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Generalized BCS Equations: A Review and a Detailed Study of the Superconducting Features of $Ba_2Sr_2CaCu_2O_8$

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Abstract

High-T_c superconductors (SCs) are most widely studied via the multiband approach (MBA) based on the work of Suhl et al. and the Nambu-Eliashberg-McMillan extension of the BCS theory. Complementing MBA and presented in a recent monograph is an approach based on the generalized BCS equations (GBCSEs), which too has been applied to a significant number of SCs. GBCSEs are obtained via a Bethe-Salpeter equation and the Matsubara technique. One of the key features of this approach is the characterization of a composite SC by Cooper pairs with different binding energies—each of which is identified with a gap (Δ) of the SC—depending on whether pairing is due to phonon exchanges with one, two, or more ion species. Another feature is the incorporation of chemical potential in the GBCSEs, which enables one to calculate the critical current density j_0 of an SC using the same parameters that determine its T_c and Δ. Following a review of the concepts of this approach, given herein, for the first time, is a detailed explanation of the multitude of reported empirical values of $\{T_c, \Delta, j_0\}$ of $Bi_2Sr_2CaCu_2O_8$. Also discussed are the currently topical issues of $s^{\pm}$-superconductivity and the isotope-like effect for high-T_c SCs.

Keywords: high-T_c superconductors, multiband approach, GBCSE-based approach, superconducting features of $Bi_2Sr_2CaCu_2O_8$, $s^{\pm}$-superconductivity, isotope-like effect for high-T_c superconductors, suggestions for further increasing the T_c of the Bi-based superconductors

1. Introduction

1.1 A historical note

The phenomenon of superconductivity was first observed by Kamerlingh Onnes [1] in 1911 while studying the electrical resistivity of metallic mercury as a function of temperature when cooled to liquid helium temperatures. He noticed that at the critical temperature T_c = 4.2 K (−269°C), the resistance vanished abruptly. This was an exciting discovery because it suggested that such materials could have immense practical applications. Since cooling a material to very low temperatures is a tedious and expensive process, Onne's discovery triggered a

search in scientific laboratories across the world for other superconducting materials having T_cs higher than that of Hg, the ultimate aim being to find a material which is superconducting at room temperature. While this search led to the discovery of many elements and alloys that are superconductors (SCs), till as late as in the 1980s, the highest T_c of any known SC was about 23 K (for Nb_3Sn). A radical change occurred in this situation in 1986 when Bednorz and Müller announced the discovery of an SC (Ba-La-Cu-O system) with $T_c \approx 38$ K, for which they were awarded the Nobel Prize in 1987. An avalanche of activity followed soon thereafter leading to the discovery of many "high-temperature SCs" such as MgB_2 ($T_c \simeq 40$ K), Fe-based SCs (highest $T_c \simeq 55$ K), and "cuprates," examples of which are Bi-, Tl-, and Hg-based SCs that contain one or more units of CuO_2 as a constituent, the T_cs of these SCs at ambient pressure being about 95, 110, and 138 K, respectively, and for the Hg-based SCs, going up to 164 K at high pressure. The confirmed current record for the highest T_c (203 K; at a pressure of 150 GPa) is held by H_3S which was discovered in 2015. It would thus appear that realization of the dream of room temperature SCs is near at hand.

On the theoretical front, a clear explanation of superconductivity evaded the efforts of many theoreticians who had been engaged in the quest. It was not until 46 years had elapsed since Onnes' discovery that John Bardeen, Leon Cooper, and Robert Schrieffer (BCS) [2] were able to provide a generally accepted explanation of it for elemental SCs, for which they were awarded the Nobel Prize in 1972. Indeed, a contributing factor for this lag between theory and experiment was the tragedy of the two world wars in the intervening period.

BCS theory explains superconductivity as due to the formation of Cooper pairs (CPs), each of which is a bound state of two electrons. A well-known explanation for the origin of the attractive interaction causing electrons to be bound together is as follows: the passage of an electron through the ion lattice of an element leaves behind a deformation trail of enhanced positive charges. If a second electron passes through the lattice while it is recovering from the effect of the passage of the first, it will experience an attractive force that may be greater than the Coulomb repulsion between them. Hence, overall, the electron-lattice-electron interaction can be a net *weak* attractive interaction that leads to the formation of CPs. Since the bound state of two particles has lower energy than the state in which they are both free, the former becomes the preferred state leading to the occurrence of superconductivity. In the language of quantum field theory, it is said that the pairing of electrons takes place because they exchange a phonon due to the effect of the ion lattice. BCS theory is hence a weak-coupling theory which works for elemental SCs because the highest T_c for this class of SCs is about 9 K (for Nb). A general perception about this theory is that it is inadequate to explain the occurrence of such high-T_cs as have been observed. This is indeed so if the one-phonon exchange mechanism (1PEM) is considered to be the sole mechanism responsible for pairing. Such a view overlooks the fact that *all* high-T_c SCs are composite materials consisting of sub-lattices of more than one ion species and therefore the theory ought to be generalized to address the situation where CPs may also be formed due to more than 1PEM. Interestingly, as noted below, such a generalization can be carried out based on three theoretical developments that took place in the same decade, 1950s, to which the BCS theory belongs.

1.2 GBCSEs and their conceptual basis

Two of the three developments alluded to the above are widely known: the invention of the Bethe-Salpeter equation (BSE) in 1951 [3] and, in analogy with quantum field theory, the creation of a finite-temperature field theory (FTFT) by

Matsubara [4] in 1955. Not so well-known is the third development that took place in 1954 due to Okubo [5], viz., the introduction of the concept of a superpropagator as the propagator for the non-polynomial field exp (gϕ) where g is a coupling constant and ϕ a scalar field.

BSE is a four-dimensional equation which was invented for the description of bound states of two relativistic particles, hordes of which were being discovered in the late 1940s and early 1950s in cosmic rays and high-energy accelerators around the world. A general perception about the BSE is that it is irrelevant for superconductivity which is a strictly nonrelativistic phenomenon.

FTFT is obtained via an adaption of quantum field theory where time is replaced by temperature as the variable in terms of which evolution of a system is studied. Because temperature is a statistical concept, this is a huge step which converts an equation setup for the bound states of two particles interacting in vacuum into an equation valid in a many-body system. A shortcut known as the Matsubara prescription or recipe for introducing temperature into a BSE is to replace in it the component q_4 of the four-vector $q_\mu = (q_4, \mathbf{q})$ as follows (e.g., [6]):

$$q_4 = (2n + 1)\pi/(-i\beta) \quad \text{for fermions}$$

$$= 2n\pi/(-i\beta) \quad\quad \text{for bosons}$$

$$\int dq_4 = \frac{2\pi}{-i\beta} \sum_{n=-\infty}^{\infty}$$

(1)

Okubo's work remained an obscure academic exercise till Salam and collaborators found that non-polynomial theories have inbuilt damping effects. It was therefore hoped that they might provide the means of renormalizing, e.g., the weak interactions. These theories were not pursued any further after the gauge theory of interactions became widely accepted as the correct theory. However, it is relevant in the context of superconductivity to note that the expression for the Feynman superpropagator corresponding to the non-polynomial interaction $1/(1 + g\phi)$ is [7]:

$$\Delta_F^{SP}(k) \approx \frac{1}{k^2 g^2} - \int\limits_0^\infty dt \frac{e^{-t}(2 - t)}{k^2 g^2 + t}.$$

(2)

The comparison of the above with the expression for one-particle Feynman propagator, viz.,

$$\Delta_F^{1-\text{Particle}}(k) = \frac{1}{k^2 - i\varepsilon}$$

(3)

reveals that a superpropagator represents a weighted superposition of multiple quanta. The significance of this remark emerges if we recall how the electron-lattice-electron interaction in an elemental SC causes formation of CPs via 1PEM. Appealing to the same picture for a composite SC, it follows that pairing can now also be caused via more than 1PEM because sub-lattices of ions of different masses are affected differently by the passage of electrons through the lattice of the SC. We are hence led to the following outline of a novel strategy for dealing with multicomponent SCs: start with a BSE for the bound states of two fermions, temperature-generalize it via the Matsubara prescription, and, for the kernel of the equation, employ a superpropagator representing exchange of multiple phonons for pairing.

