Chapter

# Cyclotomic and Littlewood Polynomials Associated to Algebras

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## Abstract

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Assume $A$ is a basic connected and triangular algebra with $n$ pairwise non-isomorphic simple modules. We consider the *Coxeter transformation* $\phi_A(T)$ as the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation $\tau$ in the derived category $D^b(\mathrm{mod}_A)$ of the module category $\mathrm{mod}_A$ of finite dimensional left $A$-modules. In this paper we study the Mahler measure $\mathbb{M}(\chi_A)$ of the Coxeter polynomial $\chi_A$ of certain *algebras* $A$. We consider in more detail two cases: (a) $A$ is said to be *cyclotomic* if all eigenvalues of $\chi_A$ are roots of unity; (b) $A$ is said to be of *Littlewood type* if all coefficients of $\chi_A$ are $-1$, 0 or 1. We find criteria in order that $A$ is of one of those types. In particular, we establish new records according to Mossingshoff's list of *Record Mahler measures* of polynomials $q$ with $1 < \mathbb{M}(q)$ as small as possible, ordered by their number of roots outside the unit circle.


**Keywords:** finite dimensional algebra, coxeter transformation, derived category, accessible algebra, characteristic polynomial, cyclotomic polynomial, littlewod type

## 1. Introduction

Assume throughout the paper that $K$ is an algebraically closed field. We assume that $A$ is a triangular finite dimensional basic $K$-algebra, that is, of the form $A = KQ/I$, where $I$ is an ideal of the path algebra $KQ$ for $Q$ a quiver without oriented cycles. In particular, $A$ has finite global dimension. The Coxeter transformation $\phi_A$ is the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation $\tau$ in the derived category $D^b(A)$ see [1]. The characteristic polynomial $\chi_A$ of $\phi_A$ is called the *Coxeter polynomial* $\chi_A$ of $A$, or simply $\chi_A$ see [15, 17]. It is a monic self-reciprocal polynomial, therefore it is $\chi_A = a_0 + a_1T + a_2T^2 + \ldots + a_{n-2}T^{n-2} + a_{n-1}T^{n-1} + a_nT^n \in \mathbb{Z}[T]$, with $a_i = a_{n-i}$ for $0 \le i \le n$, and $a_0 = 1 = a_n$.

Consider the roots $\lambda_1, \ldots, \lambda_n$ of $\chi_A$, the so called *spectrum* of $A$. There is a number of measures associated to the absolute values $|\lambda|$ for $\lambda$ in the spectrum $\mathrm{Spec}(\phi_A)$ of $A$. For instance, the *spectral radius* of $A$ is defined as $\rho_A = \max\{|\lambda| : \lambda \in \mathrm{Spec}(\phi_A)\}$ and the *Mahler measure* of $\chi_A$ defined as $\mathbb{M}(\chi_A) = \max\left\{1, \prod_{|\lambda|>1}|\lambda|\right\}$. Recently, some explorations on the relations of the Mahler measure $\mathbb{M}(\chi_A)$ and properties of the algebra $A$ have been initiated.

For a one-point extension $A = B[N]$, we show that $\mathbb{M}(\chi_B) \le \mathbb{M}(\chi_A)$. The strongest statements and examples will be given for the class of accessible algebras. We say that an algebra $A$ is *accessible from* $B$ if there is a sequence $B = B_1, B_2, \ldots, B_s = A$ of algebras such that each $B_{i+1}$ is a one-point extension (resp. coextension) of $B_i$ for some exceptional $B_i$-module $M_i$. As a special case, a $K$-algebra $A$ is called *accessible* if $A$ is accessible from the one vertex algebra $K$.

We say that $A$ is of *cyclotomic type* if the eigenvalues of $\phi_A$ lie on the unit circle. Many important finite dimensional algebras are known to be of cyclotomic type: hereditary algebras of finite or tame representation type, canonical algebras, some extended canonical algebras and many others. On the other hand, there are well-known classes of algebras with a mixed behavior with respect to cyclotomicity. For instance, in Section 6 below we consider the class of *Nakayama algebras*. Let $N(n, r)$ be the quotient obtained from the linear quiver with $n$ vertices

$$\bullet \xrightarrow{x} \bullet \xrightarrow{x} \cdots \bullet \xrightarrow{x} \bullet$$

with relations $x^r = 0$. The Nakayama algebras $N(n, 2)$ are easily proven to be of cyclotomic type, while those of the form $N(n, 3)$ are of cyclotomic type as consequence of lengthly considerations in [18]. The case $r = 4$ is more representative: $N(n, 4)$ is of cyclotomic type for all $0 \le n \le 100$ except for $n = 10, 22, 30, 42, 50, 62, 70, 82$ and $90$. Clearly, if $A$ is of cyclotomic type then $|\mathrm{Tr}\,(\phi_A)^k| \le n$, for $k \ge 0$. We show the following theorem.

**Theorem 1:** *Let M be an unimodular $n \times n$-matrix. The following are equivalent:*

a. *M is of cyclotomic type;*

b. for every positive integer $0 \le k \le n$, we have $|\mathrm{Tr}\,(M^k)| \le n$.

We also consider algebras $A$ of *Littlewood type* where $\chi_A$ has all its coefficients in the set $\{-1, 0, 1\}$. Among other structure results, we prove.

**Proposition.** *The closure $\overline{P}$ of the set P of roots of Littlewood polynomials, equals the set R of roots of Littlewood series.*

Our results make use of well established techniques in the *representation theory of algebras*, as well as results from the *theory of polynomials* and *transcendental number theory*, where Mahler measure has its usual habitat. We stress here the natural context of these investigations on the largely unexplored overlapping area of these important subjects. Hence, rather than a comprehensive study we understand our work as a preliminary exploration where examples are most valuable.

## 2. Measures for polynomials

### 2.1 Self-reciprocal polynomials

A polynomial $p(z)$ of degree $n$ is said to be self-reciprocal if $p(z) = z^n p(1/z)$. The following table displays the number $a(n)$ of polynomials $p$ of degree $n$ (for small $n$) with $p(0)$ non-zero, $b(n)$ is the number of such polynomials which are additionally self-reciprocal, and $c(n)$ is the number of those which are self-reciprocal and where $p(-1)$ is the square of an integer.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a(n)$ | 2 | 6 | 10 | 24 | 38 | 78 | 118 | 224 | 330 | 584 | 838 | 1420 | 4514 | 30,532 | 152,170 |
| $b(n)$ | 1 | 5 | 5 | 19 | 19 | 59 | 59 | 165 | 165 | 419 | 419 | 1001 | 2257 | 20,399 | 76,085 |
| $c(n)$ | 1 | 3 | 5 | 12 | 19 | 34 | 59 | 99 | 165 | 244 | 419 | 598 | 2257 | 12,526 | 76,085 |

Indeed, there is an efficient algorithm to determine such polynomials of given degree $n$, based on a quadratic bound for $n \leq 4f(n)^2$ in terms of Euler totient function, $f(n)$.

Cyclotomic polynomials $\Phi_n$ and their products are a natural source for self-reciprocal polynomials. Clearly, $\Phi_1(z) = z - 1$ is not self-reciprocal, but all remaining $\Phi_n$ (with $n \geq 2$) are. Hence, exactly the polynomials $(z-1)^{2k}\prod_{n \geq 2}\Phi_n^{e_n}$ with natural numbers $k$ and $e_n$ are self-reciprocal with spectral radio one and without eigenvalue zero.

It is not a coincidence that in the above tables we have $b(n) = c(n+1)$ for $n$ even and $b(n) = c(n)$ for n odd. Indeed, if $p$ is self-reciprocal of odd degree then $p(-1) = 0$, hence $p(z) = (z+1)q(z)$ where $q$ is also self-reciprocal.

### 2.2 Mahler measure

Let $A$ be a finite dimensional $K$-algebra with finite global dimension. The *Grothendieck group* $K_0(A)$ of the category $\mathrm{mod}_A$ of finite dimensional (right) $A$-modules, formed with respect to short exact sequences, is naturally isomorphic to the Grothendieck group of the derived category, formed with respect to exact triangles.

The Coxeter transformation $\phi_A$ is the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation $\tau$. The characteristic polynomial $\chi_A(T)$ of $\phi_A$ is called the *Coxeter polynomial* $\chi_A(T)$ of $A$, or simply $\chi_A$. It is a monic self-reciprocal polynomial, therefore it is $\chi_A(T) = a_0 + a_1T + a_2T^2 + \ldots + a_{n-2}T^{n-2} + a_{n-1}T^{n-1} + a_nT^n \in \mathbb{Z}[T]$, with $a_i = a_{n-i}$ for $0 \leq i \leq n$, and $a_0 = 1 = a_n$.

Consider the roots $\lambda_1(A), \ldots, \lambda_n(A)$ of $\chi_A$, the so called *spectrum* of $A$. In [15], a measure for polynomials was introduced. Namely, the *Mahler measure* of $\chi_A$ is $\mathbb{M}(\chi_A) = \max\{1, \prod_{i=1}^n |\lambda_i|\}$ . By a celebrated result of Kronecker [9], see also [7, Prop. 1.2.1], a monic integral polynomial $p$, with $p(0) \neq 0$, has $\mathbb{M}(p) = 1$ if and only if $p$ factorizes as product of cyclotomic polynomials. As observed in [18], $A$ is of *cyclotomic type* if and only if $\mathbb{M}(\chi_A) = 1$, that is, $\chi_A(T)$ factorizes as product of cyclotomic polynomials.

### 2.3 Spectral radius one, periodicity

If the spectrum of $A$ lies in the unit disk, then all roots of $\chi_A$ lie on the unit circle, hence $A$ has spectral radius $\rho_A = 1$. Clearly, for fixed degree there are only finitely many monic integral polynomials with this property.

The following finite dimensional algebras are known to produce Coxeter polynomials of spectral radius one:

1. hereditary algebras of finite or tame representation type;

2. all canonical algebras;

3. (some) extended canonical algebras;

4. generalizing (2), (some) algebras which are derived equivalent to categories of coherent sheaves.

We put $v_n = 1 + x + x^2 + \ldots + x^{n-1}$. Note that $v_n$ has degree $n-1$. There are several reasons for this choice: first of all $v_n(1) = n$, second this normalization yields convincing formulas for the Coxeter polynomials of canonical algebras and hereditary stars, third representing a Coxeter polynomial — for spectral radius one — as a rational function in the $v_n$'s relates to a Poincaré series, naturally attached to the setting.

| Dynkin type | Star symbol | $v$-factorization | Cyclotomic factorization | Coxeter number |
|---|---|---|---|---|
| $\mathbb{A}_n$ | $[n]$ | $v_{n+1}$ | $\prod_{d\mid n,\, d>1} \Phi_d$ | $n+1$ |
| $\mathbb{D}_n$ | $[2, 2, n-2]$ | $\frac{v_2\,(v_2 v_{n-2})}{(v_2 v_{n-2}) v_{n-1}} v_{2(n-1)}$ | $\Phi_2 \prod_{\substack{d\mid 2(n-1)\\ d\neq 1,\, d\neq n-1}} \Phi_d$ | $2(n-1)$ |
| $\mathbb{E}_6$ | $[2, 3, 3]$ | $\frac{v_2 v_3 (v_3)}{(v_3) v_4 v_6} v_{12}$ | $\Phi_3\Phi_{12}$ | 12 |
| $\mathbb{E}_7$ | $[2, 3, 4]$ | $\frac{v_2 v_3 (v_4)}{(v_4) v_6 v_9} v_{18}$ | $\Phi_2\Phi_{18}$ | 18 |
| $\mathbb{E}_8$ | $[2, 3, 5]$ | $\frac{v_2 v_3 v_5}{v_6 v_{10} v_{15}} v_{30}$ | $\Phi_{30}$ | 30 |

In the column '$v$-factorization', we have added some extra terms in the nominator and denominator which obviously cancel.

