Chapter

Investigation and Synthesis of Robust Polynomials in Uncertainty on the Basis of the Root Locus Theory

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Abstract

The root locus method is proposed in the chapter for searching intervals of uncertainty for coefficients of the given (source) polynomial with constant or interval coefficients under perturbations, which ensures its robust stability regardless of whether the given polynomial is Hurwitz or not. The method is based on introduction and application of the "extended root locus" notion. Polynomial adjustment is performed by setting up each one of its coefficients separately and sequentially and determining permissible values of coefficient variation intervals (intervals of uncertainty). The effect of each coefficient variation upon the polynomial root dynamics (behavior) is considered and analyzed separately, and this influence could be observed in the root locus portraits. Root locus method is thus generalized to the cases when the number of polynomial variable coefficients is arbitrary. The root locus parameter distribution diagram along the asymptotic stability bound is introduced and applied for observing the roots behavior regularities. On this basis, the stability conditions are derived, and analytical and graphic-analytical methods are worked out for calculating intervals of variation for the 4th order polynomial family parameters ensuring its robust stability. It also allows to extract Hurwitz subfamilies from the non-Hurwitz families of interval polynomials and to determine whether there exists at least one stable polynomial in the unstable polynomial family.

Keywords: polynomial, dynamic system, uncertainty, stability, robustness, root locus portrait, extended root locus, root locus parameter function

1. Introduction

As it is emphasized in [1, 2], the tasks of analysis and synthesis of control processes occurring in dynamic systems of different physical nature, operating in conditions of substantial plant parametric uncertainty, including the engineering ones, are currently the most urgent and challenging within the framework of the control theory. Among these tasks, one could mention the problem of flux control for the electric motor vector control systems operating in uncertainty because the flux control quality strongly affects the electromagnetic torque and speed control quality, and thus the drive power efficiency. For this reason, of great importance are the tasks of stability investigation and parametric synthesis of robust control systems (their characteristic polynomials) for the plants which parameters vary within the given or unknown intervals of values.

In the area of investigation and synthesis of dynamic system characteristic polynomials, there exists a lot of approaches and methods. For the first time, the necessary and sufficient conditions for systems up to the 3-rd order were formulated by James Maxwell in 1868. Later appeared the stability criteria of Routh–Hurwitz, Mikhajlov, Nyquist, and Bode, which made it possible to check stability of the systems of order n. The frequency Nyquist criterion was the first one that could be used for synthesis by estimation of the system degree of stability. Among the modern methods of synthesis [1, 2] together with the frequency ones, the root locus and state-space methods could be listed. In his book [1], Jurgen Ackermann gives, in particular, the algebraic approach to uncertainty considering different, including the nonlinear, types of the coefficient functions and generating stability regions in the parameter space of real physical parameters of the system (polynomial). The main results in the area of the frequency approach to analysis and synthesis of robust dynamic (control) systems are given in [3], where the stability of uncertain polynomials, including interval ones, is also considered.

The methods for analysis and synthesis of polynomial families represent a separate group. One of the most effective solutions for the task of interval polynomial family investigation within the algebraic approach has been proposed by Kharitonov [4], where in the general case, the task of polynomial stability analysis is reduced to consideration of only four specific polynomials of the whole family with constant coefficients. In [3, 5], the frequency criteria of Hurwitz robust stability are considered, which allow to define the coefficient perturbation sweep for the nominally stable polynomial and various types of uncertainties. Hurwitz robust stability is also investigated in [6–10]. In [6], the maximal deviation intervals of perturbed Hurwitz polynomial coefficients assuring strict Hurwitz property are determined on the basis of the algebraic method worked out using Kharitonov's polynomials [4]. The similar task is solved in [7] but using the Hermite-Biler theorem, which allows to reduce twice the power of investigated polynomial. The way for calculation of perturbed polynomial coefficients' maximal limit values that guarantee sector stability is given in [8]. The linear dependence of coefficient perturbation is considered by Bartlett, et al. for a class of polynomial families generated by convex polytopes in the coefficient space [9]. Here the so-called edge theorem was proved assuring derivation of the stability analysis task to investigation of root location for the finite number of the parametric families. The edge theorem allows to analyze both stability and quality characteristics of the family. A combination of the stochastic and worst-case approaches to the problem of uncertainty is proposed in [10]. It certainly widens the scope of types of treatable uncertainties and reduces conservatism. However, it works properly only in the cases permitting an arbitrarily small probability of specification violation. Thus, to the specific extent, it still bears the drawbacks of the stochastic approach to control, which guarantees only the "average" performance.

An analog of Kharitonov theorem [11] was formulated for the unstable interval polynomials' homogeneous classes of equivalence. Criteria of existence of such classes of equivalence were obtained. Based on the new interval polynomial stability criterion and Lyapunov theorem, a robust optimal proportional-integral-derivative (PID) controller is proposed in paper [12] to carry out design for different plants that contain perturbations of multiple parameters. A new stability criterion of the interval polynomial is presented to determine whether the interval polynomial belongs to Hurwitz polynomial or not. Time-delay systems involving multiple imaginary roots (MIRs) and their stability analysis, which becomes much more complicated than that in the case with only simple imaginary roots, are treated in [13]. For a class of time-delay systems, it was proved that the invariance between the multiple imaginary roots and the simple imaginary roots holds for any multiplicity as well as for the degenerate cases. In paper [14], monic complex polynomials are identified with the sets of their roots instead of being identified with the vectors of their coefficients. A proof is given that the space of Hurwitz polynomials of degree n with positive (resp. negative) coefficients is contractible and also that the space of monic (Schur or Hurwitz) aperiodic polynomials is contractible. A computational method to verify the stability of a convex combination of polynomials is considered in [15] and aimed at the robust stability analysis of a linear system. A simple algebraic test (a matrix inequality) for the stability of the segment of polynomials determined by the given two Hurwitz stable polynomials is proposed. Kučera gives a survey [16] where he navigates the area of the polynomial approach in the control system design technique. Such areas as parameterization of stabilizing controllers, called Youla–Kučera parameterization, are explained; the results on reference tracking, disturbance elimination, pole placement, deadbeat control, robust stabilization, and some others are described.

Of great interest are the problems of ensuring system stability and quality being solved in the modern statements of the problem [2] as tasks of guaranteeing system robustness, which could be solved by application of the root locus approach. The basic benefit of this approach is that its application itself, by its nature, implies parametric variations (i.e., uncertainty). The root locus approach is a powerful method used for the system synthesis [2] and is notable for its descriptiveness ensuring both calculation of the system robust parameters' values and possibility of detailed overview of the dynamic properties variation changes, the system response to uncertainties that is particularly important when investigating systems with uncertain and in particular interval parameters.