To implement the above program, we need to employ (i) the instantaneous and mean-field approximations (IA, MFA), (ii) a model for fixing the Debye

temperatures of all the ions that may cause pairing, and (iii) the Bogoliubov constraint on the interaction parameters obtained via solutions of the equations that the BSE leads to when multi-phonon exchange mechanisms are operative. An account of these concepts is as follows.

By IA is meant that we ignore the time of propagation of the quanta, the exchange of which causes the electrons to be bound together. For example, in the momentum-space transform of the one-particle propagator in Eq. (3) where $k^2 = k_0^2 - \boldsymbol{k}^2$, k_0^2 is equated to zero. This is an essential step for the application of the Matsubara prescription which introduces temperature into the theory and reduces the BSE to a three-dimensional equation which is then subjected to the nonrelativistic approximation appropriate for superconductivity.

Employing MFA in the 1PEM scenario, one approximates the propagator noted in Eq. (3) by a constant, $-V$ (as in [N(0)V], $\hbar = c = 1$), because pairing takes place in a very small region demarcated by $(E_F - k\theta \leq p^2/2m \leq E_F + k\theta)$, where E_F denotes the Fermi energy, $p^2/2m$ the energy of an electron, and θ the Debye temperature of the SC; outside of this region, $V = 0$. Similarly, in the nPEM scenario, in lieu of the expression for the superpropagator in Eq. (2), one employs the expression $- [V_1 + V_2 + ... + V_n]$, where n is the number of ion species that cause the electrons to form a pair, V_i corresponds to the phonon exchanged due to the ith species of ions of which the Debye temperature is θ_i, and $(E_F - k\theta_i \leq p^2/2m \leq E_F + k\theta_i)$ is the region of pairing due to these ions, outside of which $V_i = 0$.

Given the Debye temperature θ of any composite SC, we now need to fix the Debye temperatures of its constituent ion species, θ_1, θ_2, etc., which in general *must* be different from θ because, as was first pointed out by Born and Karman a long time ago, elastic waves in an anisotropic material travel with different velocities in different directions; for a detailed account of their work, see [8]. We now draw attention to the fact that sublattices of all the high-T_c SCs are composed of layers that contain either one or predominantly two ion species. The assumption that θ is also the Debye temperature of each of these sub-lattices then fixes the Debye temperature of the sub-lattice of layers that contains a single ion species as θ. As concerns fixing θ_1 and θ_2 for the sub-lattice comprising two ion species, we note that the following relation has been frequently used for finding the Debye temperature θ of a binary A_xB_{1-x} when the Debye temperatures θ_A and θ_B are known:

$$\theta = x\theta_A + (1 - x)\theta_B. \tag{4}$$

Since we are faced with the inverse of the above situation, i.e., given θ, we have to fix θ_A and θ_B, we now need to supplement Eq. (4) by another equation. We meet this requirement by assuming that the modes of vibration of the A and B ions simulate the weakly coupled modes of vibration of a double pendulum. This is suggested by the polariton effect where coupling between the phonon and the photon modes leads to two new modes, different from the original modes of either of them. With A as the upper bob in the double pendulum, we have our second equation as [10].

$$\frac{\theta_A}{\theta_B} = \left[\frac{1 + \sqrt{m_B/(m_A + m_B)}}{1 - \sqrt{m_B/(m_A + m_B)}} \right]^{1/2}, \tag{5}$$

where m_A and m_B are the atomic masses of the ions.

Given θ ($Bi_2Sr_2CaCu_2O_8$) = 237 K, m_{Bi} = 208.98, m_{Sr} = 87.62, and m_O = 15.999 (amu), the application of Eqs. (4) and (5) leads to

$$\theta_{Ca} = 237 \text{ K}, \theta_{Bi} = 269 \text{ K}, \theta_{Sr} = 286 \text{ K.} \qquad (6)$$

We note that (i) θ_{Ca} equals the Debye temperature of the SC because the layers containing Ca ions do not have any other constituent. (ii) The value of θ_{Bi} corresponds to Bi being the upper bob of the double pendulum and O the lower bob in the BiO layers; the other value of θ_B when Bi is the lower bob is 57 K. (iii) The value of θ_{Sr} also corresponds to Sr. being the upper bob of the double pendulum in the SrO layers; the other value of θ_{Sr} is 81 K. (iv) The values of θ_O in the two layers are not noted because they do not play a direct role in the pairing process. (v) Since we have four distinct choices for values of the pair $\{\theta_{Bi}, \theta_{Sr}\}$, we need a criterion for choosing one among them, and this is provided by the Bogoliubov constraint discussed below.

In a rigorous study concerned with renormalization of the BCS theory, starting with the assumption of a net attractive interaction between electrons, Bogoliubov found that the interaction parameter λ governing the pairing process must not exceed 0.5, otherwise the system would be unstable; for a detailed discussion, see [9]. Imposition of this constraint on the λs obtained via GBCSEs for high-T_c SCs is crucial. This is so because if one calculates λs for each of the four choices of θs as noted above, one can straightway reject those that lead to any of them that are negative and choose among the rest the ones that lead to values closest to 0.5. The values of θs in Eq. (6) have been chosen on this basis.

1.3 Plan of the chapter

This chapter is organized as follows. In Section 2.1 is given the parent Bethe-Salpeter equation from which GBCSEs for the T_c and Δs of a multicomponent SC are derived. In Section 2.2, after an outline of the key steps of their derivation, are given the GBCSEs which are constrained by the inequality $E_F \gg k\theta$. More generally, E_F-incorporated GBCSEs, which are not constrained by this inequality, are given in Section 2.3. This is followed up by an account of similar equations, i.e., subject to and not subject to the said inequality, for the dimensionless construct $y = (k\theta/P_0)\sqrt{2m/E_F}$, which is required for calculating j_0 and several other parameters of an SC. In Section 4 are listed the properties of Bi-2212, the explanation of which is taken up in Section 5. In Section 6 we draw attention to the application of the framework of GBCSEs to a diverse variety of SCs, viz., LCO, the heavy fermion, and the Fe-based SCs. Also included in this section is a brief account of a recent study concerned with the isotope-like effect for Bi-2212. Sections 6 and 7 are devoted, respectively, to the discussion of the key features of our approach and the conclusions following from them.