Inspection of the table shows the following result:

**Proposition.** *Let $k$ be an algebraically closed field and $A$ be a connected, hereditary $k$-algebra which is representation-finite. Then the Coxeter polynomial $\chi_A$ determines $A$ up to derived equivalence.* □

### 2.4 Triangular algebras

Nearly all algebras considered in this survey are triangular. By definition, a finite dimensional algebra is called *triangular* if it has triangular matrix shape

$$\begin{bmatrix} A_1 & M_{12} & \cdots & M_{1n} \\ 0 & A_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

where the diagonal entries $A_i$ are skew-fields and the off-diagonal entries $M_{ij}$, $j > i$, are $A_i, A_j$-bimodules. Each triangular algebra has finite global dimension.

**Proposition.** *Let $A$ be a triangular algebra over an algebraically closed field $K$. Then $\chi_A(-1)$ is the square of an integer.*

*Proof.* Let $C$ be the Cartan matrix of $A$ with respect to the basis of indecomposable projectives. Since $A$ is triangular and $K$ is algebraically closed, we get $\det C = 1$, yielding

$$\chi_A = \left|xI + C^{-1}C^t\right| = \left|C^{-1}\right| \cdot \left|xC + C^t\right| = \left|C^t + xC\right|.$$

Hence $\chi_A(-1)$ is the determinant of the skew-symmetric matrix $S = C^t - C$. Using the skew-normal form of $S$, see [16, Theorem IV.1], we obtain $S' = U^t S U$ for some $U \in \mathrm{GL}_n(\mathbb{Z})$, where $S'$ is a block-diagonal matrix whose first block is the zero matrix of a certain size and where the remaining blocks have the shape $\begin{bmatrix} 0 & m_i \\ -m_i & 0 \end{bmatrix}$ with integers $m_i$. The claim follows. □

*Which self-reciprocal polynomials of spectral radius one are Coxeter polynomials?* The answer is not known. If arbitrary base fields are allowed, we conjecture that all self-reciprocal polynomials are realizable as Coxeter polynomials of triangular algebras. Restricting to algebraically closed fields, already the request that $\chi_A(-1)$ is a square discards many self-reciprocal polynomials, for instance the cyclotomic polynomials $\Phi_4$, $\Phi_6$, $\Phi_8$, $\Phi_{10}$. Moreover, the polynomial $f = x^3 + 1$, which is the Coxeter polynomial of the non simply-laced Dynkin diagram $\mathbb{B}_3$, does not appear as the Coxeter polynomial of a triangular algebra over an algebraically closed field, despite of the fact that $f(-1) = 0$ is a square. Indeed, the Cartan matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

yields the Coxeter polynomial $f = x^3 + \alpha x^2 + \alpha x + 1$, where $\alpha = abc - a^2 - b^2 - c^2 + 3$. The equation $a^2 + b^2 + c^2 - abc = 3$ of Hurwitz-Markov type does not have an integral solution. (Use that reduction modulo 3 only yields the trivial solution in $\mathbb{F}_3$.)

### 2.5 Relationship with graph theory

Given a (non-oriented) graph $\Delta$, its *characteristic polynomial* $\kappa_\Delta$ is defined as the characteristic polynomial of the adjacency matrix $M_\Delta$ of $\Delta$. Observe that, since $M_\Delta$ is symmetric, all its eigenvalues are real numbers. For general results on graph theory and spectra of graphs see [4].

There are important interactions between the theory of graph spectra and the representation theory of algebras, due to the fact that if $C$ is the Cartan matrix of $A = K\left[\vec{\Delta}\right]$, then $M_\Delta$ is determined by the symmetrization $C + C^t$ of $C$, since $M_\Delta = C + C^t - 2I$. We shall see that information on the spectra of $M_\Delta$ provides fundamental insights into the spectral analysis of the Coxeter matrix $\Phi_A$ and the structure of the algebra $A$.

A fundamental fact for a hereditary algebra $A = K\left[\vec{\Delta}\right]$, when $\vec{\Delta}$ is a *bipartite quiver*, that is, every vertex is a sink or source, is that $\mathrm{Spec}(\Phi_A) \subset \mathbb{S}^1 \cup \mathbb{R}^+$. This was shown as a consequence of the following important identity.

**Proposition**. [2] *Let $A = K\left[\vec{\Delta}\right]$ be a hereditary algebra with $\vec{\Delta}$ a bipartite quiver without oriented cycles. Then $\chi_A(x^2) = x^n \kappa_\Delta(x + x^{-1})$, where $n$ is the number of vertices of $\vec{\Delta}$ and $\kappa_\Delta$ is the characteristic polynomial of the underlying graph $\Delta$ of $\vec{\Delta}$.*

*Proof.* Since $\vec{\Delta}$ is bipartite, we may assume that the first $m$ vertices are sources and the last $n - m$ vertices are sinks. Then the adjacency matrix $A$ of $\Delta$ and the Cartan matrix $C$ of $A$, in the basis of simple modules, take the form: $A = N + N^t$, $C = I_n - N$, where

$$N = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$$

for certain $m \times m$-matrix $D$. Since $N^2 = 0$, then $C^{-1} = I_n + N$. Therefore

$$\begin{aligned} \det\left(x^2 I_n - \Phi_A\right) &= \det\left(x^2 I_n + (I_n - N)(I_n + N)^t\right)\det(I_n - N^t) \\ &= \det\left(x^2 I_n - x^2 N^t + (I_n - N)\right) \\ &= x^n \det\left(\left(x + x^{-1}\right)I_n - xN^t - x^{-1}N\right) \\ &= x^n \det\left(\left(x + x^{-1}\right)I_n - A\right). \end{aligned}$$

□

The above result is important since it makes the spectral analysis of bipartite quivers and their underlying graphs almost equivalent. Note, however, that the representation theoretic context is much richer, given the categorical context behind the spectral analysis of quivers. The representation theory of bipartite quivers may thus be seen as a categorification of the class of graphs, allowing a bipartite structure.

Constructions in graph theory. Several simple constructions in graph theory provide tools to obtain in practice the characteristic polynomial of a graph. We recall two of them (see [4] for related results):

a. Assume that $a$ is a vertex in the graph $\Delta$ with a unique neighbor $b$ and $\Delta'$(resp. $\Delta''$) is the full subgraph of $\Delta$ with vertices $\Delta_0 \setminus \{a\}$ (resp. $\Delta_0 \setminus \{a, b\}$), then

$$\kappa_\Delta = x\kappa_{\Delta'} - \kappa_{\Delta''}$$

b. Let $\Delta_i$ be the graph obtained by deleting the vertex $i$ in $\Delta$. Then the first derivative of $\kappa_\Delta$ is given by

$$\kappa'_\Delta = \sum_i \kappa_{\Delta_i}$$

The above formulas can be used inductively to calculate the characteristic polynomial of trees and other graphs. They immediately imply the following result that will be used often to calculate Coxeter polynomials of algebras.

**Proposition.** *Let* $A = K\left[\vec{\Delta}\right]$ *be a bipartite hereditary algebra. The following holds:*

i. *Let a be a vertex in the graph* $\Delta$ *with a unique neighbor b. Consider the algebras B and C obtained as quotients of A modulo the ideal generated by the vertices a and a, b, respectively. Then*

$$\chi_A = (x+1)\chi_B - x\chi_C$$

ii. *The first derivative of the Coxeter polynomial satisfies:*

$$2x{\chi_A}' = n\chi_A + (x-1)\sum_i \chi_{A^{(i)}}$$

where $A^{(i)} = K\left[\vec{\Delta} \setminus \{i\}\right]$ is an algebra obtained from $A$ by 'killing' a vertex $i$.

*Proof.* Use the corresponding results for graphs and A'Campo's formula for the algebras $A$ and its quotients $A^{(i)}$. □

## 3. Important classes of algebras

In this section we give the definitions and main properties of such classes of finite dimensional algebras where information on their spectral properties is available.

### 3.1 Hereditary algebras

Let $A$ be a finite dimensional $K$-algebra. For simplicity we assume $A = K\left[\vec{\Delta}\right]/I$ for a quiver $\vec{\Delta}$ without oriented cycles and $I$ an ideal of the path algebra. The following facts about the Coxeter transformation $\Phi_A$ of $A$ are fundamental:

i. Let $S_1, \dots, S_n$ be a complete system of pairwise non-isomorphic simple $A$-modules, $P_1, \dots, P_n$ the corresponding projective covers and $I_1, \dots, I_n$ the injective envelopes. Then $\phi_A$ is the automorphism of $K_0(A)$ defined by $\Phi_A[P_i] = -[I_i]$, where $[X]$ denotes the class of a module $X$ in $K_0(A)$.

ii. For a hereditary algebra $A = K\left[\vec{\Delta}\right]$, the spectral radius $\rho_A = \rho_{\Phi_A}$ determines the representation type of $A$ in the following manner:

a. $A$ is representation-finite if $1 = \rho_A$ is not a root of the Coxeter polynomial $\chi_A$.

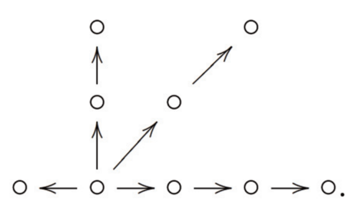
b. $A$ is tame if $1 = \rho_A \in \mathrm{Roots}(\chi_A)$.

c. $A$ is wild if $1 < \rho_A$. Moreover, if $A$ is wild connected, Ringel [20] shows that the spectral radius $\rho_A$ is a simple root of $\chi_A$. Then Perron-Frobenius theory yields a vector $y^+ \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $\Phi_A y^+ = \rho_A y^+$. Since $\chi_A$ is self reciprocal, there is a vector $y^- \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $\Phi_A y^- = \rho_A^{-1} y^-$. The vectors $y^+, y^-$ play an important role in the representation theory of $A = K\left[\vec{\Delta}\right]$, see [5, 17].

***Explicit formulas, special values***. We are discussing various instances where an explicit formula for the Coxeter polynomial is known.