Root locus approach to the problem is considered in [17–23]. Paper [17] gives a solution for a compensator synthesis on the basis of the root locus method application. The task of a stable characteristic polynomial synthesis for the interval dynamic system (IDS) by setting up coefficients of the given (initial) unstable one for the case of location of its root locus initial point (where the variable parameter is equal to zero) family within the left half-plane is solved in [21], where the stability is attained via simple setting up the interval of the free term variation.

The above analyzed literature covers various approaches to the uncertainty treatment. However, most of the theoretical works are focused on the tasks of robust stability analysis. The methods for synthesis are not that widely represented, often suffer from complexity and in most cases are enough narrow, which means that they certainly provide instruments for system synthesis, but they are mostly "closed on themselves," which means that they do not provide the complete picture in the sense of showing up what is happening "under cover," which is especially important for the qualitative robust system (polynomials) synthesis. The root locus approach is rarely applied even though it represents the dynamic picture of the system response to uncertainties in the most comprehensive way and thus seems to be the most suitable one to deal with uncertainties.

As for polynomial families, the root locus approach gives us the transparent picture of root dynamics making it possible to see as if from the inside, for example, what subfamilies constitute the whole family of uncertain polynomials in terms of their configuration and stability or some other dynamic indicators bearing significant information about the system behavior and thus leading the way for its investigation and synthesis.

In this work, the root locus methods are described for calculating intervals of uncertainty for coefficients of the given (initial) stable or unstable polynomial with coefficients subject to perturbations, which ensure its robust stability. The proposed methods are based on introduction and application of the notions "extended root locus," "diagram of the root locus parameter function values distribution along the stability bound" and can be used for both synthesis of interval stable polynomials by setting up (adjusting) the unstable ones and analysis of the polynomial behavior under coefficient perturbations. The influence of every coefficient upon the polynomial behavior could be observed.

The work further develops results represented in the papers of Anderson [22] and Kharitonov [4] where they consider the issues of analysis and synthesis of robust interval polynomial families.

2. The problem formulation

Define a polynomial like

$$g_n(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n, \tag{1}$$

where a_j are given (initial) values of real polynomial coefficients, $j = 1, 2, \ldots, n$.

In the event of coefficient perturbations, a vector of coefficients of (1), $a = (a_1, \ldots, a_{n-1}, a_n)$, belongs to some connected set $A \subset R^n$, $a \in A$; n is a degree of the polynomial (integer value); s is a complex variable, $s = \sigma + i\omega$.

Suppose that coefficients of (1) vary within the following intervals:

$$\underline{a}_j \le a_j \le \overline{a}_j, \quad j = \overline{1, \ n}. \tag{2}$$

where $\underline{a}_j$ and $\overline{a}_j$ are the minimal and maximal limit values of closed interval (2) of coefficients a_j variation correspondingly. Polynomial (1) can be both, non-Hurwitz or Hurwitz one.

After substituting $s = \sigma + i\omega$, write the root locus and parameter equations [18] correspondingly:

$$v(\sigma, \omega) = 0, \text{ and} \tag{3}$$

$$a_n = u(\sigma, \omega), \tag{4}$$

where $u(\sigma,\omega)$ and $v(\sigma,\omega)$ are the real functions of two independent variables σ and ω.

The root locus method represents a powerful and effective tool for stable and qualitative polynomial synthesis and analysis. However, as it is known, this method allows to consider polynomials with only a single variable coefficient (parameter) and cannot be applied in the cases when all coefficients are uncertain. Therefore, the task is to generalize the root locus method for the cases when the number of variable coefficients is arbitrary and thus to solve the problem of investigation of the uncertain polynomial dynamics and working out methods for synthesis of the robustly stable uncertain (interval) polynomial by setting up the given polynomial (non-Hurwitz or Hurwitz) with constant/variable coefficients and determining intervals of all its coefficients (stability intervals) assuring its robust stability.

3. Root locus portraits of uncertain polynomials

Definition 1. The algebraic equation coefficient or the parameter of the dynamic system, described by this algebraic equation, which is being varied in a definite way for generating the root locus, when it is assumed that all the rest coefficients (parameters) are constant, is called the *root locus parameter* or *free parameter*.

If the root locus parameter is a_j, it is named the *root locus relative to parameter (coefficient)* a_j.

Definition 2. The root locus relative to the algebraic equation free term is called the *free root locus*.

Definition 3. Points, where the root locus branches begin and where the root locus parameter is equal to zero are called the *root locus initial points*.

Definition 4. The family P of root loci of interval polynomial (1) with coefficients varying within (2) name as the *interval polynomial root locus portrait (interval polynomial root locus)* or *interval dynamic system root locus portrait (interval dynamic system root locus).*

Let us along with the parameter a_n vary also parameter a_{n-1} of (1). Thus, we generate a (*free*) *root locus field* F_k (k = 1. 2, ...) in the plane s of system roots, which could also be named a *two-parameter root locus field* or a (*interval*) *root locus subfamily*. Parameter a_{n-1} used for the field generation is named a *root locus field parameter*.

It is evident that the root locus Eq. (3) represents also the equation of level lines of the *free root locus field* F_k. Root locus portrait P is then represented by the family of root locus fields,

$$P = \{F_k \mid k = 1, 2, \ldots\} \tag{5}$$

that represents the infinite set of root locus fields and therefore possesses their properties, and from the mathematical point of view, all root locus fields of P feature the same qualities. Therefore, the portrait P can be investigated as a single root locus field F_k.

Hereinafter the term “root locus” is used in the sense of “Teodorchik – Ewans free root locus” [18].

4. Polynomial analysis and synthesis based on the extended root locus

4.1 Extended root locus

Introduce the following system of polynomials:

$$E_n = \begin{cases} s + a_1 = g_1(s) & (6.1) \\ s^2 + a_1 s + a_2 = g_2(s) & (6.2) \\ \ldots & \\ s^i + a_1 s^{i-1} + \ldots + a_{i-1} s + a_i = g_i(s) & (6.i) \\ \ldots & \\ s^{n-1} + \ldots + a_{n-2} s + a_{n-1} = g_{n-1}(s) & (6.(n-1)) \\ s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = g_n(s) & (6.n) \end{cases}, \tag{6}$$

where

$$g_i(s) = s^i + a_1 s^{i-1} + \ldots + a_{i1} s + a_i, \tag{7}$$

$$g_{i-1}(s) = \left(g_i(s) - a_i\right)/s; \tag{8}$$

i–sequential number of the polynomial in (6), which is equal to its degree, $i = \overline{1, n}$; a_j—coefficients, $j = \overline{1, i}$.

Every polynomial (8) of (i−1) degree is generated from the i-degree polynomial supposed that $a_i = 0$. Polynomials of (6) have common coefficients, but not common roots.