2. The framework of GBCSEs

After an outline of the key steps of their derivation, we give in this section the GBCSEs that are later employed for the study of various superconducting features of $Bi_2Sr_2CaCu_2O_8$ (Bi-2212). We refer the reader to [10] for a detailed derivation of these equations.

2.1 The parent BSE

The parent equation from which GBCSEs are derived is the following relativistic BSE [11] for the bound states of fermions 1 and 2 of equal mass (m):

$$\psi\left(p_\mu\right) = (-2\pi i)^{-1} \int d^4 q_\mu \frac{1}{\gamma_\mu^{(1)} P_\mu/2 + \gamma_\mu^{(1)} q_\mu - m + i\varepsilon}$$

$$\text{x} \frac{1}{\gamma_\mu^{(2)} P_\mu/2 + \gamma_\mu^{(2)} q_\mu - m + i\varepsilon} I\left(q_\mu - p_\mu\right)\psi\left(q_\mu\right), \tag{7}$$

where ψ is the BS amplitude for the formation of the bound state of the two particles, P_μ the total four-momentum of their center of mass, q_μ their relative four-momentum, $\gamma_\mu^{(1,2)}$ are their Dirac matrices, and $I(q_\mu - p_\mu)$ is the kernel of the equation that causes the particles to be bound together; the metric employed is time-preferred: $a_\mu b_\mu = a_4 b_4 - \mathbf{a}.\mathbf{b}$.

2.2 GBCSEs corresponding to multiphonon exchanges for pairing

We now outline how GBCSEs are derived when CPs in an SC are formed due to multi-phonon exchange mechanisms subject to the following constraint:

$$E_F >> k\theta, \tag{8}$$

where θ is the Debye temperature of the SC.
Beginning with Eq. (7), we obtain GBCSEs by:

1. Employing IA: $I\left(q_\mu - p_\mu\right) = I(\mathbf{q} - \mathbf{p})$

2. Carrying out spin reduction of the equation by multiplying it with Dirac matrices $\gamma_4^{(1)}\gamma_4^{(2)}$

3. Working in the rest frame of the two particles: $P_\mu = (E, 0)$

4. Putting $E = 2E_F + W$

5. Temperature-generalizing the equation via the Matsubara recipe given in the previous section

6. Assuming that the signature of W changes on crossing the Fermi surface

7. Choosing for the kernel a generalized form of the model BCS interaction:

$$I(\mathbf{q} - \mathbf{p}) = \sum_{i=1}^{n} \frac{-V_i}{(2\pi)^3} \quad \left(\text{for } E_F - k\theta_i \leq \frac{\mathbf{p}^2}{2m}, \frac{\mathbf{q}^2}{2m} \leq E_F + k\theta_i\right) \tag{9}$$

$$= 0 \text{ (otherwise)},$$

where n is the number of ion species due to which pairs are formed and θ_i is the Debye temperature of the ith species of ions

8. Calculating θi via the Debye temperature θ of the SC and Eqs. (4) and (5) given above

With their application to Bi-2212 in mind, given below are the equations that the above steps lead to in the scenario in which the smaller of the two gaps of an SC is due to a two-phonon exchange mechanism (2PEM) and its larger gap and the

highest T_c are due to a three-phonon exchange mechanism (3PEM) (reminder: $E_F \gg k\theta$)

$$1 = \lambda_1 \ln\left[1 + \frac{2k\theta_1}{|W_2|}\right] + \lambda_2 \ln\left[1 + \frac{2k\theta_2}{|W_2|}\right] \tag{10}$$

$$1 = \lambda_1 \ln\left[1 + \frac{2k\theta_1}{|W_3|}\right] + \lambda_2 \ln\left[1 + \frac{2k\theta_2}{|W_3|}\right] + \lambda_3 \ln\left[1 + \frac{2k\theta_3}{|W_3|}\right] \tag{11}$$

$$1 = \lambda_1 \int_0^{\theta_1/2kT_c} dx\, \frac{\tanh(x)}{x} + \lambda_2 \int_0^{\theta_2/2kT_c} dx\, \frac{\tanh(x)}{x} + \lambda_3 \int_0^{\theta_3/2kT_c} dx\, \frac{\tanh(x)}{x}, \tag{12}$$

where $\lambda_i = [N(0)V_i]$, $N(0) = (2m)^{3/2} E_F^{1/2}/4\pi^2$ ($\hbar = c = 1$).

Remark: If $\lambda_2 = 0$ in Eq. (10) and $\lambda_2 = \lambda_3 = 0$ in both Eqs. (11) and (12), then we have a one-phonon exchange mechanism (1PEM) in operation. In this case Eq. (12) becomes identical with the BCS equation for the T_c of an elemental SC, whereas Eq. (10), or Eq. (11), leads to

$$|W_1| = 2k\theta/[\exp(1/\lambda) - 1].$$

Recalling the BCS equation for the gap of an elemental SC at $T = 0$, i.e.,

$$\Delta_0 = k\theta/\sinh(1/\lambda),$$

we observe that $|W_1| = \Delta_0 = 2k\theta \exp(-1/\lambda)$, when $\lambda \to 0$. This per se suggests that one may employ the equation for $|W_1|$ in lieu of the equation for Δ_0. Based on the values of $|W_1|$ and Δ_0 obtained via a detailed study of six elemental SCs [10] by employing the empirical values of their Debye temperatures, it has been shown that $|W_1|$ is a viable alternative to Δ_0. We are thus led to assume that each of the three Ws following from Eq. (11) is to be identified with a gap of the SC and that $|W_3| > |W_2| > |W_1|$, where $|W_2|$ corresponds to the solution of the equation when one of the λs is zero and $|W_1|$ to when two of them are zero.

2.3 E_F-incorporated GBCSEs

It has been known for a long time that T_cs of some SCs—SrTiO$_3$ being a prime example—depend in a marked manner on their electron concentration n_s which is related to E_F. Recent studies—both experimental [12–14] and theoretical [15–17]—too suggest that *low* values of E_F play a fundamental role in determining the properties of high-T_c SCs. This makes it natural to seek equations for the T_cs and Δs of these SCs that incorporate E_F as a variable and are therefore not constrained by inequality (8). Such equations are in fact an integral part of the BCS theory. Their application to dilute SrTiO$_3$ by Eagles [18] perhaps marks the beginning of what is now known as BCS-BEC crossover physics. It seems to us that the focus of crossover physics has, by and large, been different from that of the studies of high-T_c SCs.