*star quivers*. Let $A$ be the path algebra of a hereditary star $[p_1, \dots, p_t]$ with respect to the standard orientation, see

$[2, 3, 3, 4]$ :



Since the Coxeter polynomial $\chi_A$ does not depend on the orientation of $A$ we will denote it by $\chi_{[p_1,\dots,p_t]}$. It follows from [11, prop. 9.1] or [2] that

$$\chi_{[p_1,\dots,p_t]} = \prod_{i=1}^{t} v_{p_i} \left( (x+1) - x \sum_{i=1}^{t} \frac{v_{p_i - 1}}{v_{p_i}} \right). \tag{1}$$

In particular, we have an explicit formula for the sum of coefficients of $\chi = \chi_{[p_1,\dots,p_t]}$ as follows:

$$\chi(1) = \prod_{i=1}^{t} p_i \left( 2 - \sum_{i=1}^{t} \left( 1 - \frac{1}{p_i} \right) \right). \tag{2}$$

This special value of $\chi$ has a specific mathematical meaning: up to the factor $\prod_{i=1}^{t} p_i$ this is just the orbifold-Euler characteristic of a weighted projective line $\mathbb{X}$ of weight type $(p_1, \dots, p_t)$. Moreover,

1. $\chi(1) > 0$ if and only if the star $[p_1, \dots, p_t]$ is of Dynkin type, correspondingly the algebra $A$ is representation-finite.

2. $\chi(1) = 0$ if and only if the star $[p_1, ..., p_t]$ is of extended Dynkin type, correspondingly the algebra $A$ is of tame (domestic) type.

3. $\chi(1) < 0$ if and only if $[p_1, ..., p_t]$ is not Dynkin or extended Dynkin, correspondingly the algebra $A$ is of wild representation type.

The above deals with all the Dynkin types and with the extended Dynkin diagrams of type $\tilde{\mathbb{D}}_n$, $n \geq 4$, and $\tilde{\mathbb{E}}_n$, $n = 6, 7, 8$. To complete the picture, we also consider the extended Dynkin quivers of type $\tilde{\mathbb{A}}_n$ $(n \geq 2)$ restricting, of course, to quivers without oriented cycles. Here, the Coxeter polynomial depends on the orientation: If $p$ (resp. $q$) denotes the number of arrows in clockwise (resp. anti-clockwise) orientation $(p, q \geq 1, p + q = n + 1)$, that is, the quiver has type $\mathbb{A}(p, q)$, the Coxeter polynomial $\chi$ is given by

$$\chi_{(p,q)} = (x-1)^2 v_p v_q. \tag{3}$$

Hence $\chi(1) = 0$, fitting into the above picture.
The next table displays the $v$-factorization of extended Dynkin quivers.

| Extended Dynkin type | Star symbol | Weight symbol | Coxeter polynomial |
|---|---|---|---|
| $\tilde{\mathbb{A}}_{p,q}$ | — | $(p, q)$ | $(x-1)^2 v_p v_q$ |
| $\tilde{\mathbb{D}}_n$, $n \geq 4$ | [2,2,n-2] | $(2, 2, n-2)$ | $(x-1)^2 v_2^2 v_{n-2}$ |
| $\tilde{\mathbb{E}}_6$ | [3, 3, 3] | $(2, 3, 3)$ | $(x-1)^2 v_2 v_3^2$ |
| $\tilde{\mathbb{E}}_7$ | [2, 4, 4] | $(2, 3, 4)$ | $(x-1)^2 v_2 v_3 v_4$ |
| $\tilde{\mathbb{E}}_8$ | [2, 3, 6] | $(2, 3, 5)$ | $(x-1)^2 v_2 v_3 v_5$ |

**Remark:** As is shown by the above table, proposition 2.3 extends to the tame hereditary case. That is, the Coxeter polynomial of a connected, tame hereditary $K$-algebra $A$ (remember, $K$ is algebraically closed) determines the algebra $A$ up to derived equivalence. This is no longer true for wild hereditary algebras, not even for trees.

### 3.2 Canonical algebras

Canonical algebras were introduced by Ringel [19]. They form a key class to study important features of representation theory. In the form of tubular canonical algebras they provide the standard examples of tame algebras of linear growth. Up to tilting canonical algebras are characterized as the connected $K$-algebras with a separating exact subcategory or a separating tubular one-parameter family (see [12]). That is, the module category mod $- \Lambda$ accepts a *separating tubular family* $\mathcal{T} = (T_\lambda)_{\lambda \in P_1 K}$, where $T_\lambda$ is a homogeneous tube for all $\lambda$ with the exception of $t$ tubes $T_{\lambda_1}, ..., T_{\lambda_t}$ with $T_{\lambda_i}$ of rank $p_i$ $(1 \leq i \leq t)$.

Canonical algebras constitute an instance, where the explicit form of the Coxeter polynomial is known, see [11] or [10].

**Proposition.** *Let $\Lambda$ be a canonical algebra with weight and parameter data $(\boldsymbol{p}, \boldsymbol{\lambda})$. Then the Coxeter polynomial of $\Lambda$ is given by*

$$\chi_\Lambda = (x-1)^2 \prod_{i=1}^{t} v_{p_i}. \tag{4} \square$$

The Coxeter polynomial therefore only depends on the weight sequence $\boldsymbol{p}$. Conversely, the Coxeter polynomial determines the weight sequence — up to ordering.

### 3.3 Incidence algebras of posets

Let $X$ be a finite partially ordered set (poset). The incidence algebra $KX$ is the $K$-algebra spanned by elements $e_{xy}$ for the pairs $x \leq y$ in $X$, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$. Finite dimensional right modules over $KX$ can be identified with commutative diagrams of finite dimensional $K$-vector spaces over the Hasse diagram of $X$, which is the directed graph whose vertices are the points of $X$, with an arrow from $x$ to $y$ if $x < y$ and there is no $z \in X$ with $x < z < y$.

We recollect the basic facts on the Euler form of posets and refer the reader to [6] for details. The algebra $KX$ is of finite global dimension, hence its Euler form is well-defined and non-degenerate. Denote by $C_X$, $\Phi_X$ the matrices of the bilinear form and the corresponding Coxeter transformation with respect to the basis of the simple $KX$-modules.

The incidence matrix of $X$, denoted $1_X$, is the $X \times X$ matrix defined by $(1_X)_{xy} = 1$ if $x \leq y$ and otherwise $(1_X)_{xy} = 0$. By extending the partial order on $X$ to a linear order, we can always arrange the elements of $X$ such that the incidence matrix is uni-triangular. In particular, $1_X$ is invertible over $\mathbb{Z}$. Recall that the Möbius function $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x,y) = (1_X)^{-1}_{xy}$.

**Lemma.** *a.* $C_X = 1_X^{-1}$.

*b. Let* $x, y \in X$. *Then* $(\Phi_X)_{xy} = -\sum_{z: z \geq x} \mu_X(y,z)$.

**Proposition.** *If* $X$ *and* $Y$ *are posets, then* $C_{X\times Y} = C_X \otimes C_Y$ *and* $\Phi_{X\times Y} = -\Phi_X \otimes \Phi_Y$.

## 4. Cyclotomic polynomials and polynomials of Littlewood type

### 4.1 Cyclotomic polynomials

We recall some facts about *cyclotomic polynomials*.

The $n$-cyclotomic polynomial $\Phi_n(T)$ is inductively defined by the formula

$$T^n - 1 = \prod_{d|n} \Phi_d(T). \tag{5}$$

The *Möbius function* is defined as follows:

$$\mu(n) = \begin{pmatrix} 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^r & \text{if } n = p_1, \dots p_r \text{ is a factorization into distinct primes.} \end{pmatrix}$$

A more explicit expression for the cyclotomic polynomials is given by

$$\Phi_n(T) = \prod_{\substack{1 \leq d < n \\ d|n}} v_{n/d}(T)^{\mu(d)} \tag{6}$$

for $n \geq 2$, where $v_n = 1 + T + T^2 + \dots + T^{n-1}$.

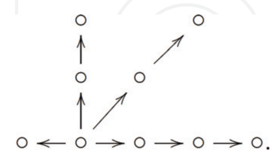
### 4.2 Hereditary stars

A *path algebra* $K\Delta$ is said to be of *Dynkin type* if the underlying graph $|\Delta|$ of $\Delta$ is one of the *ADE-series*, that is, of type, $\mathbb{A}_n$, $\mathbb{D}_n$, for some $n \geq 1$ or $\mathbb{E}_k$, for $k = 6, 7, 8$.

There are various instances where an explicit formula for the Coxeter polynomial is known.

Let $A$ be the path algebra of a hereditary star $[p_1, \ldots, p_t]$ with respect to the standard orientation, see [13].

$[2, 3, 3, 4]$ :



Since the Coxeter polynomial $\chi_A$ does not depend on the orientation of $A$ we will denote it by $\chi_{[p_1,\ldots,p_t]}$. It follows that

$$\chi_{[p_1,\ldots,p_t]} = \prod_{i=1}^{t} v_{p_i} \left( (T+1) - T \sum_{j=1}^{t} \frac{v_{p_j-1}}{v_{p_j}} \right).$$

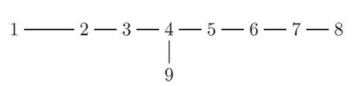
In particular, we have an explicit formula for the sum of coefficients of $\chi_{[p_1,\ldots,p_t]}$ as follows:

$$\sum_{i=0}^{n} a_i = \chi_{[p_1,\ldots,p_t]}(1) = \prod_{i=1}^{t} p_i \left( 2 - \sum_{i=1}^{t} \left( 1 - \frac{1}{p_i} \right) \right).$$

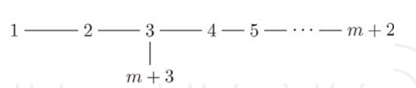
### 4.3 Wild algebras

Let $c$ be the real root of the polynomial $T^3 - T - 1$, approximately $c = 1.325$. As observed in [21], a wild hereditary algebra $A$ associated to a graph $\Delta$ without multiple arrows has spectral radius $\rho_A > c$ unless $\Delta$ is one of the following graphs:

$[2, 4, 5]$ :



$[2, 3, m]$ :



In these cases, for $m \geq 8$

$$c > \rho_{[2,4,5]} > \rho_{[2,3,m]} > \rho_{[2,3,7]} = \mu_0$$

where $\mu_0 = 1.176280\ldots$ is the real root of the Coxeter polynomial

$$T^{10} + T^9 - T^7 - T^6 - T^5 - T^4 - T^3 + T + 1$$

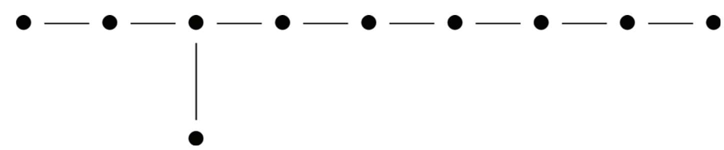
associated to any hereditary algebra whose underlying graph is $[2, 3, 7]$. Observe that in these cases, the Mahler measure of the algebra equals the spectral radius.

### 4.4 Lehmer polynomial

In 1933, D. H. Lehmer found that the polynomial

$$T^{10} + T^9 - T^7 - T^6 - T^5 - T^4 - T^3 + T + 1$$

has Mahler measure $\mu_0 = 1.176280...$, and he asked if there exist any smaller values exceeding 1. In fact, the polynomial above is the Coxeter polynomial of the hereditary algebra whose underlying graph $[2, 3, 7]$ is depicted below.