Definition 5. System of polynomials (6) name as the *extension of polynomial* (1) or *extended polynomial*.

Definition 6. Complete set of extension (6) root loci name as the *extended root loci* of (1).

Extension E_n of polynomial $g_n(s)$ could be represented by the finite set of polynomials,

$$E_n = \{g_i(s)\}. \tag{9}$$

Statement 1. In case of variation of any coefficient $a_j, j = \overline{1, (i-1)}$, of polynomial $g_i(s)$ (7) within the specific interval, $\underline{a}_j \le a_j \le \overline{a}_j$, every initial point of its free root locus (excluding the point located at the origin) moves along its unique trajectory, representing itself one of the branches of polynomial $g_{i-1}(s)$ (8) root locus, generated relative to this coefficient, and its current position is determined at a point corresponding to the current value of a_j.

Proof. As at initial points of polynomial (6) free root locus the free term a_j is equal to zero, it is evident that (8) represents the equation of initial points of the free root locus of (6), that is, when varying a_j $\left(j = \overline{1, (i-1)}\right)$, the root locus of (8) relative to a_j represents the geometric place of initial points of the root locus of (7). Therefore, every initial point of the free root locus of (7) at fixed a_j coincides in the complex plane s with one of the polynomial (8) roots at the given value of a_j. It is evident, that while varying a_j, this root (and hence, this initial point) moves in the complex plane s, generating one of the $(i-1)$ branches (trajectories) of the root loci of (8) relative to a_j. Thus, the statement has been proved.

Definition 7. Name $g_{i-1}(s)$ (8) as the *originative polynomial* relative to $g_i(s)$ (7) and the root locus of (8)—*the originative root locus* of polynomial (7) free root loci.

Every (i−1)-th polynomial of (6) is the originative one relative to i-th polynomial (6).

Consequence 1. In case of continuous variation of the polynomial $g_i(s)$ coefficient a_j, $j = \overline{1, (n-1)}$, every branch of this polynomial root locus, initiated at the specific initial point, migrates continuously along the corresponding branch of the originative root locus relative to a_{j-1}, being the trajectory of this initial point, correspondingly in direction of increase or decrease of the originative root loci parameter a_j.

Consequence 2. If polynomial $g_{i-1}(s)$ being the originative one for the polynomial $g_i(s)$ is asymptotically stable, all initial points of polynomial $g_i(s)$ free root locus, excluding zero one, are located in the left half-plane s:

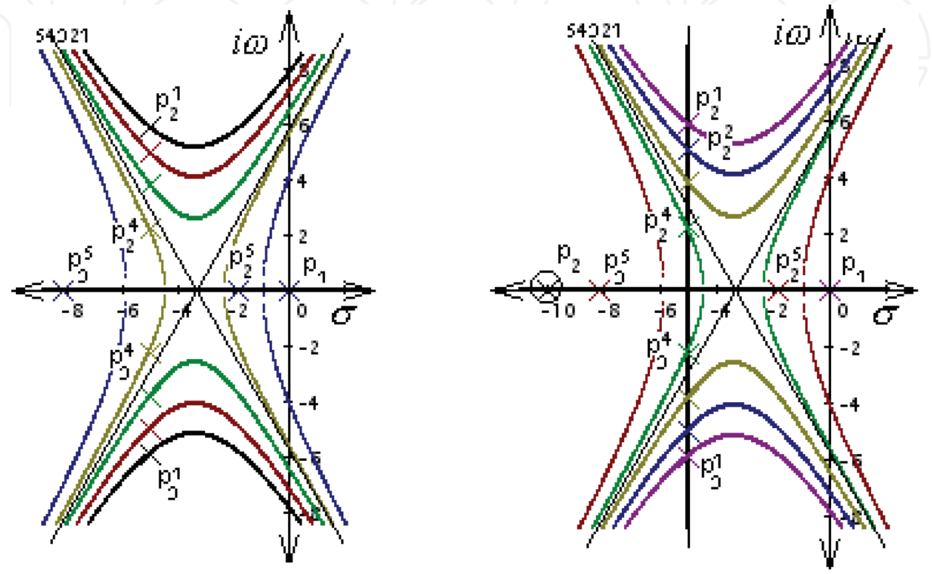
$$\forall s_\mu^{i-1}\left[\operatorname{Re} s_\mu^{i-1} < 0 \rightarrow \operatorname{Re} p_\mu^i < 0\right], \tag{10}$$

$$s_\mu^{i-1} = p_\mu^i, \tag{11}$$

where s_μ^i—roots of $g_i(s)$; p_μ^i—initial points of polynomial $g_i(s)$ free root locus; μ—root (initial point) sequential number, $\mu = \overline{1, i-1}$.

Further in the text, polynomial $g_{i\text{-}1}(s)$ free root locus is referred to as the *originative* one relative to that of $g_i(s)$ and $g_i(s)$ free root locus—as the *originated* one relative to that of $g_{i\text{-}1}(s)$.

Statement 1 is illustrated by **Figures 1** and **2**. Initial points here are designated by signs "x" (crosses) and letters "p" with the lower indexes, designating the point sequential numbers, and upper indexes, designating the sequential numbers of the corresponding root locus. The root locus sequential number is indicated by a digit next to its corresponding branch.



a) b)

Figure 1.
Polynomial (1) root locus portrait (field) at $n = 3$, $5 \le a_2 \le 45$: *(a) originated portrait and (b) originated portrait combined with its originative root locus* ($n = 2$).