Beginning with Eq. (7) and carrying out steps (1–8) noted above, without imposing inequality (8), the μ-incorporated GBCSEs corresponding to Eqs. (10)–(12) are as follows [10]:

$$\frac{\lambda_1}{2}a_1(\mu_0) + \frac{\lambda_2}{2}a_2(\mu_0) = \left[\frac{3}{4}a_3(\mu_0)\right]^{1/3} \tag{13}$$

$$\frac{\lambda_1}{2}b_1(\mu_0) + \frac{\lambda_2}{2}b_2(\mu_0) + \frac{\lambda_3}{2}b_3(\mu_0) = \left[\frac{3}{4}b_4(\mu_0)\right]^{1/3} \tag{14}$$

$$\frac{\lambda_1}{2}c_1(\mu_1) + \frac{\lambda_2}{2}c_2(\mu_1) + \frac{\lambda_3}{2}c_3(\mu_1) = \left[\frac{3}{4}c_4(\mu_1)\right]^{1/3}, \tag{15}$$

where μ_0 (μ_1) is the chemical potential at T = 0 (T = T_c) and μ_0 has been used interchangeably with E_F,

$$a_1(\mu_0) = Re\left[\int_{-k\theta_1}^{k\theta_1} d\xi \frac{\sqrt{\xi+\mu_0}}{|\xi|+|W_2|/2}\right], a_2(\mu_0) = Re\left[\int_{-k\theta_2}^{k\theta_2} d\xi \frac{\sqrt{\xi+\mu_0}}{|\xi|+|W_2|/2}\right]$$

$$a_3(\mu_0) = Re\left[\frac{4}{3}(\mu_0 - k\theta_m)^{3/2} + \int_{-k\theta_m}^{k\theta_m} d\xi\sqrt{\xi+\mu_0}\left(1 - \frac{\xi}{\sqrt{\xi^2+W_2^2}}\right)\right]$$

$$b_1(\mu_0) = Re\left[\int_{-k\theta_1}^{k\theta_1} d\xi \frac{\sqrt{\xi+\mu_0}}{|\xi|+|W_3|/2}\right], b_2(\mu_0) = Re\left[\int_{-k\theta_2}^{k\theta_2} d\xi \frac{\sqrt{\xi+\mu_0}}{|\xi|+|W_3|/2}\right]$$

$$b_3(\mu_0) = Re\left[\int_{-k\theta_3}^{k\theta_3} d\xi \frac{\sqrt{\xi+\mu_0}}{|\xi|+|W_3|/2}\right]$$

$$b_4(\mu_0) = Re\left[\frac{4}{3}(\mu_0 - k\theta_2)^{3/2} + \int_{-k\theta_m}^{k\theta_m} d\xi\sqrt{\xi+\mu_0}\left(1 - \frac{\xi}{\sqrt{\xi^2+W_3^2}}\right)\right]$$

$$c_1(\mu_1) = Re\left[\int_{-\theta_1/2T_{c3}}^{\theta_1/2T_{c3}} dx\sqrt{2kT_{c3}x+\mu_1}\,\frac{\tanh(x)}{x}\right],$$

$$c_2(\mu_1) = Re\left[\int_{-\theta_2/2T_{c3}}^{\theta_2/2T_{c3}} dx\sqrt{2kT_{c3}x+\mu_1}\,\frac{\tanh(x)}{x}\right]$$

$$c_3(\mu_1) = Re\left[\int_{-\theta_3/2T_{c3}}^{\theta_3/2T_{c3}} dx\sqrt{2kT_{c3}x+\mu_1}\,\frac{\tanh(x)}{x}\right]$$

$$c_4(\mu_1) = Re\left[2kT_{c3}\int_{-\theta_m/2T_{c3}}^{\theta_m/2T_{c3}} dx\sqrt{2kT_{c3}x+\mu_1}\{1 - \tanh(x)\}\right],$$

θ_m is the Debye temperature that has the highest value among all the Debye temperatures being used (when $\lambda_2 \neq 0$, $\theta_m = \theta_2$; when $\lambda_2 = 0$, $\theta_m = \theta_3$; when $\lambda_2 = \lambda_3 = 0$, $\theta_m = \theta_1$), and the operator Re ensures that the integrals yield real values even when expressions under the radical signs are negative.

2.4 Equation for $y = (k\theta/P_0)\sqrt{2m/E_F}$ ($E_F \gg k\theta$)

If it turns out that we need to employ the μ-incorporated Eqs. (13)–(15) to deal with high-T_c SCs, rather than Eqs. (10)–(12), as will be shown to be the case, then there will be a need for at least one more equation for an additional property of the SC, besides equations for one of its T_c and Δs, in order to fix the values of μs and λs that lead to various empirical features of the SC. We choose the critical current density j_0 of the SC at T = 0 to meet this need.

The BSE-based approach enables one to derive an equation for j_0 of an SC by employing the same values of θs and λs that determine a given value of its T_c and different values of its Δs. This comes about because one can now obtain expressions for the effective mass m^* of an electron, critical velocity v_0 at T = 0, and the number of

superconducting electrons n_s at T = 0, and hence for j_0 in terms of y and the values of the following other parameters of the SC: θ, the electronic specific heat constant $\gamma, s \equiv m^*/m_e$, v_0, n_s, and v_g, where m_e is the free electron mass and v_g the gram-atomic volume of the SC.

As concerns y, we recall that in the original BCS theory, the Hamiltonian was restricted at the outset to comprise terms corresponding to pairs having zero center-of-mass momentum. The BSE-based approach enables one to go beyond this restriction by setting $P_\mu = (E, \mathbf{P})$, rather than $(E, 0)$, where $\mathbf{P}$ is the three-momentum of CPs in the lab frame. Beginning with Eq. (7) and carrying out essentially the same steps as for the $P_\mu = (E, 0)$ case, we now obtain in the 3PEM scenario the following equation for y [10], which can be solved after the λs have been determined by solving Eqs. (10)–(12):

$$1 = \lambda_1 t(r_1, y) + \lambda_2 t(r_2, y) + \lambda_3 t(r_3, y), \tag{16}$$

where

$$t(r, y) = \left[r y \ln \left(\frac{r y}{r y - 1} \right) + \ln (r y - 1) \right], \; r_i = \theta_i/\theta.$$

Note that equating one (two) of the λs in Eq. (16) to zero, we obtain the equation for y in the 2PEM (1PEM) scenario.

In terms of y and the other parameters mentioned above, it has been shown in [19, 20] that

$$s \equiv m^*/m_e = A_1 \frac{(\gamma/v_g)^{2/3}}{E_F^{1/3}} \tag{17}$$

$$n_s(E_F) = A_2 (\gamma/v_g) E_F \tag{18}$$

$$P_0(E_F) = A_3 \frac{\theta}{y} \frac{(\gamma/v_g)^{1/3}}{E_F^{2/3}} \tag{19}$$

$$v_0(E_F) = A_4 \frac{\theta}{y} \frac{1}{(\gamma/v_g)^{1/3} E_F^{1/3}} \tag{20}$$

and hence

$$j_0(E_F) = \frac{n_s(E_F)}{2} e^* v_0(E_F) = A_5 \frac{\theta}{y} (\gamma/v_g)^{2/3} E_F^{2/3}, \quad (e^* = 2e) \tag{21}$$

where e is the electronic charge, and
$A_1 \cong 3.305 x 10^{-10} \text{ eV}^{-1/3} \text{cm}^2 \text{K}^{4/3}, A_2 \cong 2.729 x 10^7 \text{ eV}^{-2} \text{K}^2,$
$A_3 \cong 1.584 x 10^{-6} \text{ cm K}^{-1/3}, A_4 \cong 1.406 x 10^8 \text{ eV}^{2/3} \sec^{-1} \text{K}^{-5/3}, \text{ and}$
$A_5 \cong 6.146 x 10^{-4} \text{ CeV}^{-4/3} \text{K}^{1/3} \text{s}^{-1}.$

2.5 E_F-dependent equation for $y = (k\theta/P_0)\sqrt{2m/E_F}$

If the T_c and Δs of an SC are studied via Eqs. (10)–(12), which are valid when $E_F \gg k\theta$, then it is appropriate to employ Eq. (16) for y because it too is valid when the same inequality is satisfied. If it turns out that we need to employ the μ-incorporated Eqs. (13)–(15) in lieu of Eqs. (10)–(12), then consistency demands that we employ for y, in lieu of Eq. (16), an equation that too is explicitly μ-dependent.