We say that a matrix $M$ is of *Mahler type* (resp. *strictly Mahler type*) if either $\mathbb{M}(M) = 1$ or $\mathbb{M}(M) \geq \mu_0$ (resp. $\mathbb{M}(M) > \mu_0$). Earlier this year, Jean-Louis Verger-Gaugry announced a proof of Lehmer's conjecture, see https://arxiv.org/pdf/1709.03771.pdf. The key result (Theorem 5.28, p. 122) is a Dobrowolski type minoration of the Mahler Measure $\mathbb{M}(\beta)$. Experts are still reading the arguments, but there is no conclusive opinion.

### 4.5 Happel's trace formula

In [8], Happel shows that the trace of the Coxeter matrix can be expressed as follows:

$$-\mathrm{Tr}(\phi_A) = \sum_{k=0}^{\infty} (-1)^k \dim_K H^k(A) \tag{7}$$

where $H^k(A)$ denotes the $k$-th Hochschild cohomology group. In particular, if the Hochschild cohomology ring $H^*(A)$ is trivial, that is, $H^i(A) = 0$ for $i > 0$ and $H^0(A) = K$, then $\mathrm{Tr}(\phi_A) = -1$.

For an algebra $A$ and a left $A$-module $N$ we call

$$A[N] = \begin{bmatrix} A & 0 \\ N & K \end{bmatrix}$$

the *one-point extension* of $A$ by $N$. This construction provides an order of vertices to deal with *triangular algebras*, that is, algebras $KQ/I$, where $I$ is an ideal of the path algebra $KQ$ for $Q$ a quiver without oriented cycles.

### 4.6 One-point extensions

Let $B$ be an algebra and $M$ a $B$-module. Consider the one-point extension $A = B[N]$. In [19] it is shown the Coxeter transformations of $A$ and $B$ are related by

$$\phi_A = \begin{pmatrix} \phi_B & -C_B^T n^T \\ -n\phi_B & nC_B^T n^T - 1 \end{pmatrix} \tag{8}$$

where $C_B$ is the *Cartan matrix* of $B$ which satisfies $\phi_B = -C_B^{-T} C_B$ and $n$ is the class of $N$ in the Grothendieck group $K_0(B)$. In case $A = B[N]$ with $N$ an *exceptional* module, it follows that

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B)$$

We recall that the *Euler quadratic form* is defined as $q_A(x) = xC_A^t x^t$. Assume that $A = B[M]$ for an algebra $B$ and an indecomposable module $M$. In many cases, we get that $q_A(m) > 0$, for $m$ the dimension vector of $M$ (for instance, if $M$ is preprojective, or if $q_A$ coincides with the Tits form of $A$...)

**Proposition.** *Let $A$ be an accessible algebra, such that $q_A(m) > 0$ for $m$ the dimension vector of $M$, where $A = B[M]$ for certain algebra $B$ and an indecomposable module $M$. Then the following happens:*

*a.* $\mathrm{Tr}(\phi_A) \geq -1$;

*b.* if $\mathrm{Tr}(\phi_B) = -1$ and $q_B(m) = 1$, then $\mathrm{Tr}(\phi_A) = -1$.

*Proof.* Assume that $A = B[M]$ for an algebra $B$ and an indecomposable module $M$ such that $q_A(m) > 0$ for $m$ the dimension vector of $M$. Then $B$ is also accessible. By induction hypothesis, $\mathrm{Tr}(\phi_B) \geq -1$. Then

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B) + \left(mC_B^T m^T - 1\right) \geq -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) \geq -1$$

This shows (a).

For (b) assume that $\mathrm{Tr}(\phi_B) = -1$ and $q_B(m) = 1$, then

$$Tr(\phi_A) = Tr(\phi_B) + \left(mC_B^T m^T - 1\right) = -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) = -1$$

□

### 4.7 Strongly accessible algebras

**Theorem:** *A finite dimensional accessible algebra $A$ then it is strongly accessible if and only if* $\mathrm{Tr}(\phi_A) = -1$.

*Proof.* Assume $A$ is strongly accessible from $A_0$. Since $q_A(m) \geq 1$, for $A = B[M]$ a one-point extension of the subcategory $B$ of $A$ by the exceptional module $M$ (since then $q_A(m) = \dim_K \mathrm{End}_A(\mathrm{M})$). By the Proposition above

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}\left(\phi_{A_{n-1}}\right) = \ldots = \mathrm{Tr}\left(\phi_{A_0}\right) = -1$$

Conversely, assume that $\mathrm{Tr}(\phi_A) = -1$ and write $A = B[M]$ as a one-point extension of the subcategory $B$ of $A$ by the module $M$. We shall prove that $M$ is exceptional.

$$-1 = \mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B) + \left(mC_B^T m^T - 1\right) \geq -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) \geq -1$$

Equality holds and $q_B(m) = 1$, since $M$ is indecomposable, it follows that the extension ring of $M$ is trivial. □

### 4.8 Stable matrices

The following statement is Theorem 1 for stable matrices.

**Proposition.** *Suppose $M$ is a stable unimodular $n \times n$-matrix. Let $\chi_M = c_0 + c_1 T + c_2 T^2 + \ldots + c_{n-2} T^{n-2} + c_{n-1} T^{n-1} + c_n T^n$ be its characteristic polynomial.*

*Suppose that $0 < \mathrm{Tr} M^k \leq m$ for $p \leq k \leq p + n - 1$ and certain integers $1 \leq p$ and $m$.*

*Then $0 < \mathrm{Tr} M^k \leq m$ for all integers $p \leq k$.*

*In particular, $M$ is of cyclotomic type.*

*Proof.* Consider the coefficients $c_0, c_1, \ldots c_n$ of $\chi_M$. Since $M$ is stable then $c_n = 1, c_{n-1} < 0, c_{n-2} > 0$ and the signs alternate until we meet a $j$ with $c_j c_0 < 0$. Cayley-Hamilton theorem states that $\chi_M(M) = 0$. Then

$$0 = c_0 1_n + c_1 M + c_2 M^2 + \dots + c_{n-1} M^{n-1} + c_n M^n$$

Then

$$c_0 1_n + c_2 M^2 + \dots + c_{2m} M^{2m} = c_1 M + c_3 M^3 + \dots + c_{2m-1} M^{2m-1} + c_{2(m+r)-1} M^{2(m+r)-1}$$

Let $c > 0$ be the common value of the trace of this matrix.

Write $n = 2m + r$ for $r = 0$ or 1. Consider the matrices

$$P = \frac{1}{c}\left(c_0 1_n + c_2 M^2 + \dots + c_{2m} M^{2m}\right)$$
$$Q = -\frac{1}{c}\left(\left(c_1 M + c_3 M^3 + \dots + c_{2m-1} M^{2m-1} + c_{2(m+r)-1} M^{2(m+r)-1}\right)\right)$$

so that we get two expressions of $P$ as positive linear combinations of powers of $M$.

Suppose that $n = 2m + 1$. By hypothesis we have Tr $(P) \leq n$. Moreover, since $c_n = 1$ then

$$\mathrm{Tr}(M^n) \leq \mathrm{Tr}(Q) = \mathrm{Tr}(P) \leq n$$

The claim follows by induction.

Otherwise, $n = 2m$. The claim follows similarly. □

### 4.9 Theorem 1

*Proof of Theorem 1.* Observe that $M = \phi_A$ is a real unimodular matrix. One implication of the Theorem was shown before. Suppose that $|\mathrm{Tr}\left(M^k\right)| \leq n$ or equivalently, $-n \leq \mathrm{Tr}\left(M^k\right) \leq n$ for $0 \leq k \leq n$. The Proposition above yields that $M$ is cyclotomic. □

### 4.10 Polynomials of Littlewood type

An integral self-reciprocal polynomial $p(t) = p_0 + p_1 t + \dots + p_{n-1} t^{n-1} + p_n t^n$ is of Littlewood type if every coefficient non-zero $p_i$ has modulus 1. A polynomial $p(t)$ of Littlewood type with all $p_i \neq 0$, for $i = 0, 1, \dots, n$, is said to be Littlewood.

**Lemma.** *If z is a root of a polynomial of Littlewood type, then*

$$1/2 < |z| < 2$$

*Proof.* Suppose $z$ is a root of a polynomial of Littlewood type. Then

$$1 = \epsilon_1 z + \epsilon_2 z^2 + \dots + \epsilon_n z^n$$

for some $\epsilon_i \in \{-1, 0, 1\}$.

If $|z| < 1$ then $1 \leq |z| + |z|^2 + \dots + |z|^n < |z|/(1 - |z|)$ so $|z| > 1/2$. Since $z$ is the root of a polynomial of Littlewood type if and only if $z^{-1}$ is, then $1/2 < |z| < 2$.

Moreover, if $|z| > 1$, then $1/|z| < 1$ and $1/2 < 1/|z| < 2$. Hence $1/2 < |z| < 2$. □

### 4.11 Littlewood series

**Definition.** A *Littlewood series* is a power series all of whose coefficients are 1, 0 or $-1$.

Let $P = \{z \in \mathbb{C} : z$ is the root of some Littlewood polynomial $\}$.

**Remarks:**

*a.* Littlewood series converge for $|z| < 1$.

*b.* A point $z \in \mathbb{C}$ with $|z| < 1$ lies in $P$ if and only if some Littlewood series vanishes at this point.

*c.* A Littlewood polynomial is not a Littlewood series. But any Littlewood polynomial, say $p(z) = a_0 + \ldots + a_d z^d$ yields a Littlewood series having the same roots $z$ with $|z| < 1$: indeed, consider the series

$$P(z) = p(z)/\left(1 - z^{d+1}\right) = a_0 + \ldots + a_d z^d + a_0 z^{d+1} + \ldots + a_d z^{2d+1} + a_0 z^{2d+2} + \ldots$$

Thus $P \subset R$, where $R$ is the set of roots of Littlewood series. We shall show the Proposition at the Introduction.

*Proof.* Let $\mathcal{L}$ be the set of Littlewood series. Then $\mathcal{L} = \{-1, 0, 1\}^{\mathbb{N}}$, so with the product topology it is homeomorphic to the Cantor set. Choose $0 < r < 1$. Let $\mathbb{F}$ be the space of finite multisets of points $z$ with $|z| < r$, modulo the equivalence relation generated by $S \cong S \cup X$ when $|X| = r$ .

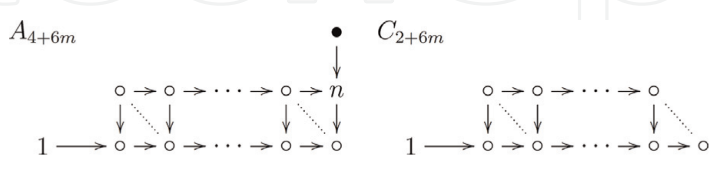
**Claim**. Any Littlewood series has finitely many roots in the disc $|z| \leq r$. The map $f : \mathcal{L} \to \mathbb{F}$ sending a Littlewood series to its multiset of roots in this disc is continuous.

