in terms of $|\alpha|$. These figures show that non-Gaussianity increases with increasing $|\alpha|$ (this behavior will be discussed in the fourth section).

3. Numerical evaluation of negativities (NPT and NWF)

In this section, we briefly review the NPT as a computable entanglement measure that possesses the proprieties of an entanglement monotone given in [21]. The NPT, $N(\hat{\rho})$ of a state $\hat{\rho}$ is defined as the absolute value of the sum of the negative eigenvalues of the partial transpose of $\hat{\rho}$ denoted $\hat{\rho}^{PT}$. We may write it as

$$N(\hat{\rho}) = \frac{1}{2}Tr\left(\sqrt{\left(\hat{\rho}^{PT}\right)^2} - \hat{\rho}^{PT}\right) = \frac{\|\hat{\rho}^{PT}\| - 1}{2}, \tag{16}$$

where $\|.\|$ denotes the trace norm [21].

The quasi-Bell coherent state (2) is defined in a non-orthonormal basis, and it is typically not possible to obtain an analytical expression for the negativity. However, as shown in the following, one can compute it numerically. First, we expand the quasi-Bell state (2) in the Fock basis:

$$\hat{\rho}_\pm = \sum_{n_1,\, n_2,\, m_1,\, m_2} \rho^\pm_{n_1,n_2,m_1,m_2} |m_1\rangle\langle n_1| \otimes |m_2\rangle\langle n_2|, \tag{17}$$

where

$$\rho^\pm_{n_1,n_2,m_1,m_2} = N_0^2 e^{-2|\alpha|}\left(\frac{\alpha^{(n_1+n_2)}}{\sqrt{n_1!n_2!}} + \frac{(-\alpha)^{(m_1+m_2)}}{\sqrt{m_1!m_2!}}\right). \tag{18}$$

The partial transpose of this state with respect to mode two is

$$\hat{\rho}_{\pm} = \sum_{n_1, n_2, m_1, m_2} \rho^{\pm}_{n_1, m_2, m_1, n_2} |m_1\rangle \langle n_1| \otimes |m_2\rangle \langle n_2|. \tag{19}$$

The eigenvalues are obtained by numerical diagonalization of the partial transpose density matrix (19). With this result, we can obtain the NPT straightforwardly using Eq. (16), and **Figure 3a** and **b** shows the numerical values of this NPT.

4. Discussion

In this section, we will discuss the different behaviors of the non-Gaussian entanglement and the variation of the negativity of the WF for the bipartite system considered early in terms of the coherent state amplitude $|\alpha|$.

Figure 2 shows the variation of the degree of non-Gaussianity for the states in Eq. (2) as a function of coherent state amplitude $|\alpha|$. We see that the non-Gaussianity δ_{NG} measured by (8) equal to 0 for small values of $|\alpha|$ increases with increasing values of the parameter $|\alpha|$ to larger values much higher than 1 and does not establish in a maximum value. On the other hand, the NPT plots are shown in **Figure 3a** and **b** for the state (2) equal to 0 for $|\alpha| = 0$ and increase with increasing values of the parameter $|\alpha|$ to reach its maximum value that is, equal to 1 for $|\alpha| \gtrsim 1.3$. Furthermore, it is seen that the entanglement for large values of α reaches its maximum value. It is worthwhile noting that, at the limit of large values of the parameter α, the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ become orthogonal; thus the behavior of quasi-Bell state (2) is, as expected, exactly that of the Bell state.