The desired equation for y in the 3PEM scenario, obtained by setting $P_\mu = (E, \mathbf{P})$ in the BSE and doing away with the earlier constraint on E_F, is [21]:

$$1 = \frac{\lambda_1}{4} T(\Sigma_1', r_1 y) + \frac{\lambda_2}{4} T(,\Sigma_2', r_2 y) + \frac{\lambda_3}{4} T(\Sigma_3', r_3 y),$$

(22)

where

$$T(\Sigma', y) = \mathbf{Re}\left[\int_0^1 dx F(\Sigma', x, y)\right], \quad F(\Sigma', x, y) = f_1(\Sigma', x, y) + f_2(\Sigma', x, y)$$

$$f_1(\Sigma', x, y) = 4\left[-u_1(\Sigma', x, y) - u_2(\Sigma', x, y) + u_3(\Sigma', x, y) + u_4(\Sigma', x, y)\right],$$

$$f_2(\Sigma', x, y) = 2\ln\left[\frac{1 + u_1(\Sigma', x, y)}{1 - u_1(\Sigma', x, y)} \frac{1 + u_2(\Sigma', x, y)}{1 - u_1(\Sigma', x, y)} \frac{1 - u_3(\Sigma', x, y)}{1 + u_3(\Sigma', x, y)} \frac{1 - u_4(\Sigma', x, y)}{1 + u_4(\Sigma', x, y)}\right]$$

$$u_1(\Sigma', x, y) = \sqrt{1 - \Sigma' x/y}, \quad u_2(\Sigma', x, y) = \sqrt{1 + \Sigma' x/y}$$

$$u_3(\Sigma', x, y) = \sqrt{1 - \Sigma'(1 - x/y)}, \quad u_4(\Sigma', x, y) = \sqrt{1 + \Sigma'(1 - x/y)}$$

$$\Sigma' = k\theta/E_F, \quad \Sigma_i' = k\theta_i/E_F, \quad r_i = \theta_i/\theta.$$

Equating one (two) of the λs in Eq. (22) to zero, we obtain the μ-dependent equation for y in the 2PEM (1PEM) scenario.

3. Superconducting features of $Bi_2Sr_2CaCu_2O_8$ addressed in this paper

The reported empirical values of $\{T_c, \Delta\}$ of the above SC are [22], p 447:

$$\{T_c \text{ (K)}, \Delta \text{ (meV)}\} = \{95, 38\}, \{86, 28\}, \{62, 18\}.$$

(23)

As concerns empirical values of j_0 of the SC, the situation is rather complicated because 14 values of this parameter have been reported in [22], p 468—depending upon the shape of the SC, how it is doped, and the values of temperature and the external magnetic field H under which they were determined. However, none of these values corresponds to the T = H = 0 scenario employed by us. The value $j_0 = 10^6$ A/cm² at T = 4.4 K and unspecified value of H being closest to it. Another value of interest for us is [23]:

$$j_0 = 3.2 \times 10^7 \text{ A/cm}^2 \ (T = 4.2 \text{ K}, H = 12 \text{ Tesla}).$$

(24)

In the following we give a quantitative explanation of the values of the parameters noted in Eqs. (23) and (24) and a plausible explanation of the multitude of the other values of these parameters.

4. An explanation of the multiple $\{T_c, \Delta, j_0\}$-values of Bi-2212 via GBCSEs

4.1 Earlier work

In our earlier work dealing with the above SC [10], we had employed GBCSEs constrained by inequality (8) and had restricted ourselves to the 2PEM scenario. The maximum value of Δ that we could then account for, subject to the Bogoliubov

constraint, was 20.4 meV corresponding to T_c = 95 K. Subsequently, even after employing μ-incorporated GBCSEs which are not constrained by inequality (8), we found that the 2PEM scenario was inadequate in explaining a value of Δ exceeding 20.4 meV [20, 21]. What remained elusive in these studies were especially the value Δ = 38 meV and the other values of T_c and Δ noted in Eq. (23). We show below how this lacuna is met via 3PEM in the present work.

4.2 GBCSEs constrained by inequality $E_F \gg k\theta$: premises and results

Since working in the 2PEM scenario, both with the μ-independent and μ-dependent GBCSEs, the maximum value of Δ satisfying the Bogoliubov constraint was found to be 20.4 meV; we assume here that among the values noted in Eq. (23),

$$\Delta_2 = 18 \text{ meV}, \Delta_3 = 38 \text{ meV}, T_{c3} = 95 \text{ K}, \tag{25}$$

i.e., as our notation suggests, we attribute the first parameter above to 2PEM and the other two parameters to 3PEM. With θs as given in Eq. (6), the solutions of Eqs. (10–12) lead to $\{\lambda_{Ca}, \lambda_{Sr}, \lambda_{Bi}\} = \{42.7, -37.6, 1.38\}$. Since these values are in conflict with the Bogoliubov constraint, in the absence of any other handle, we now systematically fine-tune the input values of the two Δs and T_c. We thus find that λs satisfying this constraint can be found only for values of Δs and T_{c3} that are substantially different from their empirical values in Eq. (23). An example, in order to obtain the set $\{\lambda_{Ca}, \lambda_{Sr}, \lambda_{Bi}\} = \{0.4652, 0.2291, 0.4856\}$ which satisfies the required constraint, we have to choose input values as $\Delta_2 = 14 meV$, $\Delta_3 = 33.5 \, meV$, and $T_{c3} = 141$ K. This is an unacceptable state of affairs, suggesting the need to have a handle other than variation of the Δs- and T_c-values of the SC to satisfy the Bogoliubov constraint. We show below how the μ-incorporated Eqs. (13)–(15), which are not constrained by inequality (8), meet this need.

4.3 Interaction parameters obtained via μ-incorporated GBCSEs

Before attempting to find the three λs satisfying the Bogoliubov constraint by employing Eqs. (13)–(15) for the same inputs as in Eqs. (6) and (25) and using μ as a handle, as a consistency check of these equations, we solve them for a value of $\mu_0 \gg k\theta$, say, $\mu_0 = 100 \, k \, \theta$. For this value of $\mu_0 = \mu_1$, or any other greater value, we find that $\lambda_{Ca} = 42.69$, $\lambda_{Sr} = -37.59$, and $\lambda_{Bi} = 1.38$, which are precisely the values we had obtained above via Eqs. (10)–(12) for the same inputs. This establishes that Eqs. (13)–(15) provide a reasonable generalization of Eqs. (10)–(12) and may therefore be used for any values of μ_0 and μ_1. If we now solve Eqs. (13)–(15) with the same inputs as before, but with progressively lower values of $\mu_0 = \mu_1$, we find that there is a marginal *increase* in the value of each of the λs; an example, for $\mu_0 = \mu_1 = 5 \, k \, \theta$, $\lambda_{Ca} = 42.90$, $\lambda_{Sr} = -37.78$, $\lambda_{Bi} = 1.39$. This suggests that the assumption that $\mu_0 = \mu_1$ may not be valid. To check this, we solve our equations again with $\mu_0 = 5 \, k \, \theta$ and $\mu_1 = 0.2 \, \mu_0$, to find that $\lambda_{Ca} = 16.46$, $\lambda_{Sr} = -15.56$, and $\lambda_{Bi} = 0.1144$. It is thus confirmed that in adopting a low value of μ_0, with μ_1 a small fraction of it, we are proceeding in the right direction. Following this course, we obtain the desired set of λ values corresponding to the T_c and Δ values in Eq. (25) as

$$\{\lambda_{Ca}, \lambda_{Sr}, \lambda_{Bi}\} = \{0.3123, 0.4993, 0.5000\}, \tag{26}$$

when

$$\mu_0 = 1.95 \text{ k}\theta \ (39.82 \text{ meV}), \mu_1 = 0.105 \, \mu_0 \ (4.18 \text{ meV}). \tag{27}$$