Since $\mathcal{L}$ is compact, the image of $f$ is closed. From this we can show that $R$, the set of roots of Littlewood series, is closed. Since Littlewood polynomials are densely included in $\mathcal{L}$ and $f$ is continuous, we get that $P$, the set of roots of Littlewood polynomials, is dense in $R$. It follows that $\overline{P} = R$, as we wanted to show. □

## 5. An example

### 5.1 Construction

For $m$ a natural number and let $n = 3 + 6m$. Let $R_n$ be an algebra formed by $n$ commutative squares. Consider the one-point extension $A_m = R_n[P_n]$ with $P_n$ the unique indecomposable projective $R_n$-module of $K$-dimension 2. Observe that $A_m$ (resp. $C_{n-1}$) is given by the following quiver with $n + 1$ vertices and commutative relations (resp. $n - 1$ vertices and relations):




We claim:

a. $\chi_{A_m} = T^n + T^{n-1} - T^3 \chi_{A_{m-1}} + T + 1$, for all $n \geq 1$. As consequence, the algebras $A_m$ and $C_n$ are of Littlewood type;

b. the number of eigenvalues of $\phi_{A_m}$ not lying in the unit disk is at least $m$;

c. $\mathbb{M}\left(\chi_{A_m}\right) \leq 8$.

*Proof.* (a): Consider $m \geq 1$, $n = 3 + 6m$ and the algebra $B_n = R_{3+6m}$ such that $A_m = B_n[P_n]$ and the perpendicular category $P_n^{\perp}$ in $D^b(B_n)$ is derived equivalent to $\mathrm{mod}(C_{n-1})$ where $C_{n-1}$ is a proper quotient of an algebra derived equivalent to $R_{2+6m}$. Therefore

$$\begin{aligned}\chi_{A_{m+1}} &= (T+1)\chi_{R_{n+6}} - T\chi_{C_{n+5}} \\ &= (T+1)\left(T^{n+6} + T^{n+5} + T + 1\right) - T^3(T+1)\chi_{R_n} - T\chi_{C_{n+5}}\end{aligned}$$

We shall calculate $\chi_{C_{2+6m}}$. Observe that $C_{2+6m}$ is tilting equivalent to the one-point extension $R_{1+6m}[P_1]$. Hence

$$\begin{aligned}\chi_{C_{2+6m}} &= (T+1)\chi_{R_{1+6m}} - T\chi_{R_{6m}} = T^{2+6m} + T^{1+6m} - T^3\left\{(T+1)\chi_{R_{1+6(m-1)}} - T\chi_{R_{6(m-1)}}\right\} \\ &\quad + T + 1 = T^{2+6m} + T^{1+6m} - T^3\chi_{C_{2+6(m-1)}} + T + 1\end{aligned}$$

which implies

$$\begin{aligned}\chi_{A_{m+1}} &= (T+1)\left(T^{n+6} + T^{n+5} + T + 1\right) - T^3(T+1)\chi_{R_n} - T\left(T^{n+5} + T^{n+4} + T + 1\right) \\ &\quad - T^3 T\chi_{C_{n-1}} = T^{n+7} + T^{n+6} - T^3\chi_{A_m} + T + 1\end{aligned}$$

as claimed.

As consequence of formula (a) we observe the following:

(a′) $L\left(\chi_{A_m}\right) = 4m + 5$.

(b) By induction, we shall construct polynomials $r_m$ representing $\chi_{A_m}$.

For $m = 0$, we have $\chi_{A_0} = T^4 + T^3 + T^2 + T + 1$, which is represented by the polynomial $r_0 = T^4 - 3T^2 + 1$.

Observe that $\left(T^{n-1} + 1\right) = v_n - Tv_{n-2}$ then $T^n + T^{n-1} + T + 1 = (T+1)\left(T^{n-1} + 1\right)$ is represented by $w_n = T(u_{n-1} - u_{n-3})$.

For $n = 4 + 6m$, we define $r_m = w_n - T^3 r_{m-1}$. We verify by induction on $m$ that $r_m$ represents $\chi_{A_m}$:

$$\begin{aligned}\chi_{A_m}\left(T^2\right) &= \left(T^2 + 1\right)\left(T^{2n-2} + 1\right) - T^6\chi_{A_{m-1}}\left(T^2\right) \\ &= T^n w_n\left(T + T^{-1}\right) - T^6 T^{n-6} r_{m-1}\left(T + T^{-1}\right) = T^n r_m\left(T + T^{-1}\right)\end{aligned}$$

For instance.

$$\begin{aligned}r_1 &= w_{10} - T^3 r_0 = T\left\{\left(T^9 - 8T^7 + 21T^5 - 20T^3 + 5T\right) - \left(T^7 - 6T^5 + 10T^3 - 4T\right)\right\} \\ &\quad - T^3\left\{T^4 - 3T^2 + 1\right\} \\ &= T^{10} - 9T^8 - T^7 + 27T^6 + 3T^5 - 30T^4 - T^3 + 9T^2\end{aligned}$$

which has $\xi(r_1) = 4$ changes of sign in the sequence of coefficients. According to *Descartes rule of signs*, $r_1$ has at most $\xi(r_1) = 4$ positive real roots. Since $r_1$ represents $\chi_{A_1}$, then $\chi_{A_1}$ has at most $2\xi(r_1) = 8$ roots in the unit circle. That is, $\chi_{A_1}$ has at least 2 roots $z$ with $|z| \neq 1$.

We shall prove, by induction, that $r_m$ has at most $\xi(r_m) = 2(m+1)$ positive real roots. Indeed, write

$$r_m = T^n - (n-1)T^{n-2} - T^3 q_m + (n-1)T^2$$

for some polynomial $q_m$ of degree $n-6$ with signs of its coefficients $+--++--\cdots\pm$ so that $\xi(q_m) = 2m$. Then

$$r_{m+1} = w_{n+6} - T^3 r_m = Tu_{n+5} - Tu_{n+3} - T^3 r_m$$

an addition of three polynomials with signs of coefficients given as follows:

$$\begin{array}{ccccccccccc} + & 0 & - & 0 & + & 0 & - & 0 & \cdots & + & 0 & 0 \\ & & - & 0 & + & 0 & - & 0 & \cdots & + & 0 & 0 \\ & & & - & + & + & - & - & \cdots & 0 & 0 & 0 \end{array}$$

Hence $r_{m+1} = T^{n+6} - (n+5)T^{n+4} - T^3 q_{m+1} + (n+5)T^2$ where the polynomial $q_{m+1}$ of degree $n$ has signs of its coefficients $+--++--\cdots\pm$ so that $\xi(q_{m+1}) = \xi(q_m) + 2 = 2(m+1)$. Hence $\xi(r_m) = 2 + \xi(q_m) = 2(m+1)$.

By the Lemma below, $\chi_{A_m}$ has at most $4(m+1)$ roots in the unit circle. Equivalently, $\chi_{A_m}$ has at least $4 + 6m - 4(m+1) = 2m$ roots outside the unit circle. Hence $\chi_{A_m}$ has at least $m$ roots $z$ satisfying $|z| > 1$.

**Lemma.** *Let $q$ be a polynomial representing the polynomial $p$. Assume $q$ accepts at most $s$ positive real roots, then $p$ has at most $2s$ roots in the unit circle.*

*Proof.* Let $\mu_1, \ldots, \mu_s$ be the positive real roots of $q$. Let $z = a + ib$ be a root of $p$ with $a^2 + b^2 = 1$. Consider $w = c + id$ a complex number with $w^2 = z$. Then $0 = p(z) = w^n q(w + w^{-1})$ where $w + w^{-1} = (c+id) + (c-id) = 2c$. Then $2c = \epsilon\lambda_j$ for some $\epsilon \in \{1, -1\}$ and $1 \le j \le s$. Hence

$$z = w^2 = \left(\frac{1}{2}\lambda_j^2 - 1\right) + i\left(2\epsilon\lambda_j\sqrt{1-\lambda_j^2}\right)$$

can be selected in two different ways. □

(c) For $n = 6m + 4$ we have $\chi_{A_m} = T^n + T^{n-1} - T^3\chi_{A_{m-1}} + T + 1$. Then

$$\chi_{A_m} = \xi_m + (-1)^{m-1}T^{2m+4}\chi_{10}, \text{ where } \xi_m = T^n + T^{n-1} - T^3\xi_{m-1} + T + 1$$

for $m \ge 2$ and $\xi_1 = 0$.

We observe that $\xi_m$ is a product of cyclotomic polynomials. Indeed, since $\xi_m(-1) = 0$ we can write

$$\xi_m = (T+1)\sigma_m \text{ and } \sigma_m = T^{n-1} - T^3\sigma_{m-1} + 1$$

for $m \ge 2$ and $\sigma_1 = 0$.

Recall $\Phi_{2^s-1} = T^{s-1} + T^{s-2} + \ldots T + 1$ and $\Phi_{2s}(T) = \Phi_s(-T)$. Moreover, $\Phi_{3p}(T) = \Phi_p(T^3)$, if $p$ is a power of 2. Altogether this yields

$$\begin{aligned}\Phi_{6\left(2^{2(m+1)}-1\right)}(T) &= \Phi_{2\left(2^{2(m+1)}-1\right)}\left(T^3\right) = \Phi_{2^{2(m+1)}-1}\left(-T^3\right) \\ &= T^{6m+3} - T^{6m} + \ldots - T^3 + 1 = \sigma_m\end{aligned}$$

hence

$$\xi_m = \Phi_2\Phi_{6\left(2^{2(m+1)}-1\right)}$$

confirming the claim.

We estimate the Mahler measure of $\chi_{A_m} = \xi_m + (-1)^{m-1}T^{2m+4}\chi_{A_{10}}$. Write $\chi_{A_m} = f_m + g_m$, where $f_m$ is the cyclotomic summand. Observe that $L(g_m) = L(\chi_{A_{10}}) = 8$ and apply Lemma (3.4) with $\mathbb{M}(f_m) = 1$ to get

$$\mathbb{M}(\chi_{A_m}) \leq \mathbb{M}(f_m)L(g_m) = 8$$

With the help of computer programs we calculate more accurate values of the Mahler measure of some of the above examples:

| No. vertices | No. roots outside unit disk | Mahler measure |
|---|---|---|
| 178 | 29 | 1.28368024451292 |
| 184 | 30 | 1.28327850483340 |
| 190 | 31 | 1.28386917621114 |
| 196 | 32 | 1.28395305512596 |

Comparing with the list of *Record Mahler measures by roots outside the unit circle* in Mossinghoff's web page we see:

i. for the entry 29 the Mahler measure is the same in both tables;

ii. the entries 30 and 31 have a smaller Mahler measure in our table, establishing new records;

iii. the entry 32 of our table seems to be new. Further entries could be calculated.

## 6. Coefficients of Coxeter polynomials

### 6.1 Derived tubular algebras

There are interesting invariants associated to the Coxeter polynomial of a triangular algebra $A = k[\Delta]/I$. For instance, the evaluation of the Coxeter polynomial $\chi_A(-1) = m^2$ for some integer $m$. Clearly, this number is a derived invariant. A simple argument yields that $m = 0$ in case $\Delta$ has an odd number of vertices. In [14], it was shown that for a representation-finite accessible algebra $A$ with gl.dim $A \leq 2$ the invariant $\chi_A(-1)$ equals zero or one. The criterion was applied to show that a canonical algebra is derived equivalent to a representation-finite algebra if and only if it has weight type $(2, p, p+k)$, where $p \geq 2$ and $k \geq 0$. In particular, the tubular canonical algebra of type $(3, 3, 3)$ is not derived equivalent to a representation-finite algebra, while the tubular algebras of type $(2, 4, 4)$ or $(2, 3, 6)$ are.