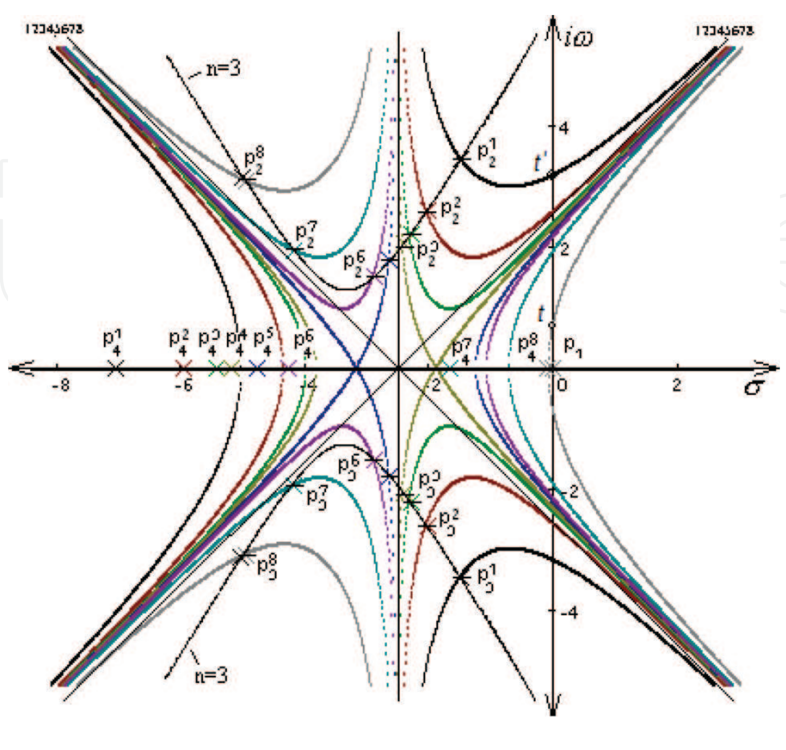


Figure 2.
Free root locus portrait (field) for polynomial $g_4(s) = s^4 + 10\,s^3 + 35\,s^2 + a_3 s + a_4$, $100 \le a_3 \le 5$ *combined with its originative root locus* ($n = 3$).

4.2 Synthesis of stable interval polynomials based on the extended root locus

Consider Eq. (3) in the sense of four following possible cases: n is uneven, $(n-1)/2$ is even/uneven, n is even, and $n/2$ is even/uneven. The root locus parameter equations (as it is in the general form see (4)) are composed in the same way.

Specify the set A_i^+ of a_i values at the cross points of polynomial (7) root locus positive branches with axis ω:

$$A_i^+ = \left\{a_i^+ l, l = \overline{1, n_i^+}\right\}, \tag{12}$$

where n_i^+ is a number of cross points.

Statement 2. If all initial points of polynomial (7) root locus, excluding a single one at the origin, are located in the left half-plane s, and this polynomial is asymptotically stabile, when the following condition holds:

$$0 < a_i < \inf A_i^+. \tag{13}$$

Proof. Based on the root locus properties [2, 18] and expressions (10) and (11), it can be stated, that provided all initial points of polynomial (7) root locus are located in the left half-plane s (excluding the initial point at the origin), the specific number n_i of root locus branches ($n_i = i - 2$ when i is even and $n_i = i - 1$ when i is uneven), initiating at these points, cross the stability bound $i\omega$ striving along the asymptotes directed to the right half-plane. As the rest of the root locus branches does not cross the stability bound, they are completely stable. For positive branches, crossing the stability bound, specify the set

$$S_i = \{S_{il}\} = \left\{\left(0, a_i^+ l\right)\right\} \tag{14}$$

of intervals S_{il} of values a_i within the segments from the initial point p_{il} (where $a_i = 0$) of every branch up to its cross point with axis $i\omega$. Thus, the maximal possible interval of a_i values, ensuring stability of (6), is equal to

$$S_{i\max} = \bigcap_{S_{il} \subset S_i} S_{il} = \inf S_i = \left(0, \inf A_i^+\right), \tag{15}$$

that proofs the statement being considered.

For the 4-th degree polynomial represented in **Figure 2**, the interval $S_{i\max} = (0, a_4(t))$.

Theorem 1. For ensuring asymptotic stability of regular or interval polynomial (1), it is enough to.

a. find among polynomials of extension (6), the stable polynomial of degree $i = k$ being the closest one to n;

b. set up sequentially every coefficient a_j of (1), beginning with $a_j = a_k + 1$, within interval $(k+1) < j \le n$ by setting up the free term a_i of the corresponding i-th polynomial of extension (5) as per condition (13) assuming $i = j$.

Proof. If polynomial $g_i(s) = g_k(s)$ is stable, then on the basis of Consequence 2 of Statement 1 (expressions (10) and (11)), the stability of $g_{i+1}(s)$ can be ensured by simple application of condition (13). Thus, stability of all polynomials $g_i(s)$ is sequentially ensured beginning with the polynomial of degree $i = k + 1$ up to the polynomial of degree $i = n$ inclusive, that is, for $i = \overline{(k+1), n}$. Thus, Theorem 1 has been proved.

An algorithm for the robustly stable regular or interval polynomial synthesis is given below.

Step 1. Composing the extension E_n (6) of the given initial nominal polynomial $g_n(s)$ (1).

Step 2. Sequential check for stability of the extension polynomials, beginning with the polynomial of degree *n,* until finding the stable polynomial of degree $i = k$.

In case of synthesis of the whole interval polynomial, begin the procedure with the 1-st degree polynomial, $i = k = 1$, specifying interval of a_1 according to the appropriate requirements or arbitrarily.

Step 3. Transfer to the polynomial of the next higher degree, $i = k + 1$.

Step 4. Calculating coordinates $\omega_{i_l}^+$ of cross points of the polynomial $g_i(s)$ free root locus positive branches with the axis *i*ω by solving its appropriate root locus Eq. (3).

Cross points $\omega_{i_l}^+$ generate on the axis *i*ω a so-called "crossing domain" W_i^+:

$$\omega_{i_l}^+ \in W_i^+ \tag{16}$$

Properties of this domain and behavior of the interval root locus portrait at the stability bound *i*ω have been investigated in [18]. On the basis of the fact, that every function of (3) represents continuous differentiable function (steadily increasing/decreasing function), it has been found in [18] that for ensuring stability of the whole interval family, it is required to calculate the parameter $a_i = a_{i_l}^+$ (13) values at only two extreme "dominating points":

$$\omega_{i\min}^+ \in \inf W_i^+, \omega_{i\min}^+ \in \sup W_i^+, \tag{17}$$

by solving the corresponding Eq. (3) after substituting preliminarily into this equation, the appropriate combination [18] of the limit values of each coefficient, from a_1 to a_{i-1}, which have been calculated already in this algorithm when generating the originative polynomial $g_{i-1}(s)$. For finding two coordinates (17), two different combinations of coefficients should be substituted into the root locus equation and thus two different equations should be solved.

Step 5. Determining the value of $\inf A_i^+$ (12) for polynomial $g_i(s)$ by calculating minimal values $a' = a''_{i\min}(\omega_i^+ \min)$ and $a'' = a_i^+ \min(\omega_i^+ \max)$ of coefficient a_i correspondingly at points $\omega_i^+ \min$ and $\omega_i^+ \max$ solving twice Eq. (4) for polynomial $g_i(s)$ at the stability bound:

$$a_i = u(\omega), \tag{18}$$

after substituting previously into (18) the corresponding combinations of coefficients (from a_1 to a_{i-1}) [18]. Thus,

$$\inf A_i^+ = \min(a'a'') = \overline{a}_i, \tag{19}$$

where $\overline{a}_i$ is the upper limit of a_i variation interval. The required interval (13) is: $0 < a_i < \overline{a}_i$.