4.4 Calculation of y via the μ-dependent Equation (22)

A consistency check of Eq. (22): While both the μ-independent Eq. (16) and the μ-dependent Eq. (22) for y require as input the values of θs and λs, the latter equation requires the additional input of the value of E_F [i.e., μ_0, as noted below Eq. (15)] for its solution. We recall that Eq. (16) was derived by assuming that $\mu_0 \gg k\theta$. Therefore if Eq. (16) is solved for any given values of θs and the λs, and Eq. (22) is solved with the same inputs together with a value of μ_0 which is much greater than $k\theta$, then the solutions of both the equations should yield the same value for y. To carry out this test in detail, we solve Eq. (22) for values of θs as in Eq. (6) and those of λs as in Eq. (26) for the following seven cases: $\lambda_2 = \lambda_3 = 0$, $\lambda_3 = \lambda_1 = 0$, $\lambda_1 = \lambda_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, and $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0$. With the same values of the θs and λs and $\mu_0 = 100\,k\theta$, solutions of Eq. (22) yield exactly the same results as were obtained via Eq. (16). Some of the values so-obtained are as follows: when $\lambda_2 = \lambda_3 = 0$ (1PEM scenario), y = 9.458; when $\lambda_2 = 0$ (2PEM scenario), y = 1.661; and when none of the λs = 0 (3PEM scenario), y = 1. 195. Having thus shown that Eq. (22) is a reasonable generalization of Eq. (16) for arbitrary values of μ_0, we now solve it for μ_0 (= E_F) as given in Eq. (27) and values of {λ, θ} for each of the seven cases just noted. The results of these calculations are given in **Table 1**.

S. no.	$\lambda_1\lambda_2\lambda_3$	$\theta_1\theta_2\theta_3$ (K)	$|W|$ (meV)	T_c (K)	y	v_0 x10^5 (cms^{-1})	j_0 x10^7 (Acm^{-2})	Remark
1	0.3123 0 0	237 - -	1.70	8.5	9.688	1.41	0.9	1PEM (Ca)
2	0 0.4993 0	- 286 -	7.40	44.4	2.726	4.99	3.2	1PEM (Sr)
3	0 0 0.5000	- - 269	7.01	28.5	2.886	4.72	3.0	1PEM (Bi)
4	0 0.4993 0.5000	- 286 269	26.3	76.8	1.326	10.3	6.6	2PEM (Sr + Bi)
5	0.3123 0 0.5000	237 - 269	17.4	54.1	1.671	8.15	5.2	2PEM (Ca + Bi)
6	0.3123 0.4993 0	237 286 -	18.0	62.4	1.613	8.44	5.4	2PEM (Ca + Sr)
7	0.3123 0.4993 0.5000	237 286 269	38	95	1.199	11.4	7.3	3PEM (Ca + Sr. + Bi)

$|W|$ and T_c are obtained by solving Eqs. (14) and (15), respectively, with the input of $\mu_0 = 1.95\,k\theta$ ($\theta = 237\,K$), $\mu_1 = 0.105\,\mu_0$, different combinations of λ-values as given in Eq. (26), and the corresponding θ-values given in Eq. (6). For each set of $\{\mu_0, \theta, \lambda\}$, y is obtained by solving Eq. (22). With $E_F = \mu_0$, $\gamma = (8/15)\,x10^{-3}\,J$, $v_g = 9.05\,cm^3/g\text{-}at$, and s = m/m$_e$ = 4.97, vide Eq. (17), and $n_s = 4.0 \times 10^{20}/cm^3$, vide Eq. (18). v_0 and j_0 corresponding to each value of y are obtained via Eqs. (20) and (21), respectively, with $\theta = 237\,K$.*

Table 1.
Values of various superconducting parameters of $Bi_2Sr_2CaCu_2O_8$ obtained via GBCSEs.

4.5 The multiple values of the set $\{|W| = \Delta, T_c, j_0\}$ obtained via GBCSEs

Given in **Table 1** are also the values of the set $\{|W| = \Delta, T_c, j_0\}$ and the constituents of j_0 corresponding to each of the seven choices of λs mentioned above and the values of μs given in Eq. (27). Among these, the values of $|W|$ and T_c are obtained via Eqs. (14) and (15), respectively. With E_F (= μ_0) fixed as in Eq. (27) and the values of γ and v_g as noted in the legend for **Table 1**, s and n_s are calculated via Eqs. (17) and (18), respectively. The values of v_0 and j_0 in each row of **Table 1** are obtained via Eqs. (20) and (21), respectively, with the input of θ = 237 K, the value of y noted in it, and the common values of γ, v_g, and E_F.

Excepting $|W|$ = 18 meV in the sixth row, and $|W|$ = 38 meV and T_c = 95 K in the last row of **Table 1**, which were employed as inputs to obtain the three λs, the values of all the other parameters in the Table are predictions following from our approach. Among these, T_c = 62.4 K, corresponding to $|W|$ = 18 meV in the sixth row, is in excellent agreement with the experimental value, whereas the values of the subset $\{T_c (K), |W| (meV)\}$ = $\{76.8, 26.3\}$ in the fourth row are in reasonable agreement with the experimental values of $\{86, 28\}$ [see Eq. (23)]. As concerns j_0, we recall that the experimental value of 3.2 x 10^7 A cm^{-2} noted for it in Eq. (24) corresponds to T = 4.2 K and H = 12 Tesla. Since our value of j_0 = 7.3 x 10^7 A cm^{-2} (in the last row of the Table) corresponds to T = H = 0, it too may be regarded as in reasonably good agreement with the experimental value. Since our approach also provides values of the constituent parameters of j_0, such as n_s and v_0, there is a need to monitor these parameters via experiment for its further validation.

We have so far been concerned with the values of $\{T_c, \Delta\}$ as noted in Eq. (23). The significance of the additional values of T_c that GBCSEs have led to, as seen from **Table 1**, is as follows: these are the temperatures at which CPs bound via one- or more-phonon exchange mechanism breakup. Any such T_c corresponds to vanishing of the associated gap noted in the same row. One would thus expect to see, via appropriately sensitized experimental setups, some of these T_cs in the resistance vs. temperature plot of the SC. We believe that this expectation is met via the experimental results reported in [24] (reproduced in [25]) where, in a plot of resistance vs. temperature for the Bi-Sr-Ca-Cu-O system obtained via a setup sensitized to determine T_cs $\geq$ 75 K, there occur discontinuities at 75, 80, 85, 108, and 114 K. While these discontinuities are generally attributed to the presence of more than one phase in the SC, they may well alternatively/additionally be due to the breakdown of one or the other mechanism responsible for pairing.