### 6.2 Strong towers

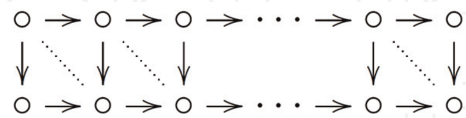
Recall from [14] that a *strong tower* $\mathbb{T} = (A_0 = k, A_1, \ldots, A_n = A)$ of access to $A$ satisfies that $A_{i+1} = A_i[M_i]$ or $[M_i]A_i$ for some exceptional module $M_i$ in such a way that, in case $A_{i+1} = A_i[M_i]$ (resp. $A_{i+1} = [M_i]A_i$), the perpendicular category $\underline{M_i}^{\perp}$ (resp. ${}^{\perp}\underline{M_i}$) of $M_i$ in $\mathrm{mod}_{A_i}$ is equivalent to $\mathrm{mod}_{C_{i-1}}$ for some accessible algebra $C_{i-1}$, $i = 1, \ldots, n-1$. In the extension situation the perpendicular category $M_i^{\perp}$ (resp. ${}^{\perp}M_i$ in the coextension situation) in $\mathrm{D}^b(\mathrm{mod}_{A_i})$ is equivalent to $\mathrm{D}^b(\mathrm{mod}_{C_{i-1}})$ and $B_i$is derived equivalent to a one-point (co-)extension of $C_{i-1}$. An algebra $C_i$ as above is called an *i-th perpendicular restriction* of the tower $\mathbb{T}$, observe that it is well-defined only up to derived equivalence. We denote by $s_i$ the number of connected components of the algebra $C_i$; in particular, $s_1 = 1$.

There are many examples of *strongly accessible algebras*, that is, algebras derived equivalent to algebras with a strong tower of access. The following are some instances:

a. A canonical algebra $C$ of weight $(p_1, \ldots, p_t)$ is strongly accessible if and only if $t = 3$, in that case, $C$ is derived-equivalent to a representation-finite algebra if and only if the weight type does not dominate $(3, 3, 3)$.

b. The following sequence of poset algebras defines strong towers of access:

$B_n$



### 6.3 Towering numbers

Consider a strong tower $\mathbb{T} = (A_0 = k, A_1, \ldots, A_n = A)$ of access to $A$ such that $A_{i+1}$ is an one-point (co)extension of $A_i$ by $M_i$ and $C_{i-1}$ the corresponding i-th perpendicular restriction of $\mathbb{T}$. Let $C_{i-1}$ have $s_{i-1}$ connected components, $i = 2, \ldots, n-1$. Define the *first towering number of* $\mathbb{T}$ as the sum $s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$.

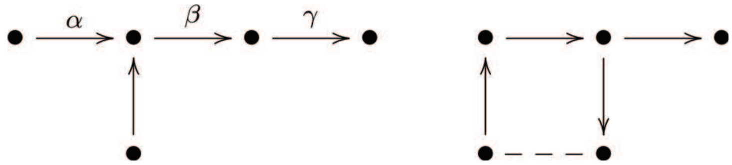
**Theorem**. *Let $A$ be a strongly accessible algebra with $n$ vertices, then the first towering number $s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$ of $\mathbb{T}$ is a derived invariant, that is, depends only on the derived class of $A$. It is $s_{\mathbb{T}}(A) = n - 1 - a_2$, where $a_2$ is the coefficient of the quadratic term in the Coxeter polynomial of $A$.*

*Proof.* Assume $A = A_n$ and $B = A_{n-1}$ such that $A = B[M]$ for $M$ an exceptional $B$-module and let $C = C_{n-2}$ be the algebra such that $\text{mod}_C$ is derived equivalent to the perpendicular category $M^{\perp}$ formed in $D^b(\text{mod}_B)$. Then $\chi_A(t) = (1+t)\chi_B(t) - t\chi_C(t)$. Write $\chi_B(t) = 1 + t + \sum_{i=2}^{n-3} b_i t^i + t^{n-2} + t^{n-1}$ and $\chi_C(t) = 1 + \sum_{i=1}^{n-3} c_i t^i + t^{n-2}$. By induction hypothesis we may assume that $s(B) = n - 2 - b_2$. Then $a_2 = b_2 + 1 - c_1$. Moreover, since $C$ is a direct sum accessible algebras, then $c_1 = \sum_{i=0}^{n-2} (-1)^i \dim_k H^i(C) = \dim_k H^0(C) = s_{n-2}$. Hence $a_2 = n - 1 - s(B) - s_{n-2} = n - 1 - s(A)$. □

**Corollary.** *Let $\mathbb{T} = (A_1 = k, \ldots, A_n = A)$ be a strong tower of access to $A$. Let $A = B[M]$ for $B = A_{n-1}$ with $M$ exceptional and $C$ a perpendicular restriction of $B$ via $M$. Consider the Coxeter polynomials $\chi_A(t) = 1 + t + a_2 t^2 + \ldots + a_{n-2} t^{n-2} + t^{n-1} + t^n$ and $\chi_B(t) = 1 + t + b_2 t^2 + \ldots + b_{n-3} t^{n-3} + t^{n-2} + t^{n-1}$, then $a_2 \leq b_2$, with equality if and only if $C$ is connected. In particular, $a_2 \leq 1$.*

*Proof.* First recall that for a connected accessible algebra the linear term of the Coxeter polynomial has coefficient 1. Let $\chi_C(t) = 1 + c_1 t + c_2 t^2 + \ldots + c_{n-4} t^{n-4} + c_{n-3} t^{n-3} + t^{n-2}$ be the Coxeter polynomial of $C$. If $C$ is the direct sum of connected accessible algebras $C_1, \ldots, C_s$, then $c_1 = s$. Therefore, $a_2 = b_2 + b_1 - c_1 = b_2 - (s-1) \leq b_2$. By induction hypothesis, we get $a_2 \leq 1$. □

Let $A$ be the algebra given by the following quiver with relation $\gamma\beta\alpha = 0$:



which is derived equivalent to the quiver algebra $B$ with the zero relation as depicted in the second diagram. Clearly, $A = A'[M]$, where $A'$ is a quiver algebra of type $\mathbb{A}_4$ and $M$ is an indecomposable module with $M^{\perp}$ the category of modules of the disconnected quiver $\bullet \to \bullet \quad \bullet$, that is $s_3(A) = 2$. Moreover $s_2(A) = s_2(A') = 1$ and $s(A) = 4$. On the other hand $B = [N]B'$ such that $B'$ is not hereditary. A calculation yields $s_3(B) = 1$ and $s_2(B) = s_2(B') = 2$, obviously implying that $s(B) = 4$.

Some properties of the invariant s:

i. Let $A$ and $B$ be accessible algebras and $A$ be accessible from $B$, then $s(B) \leq s(A)$. Equality holds exactly when $A = B$.

ii. Let $A$ be an accessible schurian algebra (that is for every couple of vertices $i, j$, $\dim_k A(i,j) \leq 1$), then for every convex subcategory $B$ we have $s(B) \leq s(A)$.

### 6.4 Totally accessible algebras

An accessible algebra $A$ with $n = 2r + r_0$ vertices, and $r_0 \in \{0, 1\}$, is said to be *totally accessible* if there is a family of (not necessarily connected) algebras $C^{(n)} = A', C^{(n-2)}, C^{(n-4)}, \ldots, C^{(r_0)}$ satisfying:

a. $A$ is derived equivalent to $A'$;

b. for each $0 \leq i = n - 2j \leq n$, there is a strong tower $\mathbb{T}^{(j)} = \left(C^{(j,1)} = k, \ldots, C^{(j,i)} = C^{(i)}\right)$ of access to $C^{(i)}$;

c. $C^{(i-2)}$ is an $i-1$-th perpendicular restriction of $\mathbb{T}^{(j)}$, that is, $C^{(i)}$ is a one-point (co)extension of $C^{(j,i-1)}$ by a module $N_{i-1}$ and $C^{(i-2)}$ is a perpendicular restriction of $C^{(j,i-1)}$ via $N_{i-1}$.

The tower $\mathbb{T}^{(j)}$ is said to be a *j-th derivative* of the tower $\mathbb{T}^{(0)}$.

*Examples* that we have encountered of totally accessible algebras are:

i. Hereditary tree algebras: since for any conneceted hereditary tree algebra $A$ with at least 3 vertices, there is an arrow $a \to b$ with $a$ a source (or dually a sink) and $A = B[P_b]$ such that the perpendicular restriction of $B$ via $P_b$ is the algebra hereditary tree algebra $C$ obtained from $A$ by deleting the vertices $a, b$.

ii. Accessible representation-finite algebras $A$ with gl.dim $A \leq 2$, since then the perpendicular restrictions of any strong tower (which exists by [14]) satisfy the same set of conditions.

iii. Certain canonical algebras: for instance the tame canonical algebra $A$ of weight type $(2, 4, 4)$ is an extension $A = B[M]$ of a hereditary algebra $B$ of extended Dynkin type $[2, 4, 4]$ by a module $M$ in a tube of rank 4, then the perpendicular restriction of $B$ via $M$ is the hereditary algebra $C$ of extended Dynkin type $[3, 3, 3]$, see for example [?](10.1). Since $C$ is totally accessible, so $A$ is. Moreover $s(A) = 8$.

iv. Let $A$ be an accessible algebra of the form $A = B[M]$ for an algebra $B$ and an exceptional module $M$ and let $C$ the perpendicular restriction of $B$ via $M$. If $A$ is totally accessible, then $B$ and $C$ are totally accessible.

The following results extend some of the features observed in the examples above.

**Proposition.** *a. Assume that $A$ is a totally accessible algebra, then $\chi_A(-1) \in \{0, 1\}$.*

*b.* Assume that $A$ is an accessible but not totally accessible algebra with gl.dim $A \leq 2$, then one of the following conditions hold:

*i.* for every exceptional $B$-module such that $A = B[M]$ and any perpendicular restriction $C$ of $B$ via $M$, then $C$ is not accessible;

*ii.* there exists a homological epimorphism $\phi : A \to B$ such that $\chi_B(-1) > 1$.

*Proof.* $(a)$: Consider the perpendicular restriction $C$ of $B$ via $M$, such that $\chi_A(t) = (1+t)\chi_B(t) - t\chi_C(t)$. Therefore $\chi_A(-1) = \chi_C(-1)$ and moreover, $C$ is totally accessible. Then by induction hypothesis, $\chi_A(-1) = \chi_{C^{(m)}}(-1)$ for a totally accessible algebra $C^{(m)}$ with number of vertices $m = 1$ or $m = 2$. Clearly, $C^{(m)}$ is either $k$, $k \oplus k$ or hereditary of type $\mathbb{A}_2$, which yields the desired result.