Step 6. If the last polynomial of extension (6), that is, that of degree *n*, has been already processed ($i = n$), the calculation is considered finished. Otherwise proceed to step 3.

4.3 Example

Synthesis of the interval polynomial of the 3-rd degree.

Consider polynomial family

$$g^0{}_3\,(s) = s^3 + a_1 s^2 + a_2 s + a_3, \tag{20}$$

where $a_j \in [\underline{a}_j, \overline{a}_j], j \in \{1,2,3\}$; $a_1 \in [10, 15]$, $a_2 \in [25, 35]$, $a_3 \in [350, 450]$.

Step 1. Compose the extended polynomial (6) for (20):

$$\begin{cases} s + a_1 = 0 & (21.1) \\ s^2 + a_1 s + a_2 = 0 & (21.2) \\ s^3 + a_1 s^2 + a_2 s + a_3 = 0 & (21.3) \end{cases}. \tag{21}$$

Step 2. As coefficients of polynomials (21.1) and (21.2) are positive, then both families of these polynomials are asymptotically stable ($i = k = 2$), and therefore, on the basis of Consequence 2 of Statement 1, the root loci family of (21.3) initial points is located in the left half-pane. Thus, for making stable, the polynomial (21.3) uses Statement 2 and Theorem 1.

Step 3. Transfer to the polynomial of the next higher degree, $i = 2 + 1 = 3$.

Step 4. Calculating coordinates (16) of the "dominating points" for polynomial $g_3(s)$. For this purpose, consider the appropriate root locus (3) and parameter (18) equations:

$$\omega^3 – a_2\omega = 0, \tag{22}$$

and parameter function (18) at the stability bound:

$$a_1\omega^2 = a_3 = f_p(\omega). \tag{23}$$

Find the 1-st order derivative of (23) and equate it zero:

$$f_p{}'(\omega) = 2a_1\omega = 0. \tag{24}$$

On the basis of (23) and (24), it can be stated that the character of parameter (23) distribution along the axis σ is steadily increasing and the single extreme point is located at the origin. Thus, there exists the only one extreme point:

$$\omega_3^+ \min = \pm\sqrt{\underline{a}_2} = \pm 5, \tag{25}$$

where function (23) gets the minimal value of the set A_3^+ (see Eq. (12)).

Step 5. Determine $\inf A_3^+$ (12) for $g_3(s)$ using (23), (25):

$\inf A_3^+ = a_3^+ \min(\omega_3^+ \min) = \underline{a}_1 \cdot (\omega_3^+ \min)^2 = 10 \cdot 5^2 = 250$. Thus, $0 < a_3 < 250$.

Step 6. As $i = 3 = n$, the algorithm is considered finished.

Thus, coefficient intervals for the resulting robustly stable polynomial $\hat{g}_3(s)$ are as follows:

$a_1 \in [10, 15]$, $a_2 \in [25, 35]$, $a_3 \in (0, 250)$.

5. Investigation of behavior at the stability bound and synthesis of interval polynomial families: root locus parameter function distribution diagram

Consider a dynamic system described by the family of interval characteristic polynomials [4, 18, 20, 22] like.

$$g_4(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4. \tag{26}$$

Coefficients of Eq. (26) to be real, positive, and variable within the intervals

$$\underline{a}_j \le a_j \le \overline{a}_j, \quad j = 1, \; \ldots, 4, \quad a_0 = 1. \tag{27}$$

Substitute $s = \sigma + i\omega = i\omega$ $(\sigma = 0)$ into (26) and rewrite:

$$\omega^4 - a_1\omega^3 i - a_2\omega^2 + a_3\omega\, i + a_4 = 0 \tag{28}$$

and on the base of (28), write the root locus equation [18, 20] at the stability boundary:

$$-a_1\omega^3 + a_3\omega = 0 \tag{29}$$

and the parameter equation (parameter function) [18, 20] at the stability boundary:

$$f(\omega) = -\omega^4 + a_2\omega^2 = a_4. \tag{30}$$

5.1 Crossing region of the polynomial root locus portrait

Functions (29) and (30) imply properties of analyticity and continuity and, thus, the points where axis $i\omega$ is crossed by the branches of the root locus family P (5), given the condition.

$$0 < a_j < +\infty, \tag{31}$$

constitute on the stability boundary, axis $i\omega$, a specific *crossing region*, D_ω^P.

Definition 8. The region at the asymptotic stability boundary $i\omega$ of the interval system root locus portrait P, described by characteristic polynomial (26), where the given portrait parameter function (30) values family is located, name the *crossing region* D_ω^P of the root locus portrait P.

The region D_ω^P is a continuous one and, thus, each root locus field F_k (5) and each branch b_{ki}, $i = 1, 2, \ldots$ of the field root loci generate specific subregions, correspondingly subregion $D_\omega^F k$ and continuous subregion $D_\omega^b i$, within the above specified region D_ω^P.

Over the symmetry of the portrait hereinafter, the only upper half-plane s is considered.

5.2 Majorant and minorant of the extremum region

Obtain the extremum parameter function values within $D_\omega^F k \subset D_\omega^P$. To do so, it is necessary to carry out investigation of this function for extremum. It is evident that the majorant parameter function (majorant) can be obtained through rewriting Eq. (30):

$$a_{4\max} = -\omega^4 + \overline{a}_2\omega^2. \tag{32}$$

Take the first-order derivative of (32) and set it to zero:

$$-4\omega^3 + 2\overline{a}_2\omega = 0. \tag{33}$$

After solving Eq. (33), obtain three points of extremum for the majorant parameter function for the field when $a_2 = \overline{a}_2$:

$$\omega_{e_{\max}} = 0, \qquad a_{4e_{\max}} = 0; \;\; \omega_{e_{\max}} = \pm\sqrt{\frac{\overline{a}_2}{2}}, \quad a_{4e_{\max}} = -\omega^4_{e_{\max}} + \overline{a}_2 \cdot \omega^2_{e_{\max}}. \quad (34)$$

Rewrite (30) for determination of a minorant parameter function (or a minorant):

$$a_{4\min} = -\omega^4 + \underline{a}_2\omega^2. \quad (35)$$

In the same way obtain three points of extremum for the minorant, when $a_2 = \underline{a}_2$:

$$\omega_{e_{\min}} = 0, \qquad a_{4e_{\min}} = 0; \;\; \omega_{e_{\min}} = \pm\sqrt{\frac{\underline{a}_2}{2}}, \quad a_{4e_{\min}} = -\omega^4_{e_{\min}} + \underline{a}_2 \cdot \omega^2_{e_{\min}}. \quad (36)$$

Evidently, for $n = 4$, Eqs. (32) and (35) are the majorant and the minorant for the whole portrait.