4.6 Are the results given in Table 1 stable?

To address the above question, which is as relevant in the present context as it is for the numerical solution of any problem, we repeated the entire exercise carried out above by taking the Δ- and the T_c-values of the SC as

$$\Delta_2 = 17.5 \text{ meV}, \Delta_3 = 37.5 \text{ meV}, T_{c3} = 94.5 \text{ K}, \tag{28}$$

which deviate slightly from those considered earlier and noted in Eq. (25). A brief account of the results following from Eq. (28) is as follows:

 i. In order to satisfy the Bogoliubov constraint, we found that we needed

$$\mu_0 = 2.4 \, k \, \theta = 49.02 \text{ meV} \, (\theta = 237K) \text{ and } \mu_1 = 0.1 \, \mu_0,$$

i. which led to

$$\{\lambda_{Ca}, \lambda_{Sr}, \lambda_{Bi}\} = \{0.3628, 0.4338, 0.4918\},$$

i. and to values of y and j0 in the 1-, 2-, and 3-PEM scenarios as

$$\lambda_2 = \lambda_3 = 0 \text{ (1PEM)}, y = 6.353, j_0 = 1.58x10^7 \text{ A/cm}^2$$
$$\lambda_1 = 0 \text{ (2PEM)}, y = 1.39, j_0 = 7.21x10^7 \text{ A/cm}^2$$
$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0 \text{ (3PEM)}, y = 1.214, j_0 = 8.24x10^7 \text{ A/cm}^2.$$

Since the sets of values given in Eqs. (25) and (28) lead to values of Δs, T_c, and j_0 in the same ball park, it is seen that our results are stable with respect to small variations in the input variables.

5. Some other applications of GBCSEs

5.1 La$_2$CuO$_4$

La$_2$CuO$_4$ is unique not only because it heralded the era of high-T_c superconductivity but also because explanation of its $T_c \simeq 38$ K via GBCSEs requires invoking 2PEM, whereas it has only kind of ions (La) that can cause pairing. This paradoxical situation is resolved by appealing to the structure of the unit cell of the SC and employing Eqs. (4) and (5) to determine θ_{La}. Since the unit cell comprises layers of LaO and OLa, if La is considered as the upper bob of the double pendulum in one of these, then it must be the lower bob in the other. Eqs. (4) and (5) therefore lead to two values of θ_{La}. We thus have 2PEM in operation due to $\{\lambda, \theta_{La1}\}$ and $\{\lambda, \theta_{La2}\}$. Following this approach [10], we were able to account for all the values of $(2\Delta_0/K_B T_C)$, i.e., 4.3, 7.1, and 9.3, which were reported by Bednorz and Müller in their Nobel Lecture in 1987.

5.2 Heavy fermion SCs (HFSCs)

Superconductivity in HFSCs—so-named because a conduction electron in them behaves as if it has an effective mass up to three orders of magnitude greater than its free mass—was discovered by Steglich et al. [26]. Even though the highest T_c reported so far for this family of compounds is only 2.3 K (for CeCoIn$_5$), these SCs have been the subject of avid study by theoreticians because of their following unusual properties:

$$E_F < k\theta, \, kT_c < E_F, \, T_c/T_F \approx T_F/\theta \approx 0.05, \quad (29)$$

where T_F is the Fermi temperature. Besides the properties noted above, HFSCs are characterized by (a) large heat capacities of conduction electrons, much larger than those found for elemental SCs and about as large as those associated with fixed magnetic momenta, and (b) anisotropy of their gap structures. These features caused the term *exotic* or *unconventional* to be coined for them and led to the revival of an old idea that superconductivity may also arise due to the exchange of magnons, rather than just phonons. As a follow-up of this idea, it was shown in three well-known papers [27–29] that several experimental features of HFSCs can be explained if one assumes that magnetic fluctuations are the cause of d-wave

pairing in them. By and large, this seems to be the currently popular view about these SCs.

One of the reasons for regarding HFSCs as outside the purview of BCS theory is the conflict between inequality (8), which is a tenet of the theory, and inequalities (29) which the HFSCs are found to satisfy. Since μ-incorporated GBCSEs are not constrained by inequality (8), we employed them for a detailed study of $CeCoIn_5$, which is a prominent member of the HFSC family. Salient features of this study are as follows. (i) The pairing mechanism was assumed to be 1PEM because we are dealing with a low value of T_c. (ii) Since 1PEM can be due to either Co or Ce ions, we considered both of these possibilities. (iii) Using Eqs. (4) and (5) and θ ($CeCoIn_5$) = 161 K, we found $θ_{Ce}$ = 73 (276) K, corresponding to Ce as the lower (upper) bob in the double pendulum in the layer containing $CeIn_3$, and $θ_{Co}$ = 98.7, (294) K (in the layer containing $CoIn_3$). (iv) Upon solving the μ-incorporated GBCSE for Δ, and different values of μ in an appropriate range determined via a consideration of inequalities (29), we were led to a multitude of Δ values in the range (2.76–3.49) × 10^{-4} eV corresponding to the single value of T_c = 2.3 K. For the SC under consideration, this is in qualitative accord with the experimental finding via the Bogoliubov quasiparticle interference (QPI) technique [the highest Δ value thus found is $(5.5 ± 0.05) × 10^{-4}$ eV, whereas our corresponding value based on the mean-field approximation is $3.49 × 10^{-4}$ eV]. (v) Some other results $\left[n_s = (1/3π^2)(2mμ_1/ℏ^2)^{3/2} \right]$ are as follows:

$$7.84x10^{21} (μ_1 = 23.81 \text{ meV}, s = 60.4) ≥ n_s (\text{cm}^{-3}) ≥ 1.07x10^{20} (μ_1 = 0.324 \text{ meV}, s = 253)$$
$$2.59x10^{22} (μ_1 = 25.34 \text{ meV}, s = 126) ≥ n_s (\text{cm}^{-3}) ≥ 3.31x10^{20} (μ_1 = 0.324 \text{ meV}, s = 539),$$

where the upper row corresponds to pairing via the Ce ions and the lower row to pairing via the Co ions.

It follows from even the brief account given above that the framework of μ-incorporated GBCSEs provides a viable explanation of the empirical features of $CeCoIn_5$ as an alternative to the "popular view" about them; for a more detailed treatment of the SC, we refer the reader to [30].

5.3 Fe-based SCs

The salient superconducting properties of $Ba_{0.6}K_{0.4}Fe_2As_2$, which are generic of the Fe-based SCs (also called iron-pnictide SCs), are as follows (for sources of these properties, see [31]): (i) Superconductivity in this SC is due to the $s^{±}$-wave state, which signifies that a gap below the Fermi surface and the one corresponding to it above the Fermi surface, $Δ_h$ and $Δ_e$, respectively, has opposite signs. (ii) In general, $Δ_h ≠ Δ_e$. (iii) T_c = 37 K. (iv) The highest T_c reported for this class of SCs > 50 K. (v) Prominent Δ-values: 6, 12 meV. (vi) Near-zero values of Δ: Δ for the SC can fall to zero along lines of its Fermi surface. (vii) Some other reported values of Δ (meV): 2.5, 9.0; 3.3, 7.6; 3.6, 8.5, 9.2; 4, 7, 12, 9.5. (viii) j_0: exceeds 0.1 x 10^6 Acm^{-2} at T = 4.2 K and H = 0; 1.1 x 10^7 Acm^{-2} at T = 2 K and H = 0. (ix) A characteristic ratio for the SC: E_F/kT_c = 4.4. (x) One of the s = m^*/m_e values: 9.0 (based on ARPES and a four-band model). (xi) Coherence length ξ = 9–14 Å. (xii) The T_c of the SC plotted against a tuning variable has a dome-like structure.