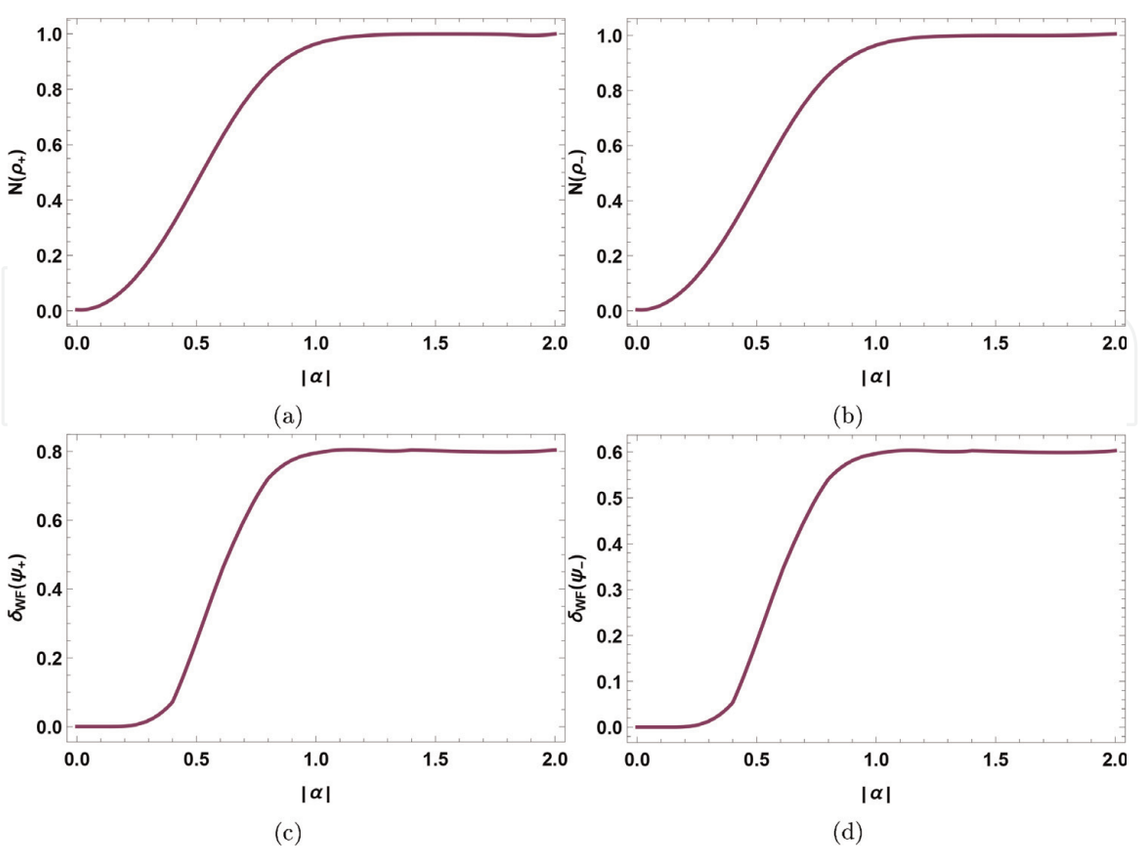

Figure 3.
Negativity of the partial transpose versus $|\alpha|$: for the quasi-Bell state ψ_+ (a) and ψ_- (b). Negativity of the Wigner function versus $|\alpha|$: for the quasi-Bell state ψ_+ (c) and ψ_- (d).

The plot in **Figure 3c** and **d** shows the behavior of the NWF as a function of $|\alpha|$ for the non-Gaussian system (2). These two plots show that the NWF δ_{WF} has the same behavior as the NPT. This allows to show that they behave identically and they have the same inflection points. Which confirms that the NWF is a direct computable measure of non-Gaussian bipartite entanglement that posses the proprieties of an entanglement quantifier [21].

For our measure, $1 \geq \delta_{WF} \geq 0$, equal to zero when α became null and the state in Eq. 2 is now nothing but a two-vacuum product state, and it is maximal for large values of α where the state (2) is maximally entangled (Bell state).

5. Conclusion

In this work, we have evaluated the negativity of Wigner function and the negativity of the partial transpose in non-Gaussian states formed by two modes of field coherent states. We have shown that the negative parts of the Wigner function can be used as a detector of non-Gaussian entanglement. Interestingly, as used in this work, the degree of Wigner function negativity can be used as a direct quantifier of non-Gaussian bipartite entanglement.

This work allows us to describe the best characterization of the non-Gaussian Wigner function and the important use of its negativity in bipartite non-Gaussian systems, which gives more efficiency in CV quantum information theory, particularly in quantum computing [30], because the Wigner function can be measured experimentally [31, 32], including the measurements of its negative values [33]. The interest put on such experiments has triggered a search for operational definitions of the Wigner functions, based on the experimental setup [34, 35]. It does represent a major step forward in the detection and the quantification of non-Gaussian entanglement in bipartite systems.

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References

[1] Nielsen MA, Chuang I. Quantum Computation and Quantum Information. 10th Anniversary Edition. New York, NY: Cambridge University Press; 2011

[2] Pati AK, Braunstein SL. Quantum information with continuous variables. Deutsch-Jozsa Algorithm for Continuous Variables. Dordrecht: Springer; 2003. pp. 31-36

[3] Horodecki R, Horodecki P, Horodecki M, Horodecki K. Quantum entanglement. Reviews of Modern Physics. 2009;**81**(2):865

[4] Braunstein SL, Kimble HJ. Teleportation of continuous quantum variables. Physical Review Letters. 1998; **80**(4):869

[5] Furusawa A, Sørensen JL, Braunstein SL, Fuchs CA, Kimble HJ, Polzik ES. Unconditional quantum teleportation. Science. 1998;**282**(5389):706-709

[6] Zhang TC, Goh KW, Chou CW, Lodahl P, Kimble HJ. Quantum teleportation of light beams. Physical Review A. 2003;**67**:033802

[7] Bowen WP, Treps N, Buchler BC, Schnabel R, Ralph TC, Bachor H-A, et al. Experimental investigation of continuous-variable quantum teleportation. Physical Review A. 2003; **67**:032302