Definition 9. *Extremum region* D^e_ω of the interval system root locus portrait described by the characteristic polynomial (26) is a region $[0, \omega_{e\max}]$ at the system asymptotic stability boundary $i\omega$ where the given portrait parameter function (30) extremum values, $a_{4e\max}$ (34) and $a_{4e\min}$ (36), family is located provided all coefficients a_j vary within limits (31).

5.3 Diagram of the parameter function distribution along the stability boundary

Figure 3 represents the character (diagram) of the parameter function (30) distribution along the boundary of stability by its majorant (32) and minorant (35). For better understanding and descriptiveness, the diagram in **Figure 3** is shown by strait lines, although it constitutes curves. Region D^P_ω constitutes three subregions (see **Figure 3**):

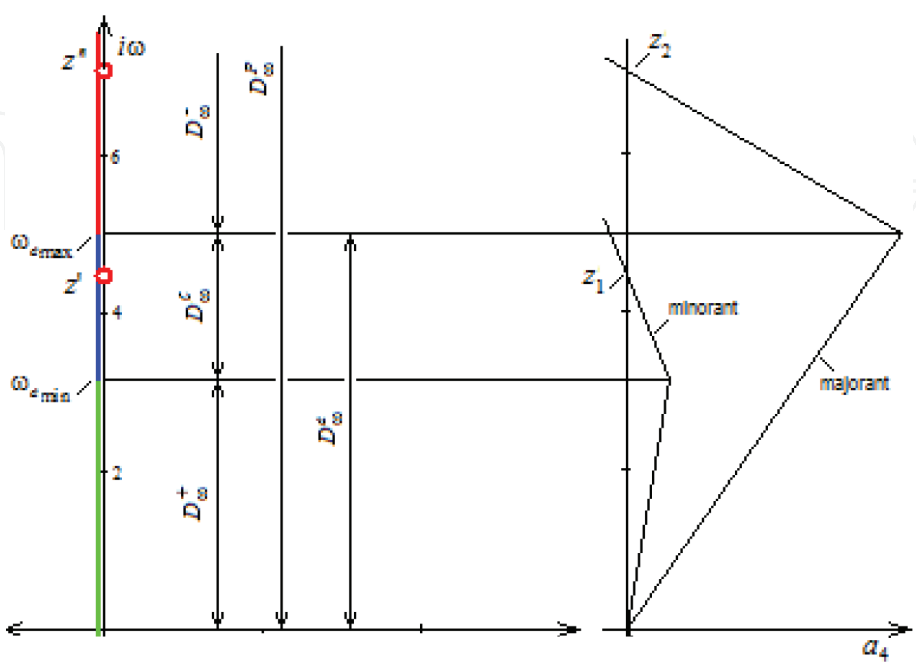


Figure 3.
A diagram for distribution of the interval system root locus portrait parameter function along the asymptotic stability boundary.

- D_ω^+ where the parameter function is getting increased (*increase region*);
- $D\omega^-$ where the parameter function is getting decreased (*decrease region*);
- $D\omega^c$ where increase and decrease regions combine (*mixed region*).

Analyze the region Z_ω with the interval $[z', z''] \subseteq Z_\omega$ where the initial points of the root locus portrait migrate through the stability boundary to the right half-plane. In the diagram, zero points z', z'' are mapped by points z_1, z_2.

Within interval $[0, z']$, covering completely region D_ω^+ and partly region D_ω^c ($D_\omega^+ \subset [0, z']$, $[0, z'] \cap D_\omega^c$), only the positive branches cross the stability boundary, and here the whole family Z of initial points is located in the left half-plane L,

$$Z \subset L. \tag{37}$$

But specific pieces of the positive branches are situated within the right half-plane. For this reason, in some cases, the unstable polynomials could have been found within the whole family (26). However, there certainly could always be found the intervals (27) of stability where the whole family is stable. Name the interval $[0, z']$ the *system stability region*.

The interval $[z', z'']$ covers some piece of the region D_ω^c and some of the region D_ω^-, $[z', z''] \cap D_\omega^c$, $[z', z''] \cap D_\omega^-$. In this case, axis $i\omega$ is crossed by combination of both positive and negative branches, and the root locus portrait certainly includes a series of initial points, and thus the whole branches, that have migrated over the boundary to the right half-plane. Therefore, this case always gives us the family (26) that includes combination of stable and unstable polynomials. Name the interval $[z', z'']$ the *system instability region*.

If the interval $[z'', \infty]$ completely belongs to the region D_ω^-,

$$[z'' \infty] \subset D_\omega^-, \tag{38}$$

only the negative branches cross the stability boundary $i\omega$, and the family Z together with the corresponding positive branches are located in the right half-pane,

$$Z \subset R. \tag{39}$$

No stable polynomial could be found in (26). This region name the *system complete instability region*.

5.4 Real crossing region of the portrait

Specify the region D_ω^R where the branches of the given real root locus portrait cross the stability boundary. To find its limits, consider Eq. (29) and determine the values of its roots. When $\omega>0$.

$$\omega_{\max} = \sqrt{\frac{\overline{a}_3}{\underline{a}_1}}, \omega_{\min} = \sqrt{\frac{\underline{a}_3}{\overline{a}_1}}, \tag{40}$$

where $\omega_{\max}$, $\omega_{\min}$ represent the *real crossing region*.

Definition 10. The region $[\omega_{\min}, \omega_{\max}]$ at the stability boundary $i\omega$, where the polynomial (26) root locus portrait branches migrate through to the right half-plane, name the *real crossing region* D_ω^R of the system root locus portrait:

$$[\omega_{\min}, \omega_{\max}] \subseteq D_\omega^R. \tag{41}$$

5.5 Graphic-analytical stability conditions for interval polynomials

Define below three possible ways of the real crossing region location and the corresponding stability conditions.

5.5.1 Real crossing region belongs to the increase region $D_\omega{}^+$

$$D_\omega^R \subset D_\omega{}^+. \tag{42}$$

In this case $\omega_{\max} < \omega_{e_{\min}}$.