$(b)$: Assume $A$ is an accessible algebra with gl.dim $A \leq 2$ and such that for every homological epimorphism $\phi : A \to B$ we have $\chi_B(-1) \in \{0, 1\}$. Let $A = B[M]$ for an accessible algebra $B$ and an exceptional $B$-module $M$ such that $C$ is a perpendicular restriction of $B$ via $M$. Since gl.dim $A \leq 2$ then there is a homological epimorphism $A \to C$ and gl.dim $C \leq 2$. Observe that for every homological epimorphism $\psi : B \to B'$ (resp. $\psi : C \to C'$) there is a homological epimorphism $\phi : A \to B'$ (resp. $\phi : A \to C'$), hence $\chi_{B'}(-1)$ (resp. $\chi_{C'}(-1)$) is 0 or 1. By induction hypothesis, $B$ is totally accessible. Moreover if $C$ is accessible, then the induction hypothesis yields that $C$ is totally accessible and also $A$ is totally accessible, a contradiction. Therefore $C$ is not accessible. □

## 7. On the quadratic coefficient of the Coxeter polynomial of a totally accessible algebra

### 7.1 Derived algebras of linear type

Recall that an *extended canonical* algebra of weight type $\langle p_1, \ldots, p_t \rangle$ is a one-point extension of the canonical algebra of weight type $[p_1, \ldots, p_t]$ by an indecomposable projective module. As in (1.3), the extended canonical algebras of type $\langle p_1, p_2, p_3 \rangle$ is strongly accessible. Moreover, the extended canonical algebra $A$ of type $\langle 3, 4, 5 \rangle$ (with 12 points) has Coxeter polynomial $1 + t + t^2 + \ldots + t^{12}$ which is also the Coxeter polynomial of a linear hereditary algebra $H$ with 12 vertices. Clearly $A$ and $H$ are not derived equivalent.

The following generalizes a result of Happel who considers the case of Coxeter polynomials associated to hereditary algebras [8].

**Theorem 1.** *Let $A$ be a totally accessible algebra with $n$ vertices and let $\chi_A(t) = \sum_{i=0}^{n} a_i t^i$ be the Coxeter polynomial of $A$. The following are equivalent:*

i. $a_2 = 1$;

ii. *let $\mathbb{T} = (A_1 = k, \ldots, A_{n-1}, A_n = A)$ be a strong tower of access to $A$ and $C_i$ the $i$-th perpendicular restriction of $\mathbb{T}$, for all $1 \leq i \leq n-2$, then the algebras $C_i$ are connected;*

iii. *$A$ is derived equivalent to a quiver algebra of type $\mathbb{A}_n$.*

*Proof.* $(i) \Leftrightarrow (ii)$: Let $\mathbb{T} = (A_1 = k, \ldots, A_n = A)$ be a strong tower of access to $A$. In case each $C_i$ is connected, then $s(A) = n-2$, that is $a_2 = 1$. If $a_2 = 1$, then $n - 2 = s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$ with each $s_i \geq 1$. $(i) \Leftrightarrow (iii)$: We know that an algebra $A$ derived equivalent to a quiver algebra of type $\mathbb{A}_n$ has $\chi_A(t) = \sum_{i=0}^{n} t^i$, in particular, $a_2 = 1$. Assume that an accessible algebra $A$ has the quadratic coefficient of its Coxeter polynomial $a_2 = 1$. Let $A = B[M]$ for an accessible algebra $B = A_{n-1}$ and an exceptional module $M$. Since $B$ is also totally accessible with a tower $\mathbb{T}' = (A_1 = k, \ldots, A_{n-1} = B)$ satisfying $(ii)$, then the quadratic coefficient of the Coxeter polynomial of $B$ is $b_2 = 1$ and we may assume that $B$ is derived equivalent to a quiver algebra of type $\mathbb{A}_{n-1}$. In particular, $B$ is representation-finite with a preprojective component $\mathcal{P}$ such that the orbit graph $\mathcal{O}(\mathcal{P})^{\tau}$ is of type $\mathbb{A}_{n-1}$ (recall that the orbit graph has vertices the $\tau$-orbits in the quiver $\mathcal{P}$ with Auslander-Reiten translation $\tau$ and there is an edge between the orbit of $X$ and the orbit of $Y$ if there is some numbers $a, b$ and an irreducible morphism $\tau^a X \to \tau^b Y$). Observe that for any $X$ in $\mathrm{D}^b(\mathrm{mod}_A)$ not in the orbit of $M$, there is some translation $\tau^a X$ belonging to $M^{\perp}$, implying that in case $M^{\tau}$ has two neighbors in the orbit graph then $M^{\perp}$ is not connected, that is $s_{n-2} > 1$ and $a_2 = n - 1 - s(A) \leq 0$, a contradiction. Therefore, $M^{\tau}$ has just one neighbor in $\mathcal{O}(\mathcal{P})^{\tau}$, hence $A$ is derived of type $\mathbb{A}_n$. □

### 7.2 Theorem 2

Consider a tower $A_1, \ldots, A_n = A$ of accessible algebras where $A_{i+1}$ is a one-point (co)extension of $A_i$ by the indecomposable $M_i$ and $C_i$ is such that $M_i^{\perp}$ is derived equivalent to $\mathrm{D}^b(\mathrm{mod}_{C_i})$. Assume that $C_i^{(j)}$, for $1 \leq j \leq s_i$, are the connected components of the category $C_i$. Consider the corresponding Coxeter polynomials:

$$\chi_{A_i}(t) = 1 + t + \sum_{j=2}^{i-2} a_j^{(i)} t^j + t^{i-1} + t^i,$$

$$\chi_{C_i}(t) = 1 + s_i t + \sum_{r=2}^{n_i-2} c_{i,r} t^r + s_i t^{n_i-1} + t^{n_i},$$

$$\chi_{C_i^{(j)}}(t) = 1 + t + \sum_{s=2}^{n_{i,j}-2} c_{i,s}^{(j)} t^s + t^{n_{i,j}-1} + t^{n_{i,j}},$$

where clearly, $\sum_{j=1}^{s_i} n_{i,j} = n_i$.

**Lemma.** *$(\alpha)$ For every $1 \leq j \leq i-2$, we have $a_j^{(i)} \leq 1$.*

*$(\alpha\alpha)$ For every $1 \leq j \leq i-2$, we have $a_j^{(i)} \leq c_{i,j}$ and $a_j^{(i)} \leq a_j^{(i-1)}$.*

*Proof.* We shall check that $(\alpha)$ implies $(\alpha\alpha)$, then we show that (a') holds by induction on $j$.

Indeed, assume that $(\alpha)$ holds and proceed to show $(\alpha\alpha)$ by induction on $j$. If $j = 0, 1$, then $a_j^{(i)} = 1 = a_{i-j}^{(i)}$. Assume that $2 \leq j \leq i-2$ and $a_j^{(i)} \leq c_{i,j}$ and $a_j^{(i)} \leq a_j^{(i-1)}$. Then

$$a_{j+1}^{(i)} = a_{j+1}^{(i-1)} + \left(a_j^{(i-1)} - c_{j,i-1}\right) \leq a_{j+1}^{(i-1)} \leq \ldots \leq a_{j+1}^{(j+1)} = 1.$$

Let $0 \leq j \leq i-2$. If $j = 0, 1$ we have $a_0^{(i)} = 1 = c_{i,0}$ and $a_1^{(i)} \leq s_1(A) = c_{i,1}$. Moreover $a_1^{(i)} = a_1^{(i-1)}$. Assume $(\alpha)$ holds for $j \geq 2$, then.

$$a_{j+1}^{(i)} = a_{j+1}^{(i-1)} + \left(a_j^{(i-1)} - c_{j,i-1}\right) \leq a_{j+1}^{(i-1)},$$
$$a_{j+1}^{(i)} - c_{i,j+1} = a_{j+2}^{(i)} - a_{j+2}^{(i-1)} \leq 0. \qquad \square$$

**Theorem 2.** *Let $A$ be a totally accessible algebra with Coxeter polynomial $\chi_A(t) = 1 + t + a_2t^2 + \ldots + a_{n-2}t^{n-2} + t^{n-1} + t^n$, then:*

a. $a_j \leq 1$, *for every* $2 \leq j \leq n-2$;

b. *if for some* $2 \leq j \leq n-2$, *we have* $a_j = 1$ *then* $A$ *is derived equivalent to a hereditary algebra of type* $\mathbb{A}_n$.

*Proof.* Keep the notation as in (4.1). Then ($a$) is the case $i = n$ of the Lemma above.

We shall prove (b) by induction on $n$ the number of vertices of $A$. Let $j = 2$ and assume $a_2 = 1$, then (3.1) implies that $A$ is derived equivalent to $\mathbb{A}_n$. Consider now $2 < j < n-2$ and assume that $a_j = 1$, we get:

$$1 = a_j^{(n)} = a_j^{(n-1)} + \left(a_{j-1}^{(n-1)} - c_{n-1,j-1}\right) \leq a_j^{(n-1)} \leq 1$$

The last inequality due to (a), hence $a_j^{(n-1)} = 1$. Induction hypothesis yields that $A_{n-1}$ is derived equivalent to $\mathbb{A}_{n-1}$ and its Auslander-Reiten quiver consists of a preprojective component $\mathcal{P}$. In particular, $a_2^{(n-1)} = 1$, which implies that $s_{n-3}(A_{n-1}) = 1$, that is, $A = A_{n-1}[M]$ for some exceptional module $M$ such that $M^{\perp}$ is derived equivalent to $\mathrm{mod}_C$ for a connected algebra $C$, that is, $s(A) = n-2$ and by (3.1), $A = B[M]$ is derived equivalent to a hereditary algebra of type $\mathbb{A}_n$. $\square$

### 7.3 Examples

If $A$ is a representation-finite accessible algebra with gl.dim $A \leq 2$, then $A$ is totally accessible. On the other hand the algebra $B$ with quiver:

$$1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} 4 \ldots \xrightarrow{x} 11 \xrightarrow{x} 12$$

and $x^3 = 0$ is representation-finite and accessible (but not gl.dim $B \leq 2$). The Coxeter polynomial of $B$ is:

$$\chi_B(t) = 1 + t - t^3 - t^4 + t^6 - t^8 - t^9 + t^{11} + t^{12}.$$

Then observe that the 6-th coefficient is 1 but the algebra $B$ is not derived equivalent to Dynkin type $\mathbb{A}_{12}$.

## 8. On the traces of Coxeter matrices

Let $A$ be an algebra such that not all roots of $\chi_A$ are roots of unity. By the result of Kronecker [36], not all of the spectrum of $A$ lies in the unit disk. Equivalently, the spectral radius $\rho_A = \max\{|\lambda| : \lambda \text{ eigenvalue of } \phi_A\} > 1$. Arrange the eigenvalues of $\phi_A$ so that $\mu_1, \mu_2, \ldots, \mu_n$ have absolute values $\rho_A = r_1 > r_2 > \ldots > r_s$ and multiplicities $m_1, \ldots, m_s$, respectively. Therefore $s \geq 2$ and

$$|\det\phi_A| = r_1^{m_1} r_2^{m_2} \dots r_s^{m_s} = 1.$$

We define the *critical power* $\kappa(A)$ as the minimal $k$ such that

$$|\mathrm{Tr}\,(\phi_A^k)| > n$$

Since $r_1$ is a simple eigenvalue of $\phi_A$, then it follows that $\kappa(A)$ is well defined due to the existence of $k$ satisfying the following chain of inequalities:

$$|\mathrm{Tr}\,(\phi_A^k)| = |\sum_{j=1}^{n} \mu_j^k| \geq r_1^{km_1} - \sum_{j=2}^{s} r_j^{km_j} \geq r_1^k - (n-1) r_2^k > n.$$

The following is a reformulation of Theorem 2.