[8] Li X, Pan Q. Quantum dense coding exploiting a bright Einstein-Podolsky-Rosen beam. Physical Review Letters. 2002;**88**:047904

[9] Mizuno J, Wakui K, Furusawa A, Sasaki M. Experimental demonstration of entanglement-assisted coding using a two-mode squeezed vacuum state. Physical Review A. 2005;**71**(1):012304

[10] Takei N. High-fidelity teleportation beyond the no-cloning limit and entanglement swapping for continuous variables. Physical Review Letters. 2005;**94**:220502

[11] Gomes R, Salles A, Toscano F, Ribeiro PS, Walborn S. Quantum entanglement beyond Gaussian criteria. Proceedings of the National Academy of Sciences. 2009;**106**(51):21517-21520

[12] Strobel H, Muessel W, Linnemann D, Zibold T, Hume DB, Pezzè L, et al. Fisher information and entanglement of non-Gaussian spin states. Science. 2014; **345**(6195):424-427

[13] Walschaers M, Fabre C, Parigi V, Treps N. Entanglement and wigner function negativity of multimode non-Gaussian states. Physical Review Letters. 2017;**119**(18):183601

[14] Eisert J, Scheel S, Plenio MB. Distilling gaussian states with Gaussian operations is impossible. Physical Review Letters. 2002;**89**(13):137903

[15] Giedke G. Characterization of Gaussian operations and distillation of Gaussian states. Physical Review A. 2002;**66**:032316

[16] Abdul-Rahman H, Nachtergaele B, Sims R, Stolz G. Entanglement dynamics of disordered quantum xy chains. Letters in Mathematical Physics. 2016;**106**(5):649-674

[17] Eisert J, Cramer M, Plenio MB. Colloquium: Area laws for the entanglement entropy. Reviews of Modern Physics. 2010;**82**(1):277

[18] Lami L, Serafini A, Adesso G. Gaussian entanglement revisited. New Journal of Physics. 2018;**20**(2):023030

[19] Nachtergaele B, Sims R, Stolz G. An area law for the bipartite entanglement of disordered oscillator systems. Journal of Mathematical Physics. 2013;**54**(4): 042110

[20] Vidal G. Entanglement monotones. Journal of Modern Optics. 2000;**47**:355

[21] Vidal G, Werner RF. Computable measure of entanglement. Physical Review A. 2002;**65**(3):032314

[22] Ziane M, El Baz M. Direct measure of genuine tripartite entanglement independent from bipartite constructions. Quantum Information Processing. 2018;**17**(8):196

[23] Ziane M, Siyouri F-Z, Elbaz M, Hassouni Y. The negativity of partial transpose vs the negativity of wigner function. International Journal of Geometric Methods in Modern Physics. 2019

[24] Agarwal GS, Tara K. Nonclassical properties of states generated by the excitations on a coherent state. Physical Review A. 1991;**43**:492-497

[25] Agarwal GS, Puri RR, Singh RP. Vortex states for the quantized radiation field. Physical Review A. 1997;**56**: 4207-4215

[26] Sen AK. Phase distribution in a disordered chain and the emergence of a two-parameter scaling in the quasiballistic to the mildly localized regime. Modern Physics Letters B. 1997; **11**(13):555-564

[27] Jiang N-Q. The n-partite entangled wigner operator and its applications in Wigner function. Journal of Optics B: Quantum and Semiclassical Optics. 2005;**7**(9):264

[28] Genoni MG, Paris MG, Banaszek K. Measure of the non-Gaussian character of a quantum state. Physical Review A. 2007;**76**(4):042327

[29] Genoni MG, Paris MG. Quantifying non-Gaussianity for quantum information. Physical Review A. 2010; **82**(5):052341

[30] Forcer T, Hey A, Ross D, Smith P. Superposition, entanglement and quantum computation. Quantum Information and Computation. 2002; **2**(2):97-116

[31] Smithey D, Beck M, Raymer M, Faridani A. Measurement of the wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum. Physical Review Letters. 1993;**70**(9):1244

[32] Banaszek K, Radzewicz C, Wódkiewicz K, Krasiński J. Direct measurement of the Wigner function by photon counting. Physical Review A. 1999;**60**(1):674

[33] Kenfack A, yczkowski K. Negativity of the wigner function as an indicator of nonclassicality. Journal of Optics B: Quantum and Semiclassical Optics. 2004;**6**(10):396

[34] Lougovski P, Solano E, Zhang Z, Walther H, Mack H, Schleich W. Fresnel representation of the Wigner function: An operational approach. Physical Review Letters. 2003;**91**(1): 010401

[35] Leonhardt U. Measuring the Quantum State of Light. Vol. 22. Cambridge University Press; 1997