Statement 3. When the dynamic system root locus portrait, described by polynomial (26), satisfies relationship (42), the whole family Z of the portrait initial points is located in the left half-plane L,

$$Z \subset L. \tag{43}$$

Then, define the set S of the root locus portrait P branches' intervals s_i:

$$S = \{s_i = [0, a_4(\omega_i)],\ \ i = 1, 2, \dots\}. \tag{44}$$

$a_4(\omega_i)$ represents the parameter function (30) at points with the coordinates ω_i; $S \subset P$ and $S \subset L$ (40). Thus, from (42) and (43) obtain:

$$\bigcap_{i=1}^{\infty} s_i = \inf S = \left[0, \underline{a}_4(\omega_{\min})\right], \tag{45}$$

where $\underline{a}_4(\omega_{\min})$—function (30) minimal value at point $\omega_{\min}$ (40). Hence,

$$\forall\ a_4 \in \left[\underline{a}_4, \overline{a}_4\right] \quad \left[a_4 \in \left[0, \underline{a}_4(\omega_{\min})\right] \ \rightarrow\ a_4 \in S \& P \subset L\right], \tag{46}$$

$$\forall\ a_4 \in \left[\underline{a}_4, \overline{a}_4\right] \quad \left[a_4 \notin \left[0, \underline{a}_4(\omega_{\min})\right] \ \rightarrow\ a_4 \notin S \& P \not\subset L\right]. \tag{47}$$

The following statement can be formulated on the basis of expressions (42) and (47).

Statement 4. The dynamic system, described by the interval characteristic polynomial family (26) and satisfying expression (42), is asymptotically stable if

$$\overline{a}_4 < \underline{a}_4(\omega_{\min}). \tag{48}$$

Definition 11. One or more stable polynomials with constant coefficients within the family (26) that guarantee stability of the whole family name the d*ominating polynomials*.

From Statement 4 and the previous conclusions, the following stability condition goes.

Stability condition 1. The asymptotic stability of the interval system family, described by the root locus portrait P (5) satisfying expression (42), is guaranteed if polynomial

$$s^4 + \overline{a}_1 s^3 + \underline{a}_2 s^2 + \underline{a}_3 s + \overline{a}_4 = 0 \tag{49}$$

of the family is stable. Polynomial (49) represents the dominating one.

Stability is verified using the Stability condition 1. The polynomial parameters are calculated with application of the Statement 4.

5.5.2 Real crossing region belongs to the decrease region $D_\omega{}^-$

$$D_\omega^R \subset D_\omega{}^- . \tag{50}$$

It happens in case if $\omega_{\min} \geq \omega_{e_{\max}}$.

The above made conclusions allow to formulate the following statement.

Statement 5. If the interval system root locus portrait P satisfies condition (50), the whole family Z of its initial points satisfies Eq. (39), and the system is asymptotically unstable.

5.5.3 Real crossing region completely or partially belongs to the mixed region $D_\omega{}^c$

$$D_\omega^R \subset D_\omega{}^c \ \vee \ D_\omega^R \cap \ D_\omega{}^c . \tag{51}$$

We have this when the following conditions are not satisfied: $\omega_{\max} < \omega_{e_{\min}}$, $\omega_{\min} \geq \omega_{e\max}$.

For this case

$$P = P^+ + P^-, \tag{52}$$

We have already discussed the increase part of (52), when $P^- = \varnothing$. Hence, this section considers the decrease part, P^-. Consider first the family Z of the root locus portrait P^-.

Statement 6. If condition (51) holds, family Z of initial points of the dynamic system root locus portrait, described by characteristic polynomial (26), can be located in both left half-plane L and right half-plane R, that is, the following options of Z location may take place:

$$Z \subset L, \tag{53}$$

$$Z \subset (L + R), \tag{54}$$

$$Z \subset R. \tag{55}$$

Evidently, options (54) and (55) take place when

$$D_\omega^R \subset D_\omega{}^- \tag{56}$$

$$\text{or } D_\omega^R \cap D_\omega{}^- . \tag{57}$$

As options (54)–(57) deliberately indicate instability of the system in whole, consider below only option (53) of the system poles location,

$$\omega_{\max} < \omega(z'), \tag{58}$$

where $\omega(z')$ is coordinate ω at point z' (**Figure 3**).

In this case proceed just as in (44)–(47) but only substituting $\omega_{\max}$ instead of $\omega_{\min}$.

Statement 7. The asymptotic stability of the dynamic system, described by polynomial family (26) and satisfying expression (51), is ensured when the following condition holds:

$$\overline{a}_4 < \min\left\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\right\}. \tag{59}$$

From condition (59) follows that the system asymptotic stability for part P^- of portrait (52), provided that condition (53) holds, is defined by the value of

$\underline{a}_4(\omega_{\max})$. Therefore, for checking stability of P^- (52), it is enough to check the only one following dominating polynomial of (26):

$$s^4 + \underline{a}_1 s^3 + \underline{a}_2 s^2 + \overline{a}_3 s + \overline{a}_4 = 0. \tag{60}$$

Because in this case, the portrait represents the compound one (52), check the stability by checking both polynomials, (49) and (60).

Stability condition 2. If the interval dynamic system root locus portrait P (52), describing the family of characteristic polynomials (26), satisfies expression (51), the system asymptotic stability is ensured when the following dominating polynomials

$$s^4 + \overline{a}_1 s^3 + \underline{a}_2 s^2 + \underline{a}_3 s + \overline{a}_4 = 0, \tag{61}$$

$$s^4 + \underline{a}_1 s^3 + \underline{a}_2 s^2 + \overline{a}_3 s + \overline{a}_4 = 0 \tag{62}$$

of family (26) are both stable.

From the results obtained above also goes that in case (51) the system asymptotic stability can be verified by only a single polynomial of (26) having constant coefficients. The equation to choose depends of condition (49) verification results. If the verification shows that $\min\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\} = \underline{a}_4(\omega_{\min})$, then Eq. (61) is applied for the stability check. If it shows that $\min\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\} = \underline{a}_4(\omega_{\max})$, then the stability is verified by (62).

To determine the coefficients of (26), ensuring satisfaction of expressions (53) and (58), Eqs. (30) and (31) are applied. Thus, coefficients a_1 and a_3 must satisfy the inequality:

$$\sqrt{\frac{\overline{a}_3}{\underline{a}_1}} < \omega\left(z'\right), \overline{a}_3 < \underline{a}_1 \omega^2\left(z'\right). \tag{63}$$

To verify the system stability, the stability conditions 1 and 2 are used. For calculation of the system (polynomial) parameters, expressions (48), (49) and (63) are used.