By adopting the framework of GBCSEs as outlined above, it has been shown in [31] that one can quantitatively explain *all* the above features of $Ba_{0.6}K_{0.4}Fe_2As_2$—without invoking a new superconducting state for the SC. This is an important remark because, on the basis of the multiband approach, it has been suggested in a recent review article [32] that superconductivity in Fe-based SCs is manifestation of

a *new* state. One of the reasons for this could well be the fact that the BCS equation for Δ is quadratic in Δ and is therefore unaffected when $\Delta \rightarrow -\Delta$. On the other hand, GBCSE for W_1 is linear in this variable and has been derived by assuming that W_1 undergoes a change in signature upon crossing the Fermi surface. This is also a feature of GBCSEs for W_2, etc. Since $s^{\pm}$-wave is an inbuilt feature of GBCSEs, one does not have to invent a new state for any SC, as has been suggested in [32].

5.4 Isotope-like effect for composite SCs

The following relation between T_c and the average mass of ions M in an elemental SC is well-known as the isotope effect:

$$T_c \infty M^{-\alpha}. \tag{30}$$

While in the BCS theory $\alpha = 0.5$, values significantly different from it have also been found for some elements such as Mo, Os, and Ru, for which $\alpha = 0.33$, 0.2, and 0, respectively. Therefore, while Eq. (30) does not have the status of a *law*, it nonetheless helped in the formulation of BCS theory because it sheds light on the role of the ion lattice in the scenario of 1PEM, which has been shown to be the operative mechanism for elemental SCs.

We now draw attention to the fact that when Bi and Sr. in $Bi_2Sr_2CaCu_2O_8$ are replaced by Tl and Ba, respectively, the T_c of the SC increases from 95 to 110 K. Consequent upon these substitutions, the only property that can be unequivocally determined is the mass of the SC. Because the operative mechanism for pairing in composite SCs is not 1PEM, it becomes interesting to ask if Eq. (30) can be generalized to address the scenarios of 2PEM and 3PEM. A generalization of Eq. (30) that suggests itself naturally for the 2PEM scenario is

$$T_c = p(M_1 M_2)^{-\alpha}, \tag{31}$$

where p is the constant of proportionality and M_1 and M_2 are the masses of ion species that cause pairing. Assuming that the value of $T_c = 95$ K for $Bi_2Sr_2CaCu_2O_8$ is due to 2PEM as in [25], an explanation for the increase in the T_c of $Bi_2Sr_2CaCu_2O_8$ when $Tl_2Ba_2CaCu_2O_8$ is obtained from it via substitutions was given in [33]. Noting that Bi and Tl belong to the same period and Sr. and Ba to the same group of the periodic table, several suggestions were made in [33] to further increase the T_c of the Bi-based SC. An example, following from our study, it was shown that T_c ($Bi_2 Mg_2 CaCu_2O_8$) should be 171 K.

6. Discussion

The T_cs and Δs of most of the hetero-structured, multi-gapped SCs which were studied above via the GBCSE-based approach have also been studied via the more-widely followed multiband approach (MBA) which originated with the work of Suhl et al. [34]. Because the former approach sheds light on additional features of these SCs, such as j_0, s, n_s, etc., as also because of its distinctly different conceptual basis, it complements the latter approach. Since the conceptual bases of both the approaches have been dealt with in detail in a recent paper [35], we confine our-selves here to the following remarks:

    i. Employing the concept of a superpropagator, the GBCSE-based approach invariably invokes a λ for each of the ion species in an SC that may cause

pairing. One then has the same λs in the equations for any Δ and the corresponding T_c of the SC—as is the case for elemental SCs. Multiple gaps arise in this approach because different combinations of λs operate on different parts of the Fermi surface due to its undulations. For a discussion of the moot question concerning the assumption of *locally* spherical values characterizing the Fermi surface of a composite SC, see [33].

On the other hand, the number of bands employed in MBA for the same SC differs from author to author. In this approach, in the two-band models, multiple gaps arise because the Hamiltonian is now postulated to have a term for pairing in each of the bands and another term corresponding to crossband pairing. For the T_c of the SC, one often employs the Migdal-Eliashberg-McMillan approach [36] which, even though based on 1PEM, allows λ to be greater than unity because it is based on an integral equation and the expansion parameter of which is not λ but m_e/M, where M is the mass of an ion species.

ii. Because of the feature of the GBCSE-based approach discussed above, with the input of the values of any two gaps of an SC and a value of T_c, it goes on to shed light on several other values of these parameters. This is not so for MBA.

7. Conclusions

i. Besides the SCs dealt with above, the GBCSE-based approach has been applied for the study of several elemental SCs, MgB_2, YBCO and Tl-2212 [10, 19, 20], $SrTiO_3$ [10], and NbN [21]. It is hence seen that a striking feature of this approach is its versatility: it is applicable to a wide variety of SCs that includes the so-called exotic or unconventional SCs such as the HFSCs and the Fe-based SCs, without invoking a new superconducting state for the latter.

ii. We believe to have shown that in explaining the empirical features of high-T_c SCs, the GBCSE-based approach goes farther than MBA or, so far as we are aware, any other approach, e.g., [37].

iii. The fact that T_c ($Bi_2Sr_2Ca_2Cu_3O_{10}$) > T_c ($Bi_2Sr_2CaCu_2O_8$) > T_c ($YBa_2Cu_3O_7$) > T_c (MgB_2), etc., obviously suggests that the greater the complexity of structure of the SC, the greater is its T_c owing to the increase in the number of channels that may cause formation of CPs. While this is a situation that the GBCSE-based approach can easily deal with, in practice, going beyond $Bi_2Sr_2Ca_2Cu_3O_{10}$, for example, is most likely to lead to an unstable and hence unrealizable SC. This is a problem that we believe belongs to the realm of chemical engineering.

iv. It was noted above that 14 different values of j_0 have been reported for Bi-2212. For none of these values were reported the values of the other superconducting parameters of the SC. On the basis of [19, 20], we believe that, in order to help theory to suggest means to increase the T_c of an SC, the report of any of its properties, for example, j_0, should be accompanied by, in so far as it is experimentally feasible, a list of all its other superconducting properties, viz., θ, T_c, Δs, m^*, v_0, n_s, γ, and v_g.

v. The E_F-incorporated GBCSEs in this communication correspond to the T_c of a composite SC and its Δ-values at T = H = 0. These equations can be further

generalized to deal with the situation when the SC is in heat bath in an external magnetic field, i.e., when $T \neq H \neq 0$, a procedure for which has been given in [38]. The import of this remark is that not only will such an undertaking enable theory to address the j_0 values of an SC which are generally reported for such values of T and H but also that it may shed new light on the Volovik effect for the Fe-based SCs, as discussed in, e.g., [32].

Conflict of interest

The authors declare that there is no conflict of interest with regard to this submission.

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