**Theorem.** *Let A be an algebra such that not all roots of $\chi_A$ are roots of unity. We have $\kappa(A) \leq n$.*

*Proof.* Indeed, suppose that $A$ is not of cyclotomic type and $\kappa(A) > n$, that is, $|\mathrm{Tr}\,(\phi_A^k)| \leq n$ for all $0 \leq k \leq n$. Observe that $M = \phi_A$ is a unimodular matrix and therefore, Theorem 2 implies that $M$ is of cyclotomic type, which yields a contradiction. □

**Remark:** We consider explicitly the case $n = 2$ in the above Theorem. Obviously, the Cartan matrix of $A$ is of the form

$$C = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad \phi_A = -C^{-1}C^T = \begin{pmatrix} a^2 - 1 & a \\ -a & -1 \end{pmatrix}$$

for some $a \geq 1$. Then $\phi_A$ has the indicated shape. If $A$ is not cyclotomic, then $a \geq 3$ and $\mathrm{T}r\,(\phi_A^2) = (a^2 - 2)^2 - 2 > 2$.

## 9. Stability of a real matrix

### 9.1 Stability of matrices and the Lyapunov criterion

Let $M$ be a real invertible $n \times n$-matrix with eigenvalues $\lambda_j = r_j e^{i\theta_j}$, for some numbers $\theta_j \in [0, 2\pi)$ and $j = 1, \dots, n$. We will say that $M$ is *stable* (resp. *semi-stable*) if the real part $\mathrm{Re}(e^{i\theta_j}) = \cos\theta_j$ of the argument of the eigenvalue $\lambda_j$ is positive (resp. non-negative), for every $j = 1, \dots, n$. The following is well-known, we sketch a proof for the sake of completeness.

**Proposition.** *Let M be a stable (resp. semi-stable) $n \times n$-matrix. Then the characteristic polynomial $\chi_M = T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0$ has coefficients satisfying $(-1)^{n-j} a_j > 0$ (resp. $\geq 0$), for $j = 0, 1, \dots, n$;*

*Proof.* Observe that $(-1)^n p(-T)$ is the product of polynomials $T - \alpha$ with $\alpha \in \mathbb{R}$ and $(T - (\alpha + i\beta))(T - (\alpha - i\beta)) = T^2 - 2\alpha T + (\alpha^2 + \beta^2)$ with $0 \neq \beta, \alpha \in \mathbb{R}$. Stability (resp. semi-stability) implies that $\alpha < 0$ (resp. $\alpha \leq 0$) above. Therefore, $(-1)^n p(-T)$ is product of polynomials with positive coefficients. □

**Remark:** In most of the literature the stability concept we use goes by the name of *positive stability*, while the *stability* name is used also as *Hurwitz stability*, or *Lyapunov stability*.

The system of differential equations

$$y'(t) = -My(t)$$

is said to be *stable* if for every vector $d = (d_1, \ldots, d_n)$, the solution $v(t) = e^{-tM}d$ of the above system has the property that $\lim_{t\to\infty} v(t) = 0$.

We recall here the celebrated.

**Lyapunov criterion**: The system $y'(t) = -My(t)$ is stable if and only if $M$ is a stable matrix, equivalently there is a real positive definite matrix $P$ such that

$$M^T P + PM = I_n.$$

It is not hard to see that given $M$, the corresponding $P$ is unique. A proof of the criterion and its equivalence to other stability conditions are considered in [13].

### 9.2 Semi-stable powers

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of the real matrix $M$ with $\mu_j = \rho_j e^{2\pi i \theta_j}$ in polar form. Observe that $\mu_j^k$, for $j = 1, \ldots, n$, are the eigenvalues of $M^k$ and

$$\mathrm{Tr} M^k = \sum_{j=1}^{n} \rho_j^k \cos\left(k\theta_j\right) \le \sum_{j=1}^{n} |\mu_j^k| |\cos\left(k\theta_j\right)|$$

**Lemma.** *For a positive integer $k \geq 1$ the following assertions are equivalent:*

*a.* $M^k$ is a semi-stable matrix;

*b.* $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^n \left|\mu_j\right|^k |\cos\left(k\theta_j\right)|$.

*Proof.* If $M^k$ is a semi-stable matrix, then $\mu^k = \rho_j^k\left(\cos\left(k\theta_j\right) + i \sin\left(k\theta_j\right)\right)$ has $\cos\left(k\theta_j\right) \geq 0$. Since $M$ is a real matrix then $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^n \rho_j^k \cos\left(k\theta_j\right) \geq 0$. Therefore

$$\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^{n} \rho_j^k |\cos\left(k\theta_j\right)|.$$

Assume that $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^n |\lambda_j|^k |\cos\left(k\theta_j\right)|$. Since $|\lambda_j^k| \geq \rho_j^k \cos\left(k\theta_j\right)$ for $j = 1, \ldots, n$, adding up, we get

$$\mathrm{Tr}\left(M^k\right) \geq \sum_{j=1}^{n} \rho_j^k \cos\left(k\theta_j\right) = \mathrm{Tr}\left(M^k\right)$$

Hence we have equalities $|\lambda_j^k| |\cos\left(k\theta_j\right)| = \rho_j^k \cos\left(k\theta_j\right)$ for $j = 1, \ldots, n$. Then $M^k$ is semi-stable. $\square$

We say that $k$ is a *stable power* (resp. *semi-stable power*) of $M$ if $M^k$ is a stable (resp. semi-stable) matrix.

## 10. Nakayama algebras

### 10.1 Cyclotomic Nakayama algebras

As a well-understood example the representation theory of the *Nakayama algebras* stands appart. Let $N(n, r)$ be the quotient obtained from the linear quiver with $n$ vertices with radical $\mathrm{rad}_A$ of nilpotency index $r$.

For instance, for $A = N(6, 3)$ the Cartan matrix $C$ and Coxeter matrix $\phi$ are:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ and } \phi = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

whose characteristic polynomial is cyclotomic as we know from [18] or might be verified calculating $\mathrm{Tr}\left(\phi_B^k\right) \leq n$, for $1 \leq k \leq 72$ and applying the criterion of Theorem 1. Indeed, for.

| $k =$ | $\mathrm{Tr}\chi_A^k =$ |
|---|---|
| 11 | $-1$ |
| $1, 2, 5, 7, 9, 10, 13, 14, 17$ | $= 1$ |
| $3, 6, 15$ | $= 2$ |
| $4, 8, 16$ | $= 3$ |
| 12 | $= 6$ |

Starting with $k = 17$ the sequence of traces repeats cyclically. Therefore, $\mathrm{Tr}\left(\chi_A^k\right) \leq 6$ for all $0 \leq k$. Then $N(6, 3)$ is of cyclotomic type.

### 10.2 An example

We recall in some length the argument given in [18] for the cyclotomicity of $N(n, 3)$, for all $n \geq 1$.

Consider the algebra $R_{2n}$ with $2n$ vertices and whose quiver is given as

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & 2 \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & 3 \arrow[r] \arrow[d] & \cdots \arrow[r] & n \arrow[d] \\ 1' \arrow[r] & 2' \arrow[r] & 3' \arrow[r] & \cdots \arrow[r] & n' \end{tikzcd}$$

with all commutative relations. The corresponding Coxeter polynomial

$$\chi_{R_{2n}} = \chi_{\mathbb{A}_n} \otimes \chi_{\mathbb{A}_2} = v_{n+1} \otimes v_3$$

is a product of cyclotomic polynomials, therefore $\chi_{R_{2n}}$ is a cyclotomic polynomial. In fact $R_{2n} = \mathbb{A}_n \otimes \mathbb{A}_2$, where $\mathbb{A}_s$ is the hereditary algebra associated to the linear quiver $1 \to 2 \to \cdots \to s$.

For $2m + 1$ odd, we consider.

$$R_{2m+1} \qquad \begin{tikzcd} & \circ \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & \circ \arrow[r] \arrow[d] & \cdots \arrow[r] & \circ \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

The following holds for the sequence of algebras $R_n$ and its Coxeter polynomials $\chi_{R_n}$:

a. $R_n$ is derived equivalent to $N(n, 3)$.

b. $\chi_{R_n} = T^n + T^{n-1} - T^3\chi_{R_{n-6}} + T + 1$, for all $n \geq 6$;

c. $\mathbb{M}(\chi_{R_n}) = 1$.

Observe that the sequence of algebras $(R_n)$ forms an *interlaced tower of algebras*, that is, it is a sequence of triangular algebras $R_1, \ldots, R_n$, such that $R_s$ is a basic algebra with $s$ simple modules and, among others, the condition

$$\chi_{R_{s+1}} = (T+1)\chi_{R_s} - T\chi_{R_{s-1}}$$

is satisfied for $s = 1, \ldots, n-1$. Moreover, $A_{s+1}$ is a one-point extension (or coextension) of an accessible algebra $A_s$ by an exceptional $A_s$- module $M_s$ such that the perpendicular category $M_s^{\perp}$ formed in the derived category is triangular equivalent to $\mathrm{mod}(A_{s-1})$, for $s = m+1, \ldots, n-1$.

The following was shown in [18]: *Consider an interlaced tower of algebras $A_m, \ldots, A_n$ with $m \leq n-2$. If $\mathrm{Spec}\,\phi_{A_n}$ is contained in the union of the unit circle and the semi-ray of positive real numbers then either all $A_i$ are of cyclotomic type or $M(\chi_{A_m}) < M(\chi_{A_n})$. In the latter case, $M(\chi_{A_n}) < \prod_{s=m}^{n-1} M(\chi_{A_s})$.*

Since we know that $M(\chi_{R_{2n}}) = 1$, for all $n \geq 0$, we conclude that $M(\chi_{R_n}) = 1$, for all $n \geq 0$. That is the Nakayama algebras of the form $N(n,3)$ are of cyclotomic type.

### 10.3 Non-cyclotomic Nakayama algebras

Calculation of $\mathrm{Tr}\,\phi_A^k$ for $A = N(n,r)$ and $k$ in intervals, for data sets $(n,r,k)$, yield interesting information. Namely,

a. Many Nakayama algebras are of cyclotomic type;

b. Not all Nakayama algebras are of cyclotomic type. The case $r = 4$ illustrates this claim:

   $N(n,4)$ is of cyclotomic type for all $0 \leq n \leq 100$ except for $n = 10, 22, 30, 42, 50, 62, 70, 82$ and $90$

c. A canonical algebra $C$ of weight $(p_1, \ldots, p_t)$ is strongly accessible if and only if $t = 3$, in that case, $C$ is derived-equivalent to a representation-finite algebra if and only if the weight type does not dominate $(3,3,3)$.

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