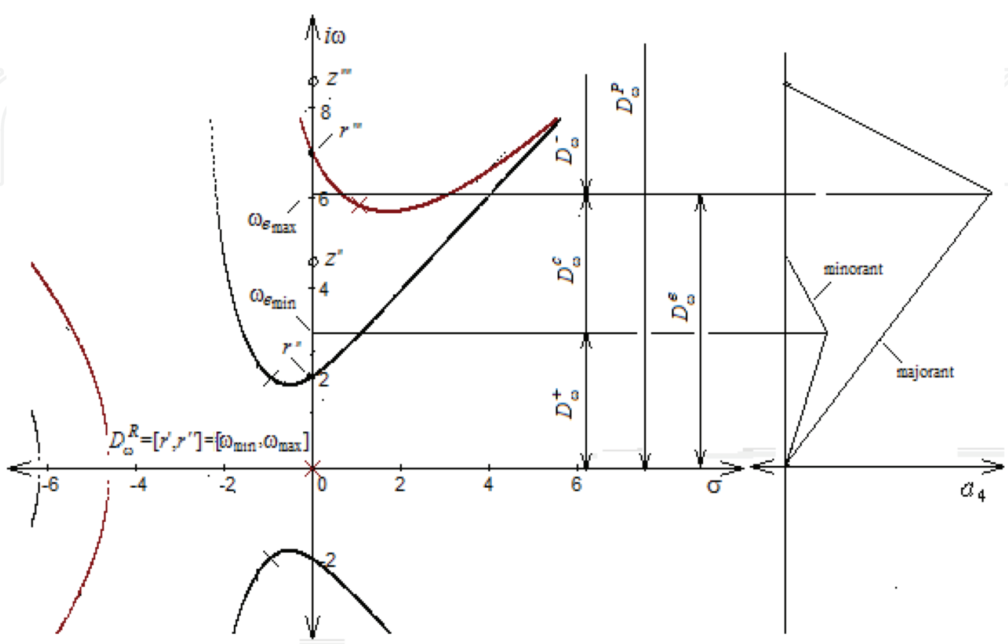


Figure 4.
Dynamics of the interval system root locus portrait at the asymptotic stability boundary.

Polynomial stability could be estimated graphically directly from the plots (see **Figures 3** and **4**).

5.6 Example

Coefficients of the given polynomial (26): $a_1 \in [5, 10]$, $a_2 \in [5, 10]$, $a_3 \in [5, 10]$, $a_4 \in [5, 10]$.

Extremum region: D_ω^e: $\omega_{e\ \min} = 3{,}16$; $\omega_{e\ \max} = 5{,}92$; $a_{4e\ \min} = 100{,}04$; $a_{4e\ \max} = 1223{,}96$.

Real region: D_ω^R: $\omega_{\min} = 2$; $\omega_{\max} = 7{,}1$; $\underline{a}_4(\omega_{\min}) = 64$; $\underline{a}_4(\omega_{\max}) = -1532{,}97$.

$[z', z'']$: $\omega(z') = 4{,}47$; $\omega(z'') = 8{,}37$.

In **Figure 4**, the above indicated regions are shown. The points, corresponding to the dominating polynomials (61), (62), are designated by r' and r''. The real crossing region in this case completely covers the extremum region, $D_\omega^e \subset D_\omega^R$, $[r', r''] \subseteq D_\omega^R$.

It is evident that the given polynomial family in whole is unstable. Within region $Z_\omega = [z', z'']$, there exist poles that have migrated to the right half-plane (see (54)), which is confirmed by the negative value of the parameter $\underline{a}_4(\omega_{\max})$I.

Dominating polynomials of the family are the following:

$$s^4 + 10s^3 + 20s^2 + 40s + 30 = 0. \tag{64}$$

$$s^4 + 5s^3 + 20s^2 + 250s + 30 = 0. \tag{65}$$

Polynomials stability check shows that polynomial (6), which root loci crosses the stability boundary at point $\underline{a}_4(\omega_{\min})$, is stable, and polynomial (66), which root loci crosses the stability boundary at point $\underline{a}_4(\omega_{\max})$, has two roots with positive real parts.

Extraction of the stable polynomial subfamily of the given unstable family:

The stable root locus family, satisfying conditions (58) and (59), should cross the stability boundary within the region bounded by interval $[r', z']$ as in this case all initial points of the root locus family are located in the left half-plane (53) (see Section 5.5).

To calculate the maximal value of a_3 that defines the stable subfamily within the given root locus portrait, apply formula (63):

$$\overline{a}_3 < \underline{a}_1 \cdot 4,47^2, \quad \overline{a}_3 < 99,9. \tag{66}$$

Based on (66), accept $\overline{a}_3 = 80$.

Based on (59), accept: $\overline{a}_4 < \underline{a}_4(\omega_{\min})$, $\overline{a}_4 = 60$ and write the dominating polynomials:

$$s^4 + 10s^3 + 20s^2 + 40s + 60 = 0,$$

$$s^4 + 5s^3 + 20s^2 + 80s + 60 = 0.$$

As per stability condition 2, the root locus portrait subfamily having new modified values of a_3 and a_4 ($\overline{a}_3 = 80, \overline{a}_4 = 60$) is asymptotically stable.

6. Conclusions and future developments

A method has been worked out for synthesis of asymptotically stable regular or interval polynomial from the given Hurwitz or non-Hurwitz source polynomial with constant/interval coefficients by setting up coefficients of the given one. The root locus approach is used. The task is solved by introduction of notions of the "extended polynomial" ("generalized polynomial") and the polynomial "extended root locus," which allows to obtain a descriptive picture of the polynomial root dynamics under coefficient variations and to disclose on this basis the cause of instability. The intervals of uncertainty for each coefficient being set up are specified along the root locus branches.

The above described method based on the "extended root locus" notion is new and allows to extend the application sphere of the root locus method, which is traditionally considered to be the method of system synthesis by only a single parameter (coefficient) variation and with only one variable parameter (coefficient), in both directions: system synthesis by many parameter variations and system synthesis with many parameter variations.

Investigation of the fourth power dynamic system behavior in conditions of the interval parameter variations has also been carried out on the basis of root locus portraits and introduction of the notion of the "diagram of the root locus parameter function values distribution along the stability bound." Behavior regularities for interval system root locus portraits at the stability boundary have been formulated. On this basis, the stability conditions have been derived, and graphic-analytical method has been worked out for calculating intervals of parameter variation ensuring the system robust stability.

In continuation of the results of Anderson [22] and Kharitonov [4] in this work, it is proved that for the 4th power interval system family asymptotic stability analysis, it is enough to use the only one polynomial of this kind. It is also shown, how to find and extract the stable families from the unstable ones.

The above discussed topic is certainly worth further investigation in the light of continuous progress of both theory and technology. When speaking of the practical implementations, it could be noted that most of the control system synthesis tasks, especially those in the area of robust control, are currently still being solved in a somewhat "local domestic" way, when a designer each time tries to invent a solution to be suitable for the specific application experiencing the lack of more generalized methods. Besides this, a great deal of existing robust control methods share and suffer complexity. In this connection, further in-depth investigation of the uncertain polynomials' root locus portraits seems helpful, especially the analysis of its composition in terms of configurations variety, constituting subfamilies, placement of various root domains within the prescribed regions in the complex plane and, of course, dynamics. They also could be distinguished for their undoubted descriptiveness.

Polynomial equation approach in the design technique [16], and root locus technique in particular, is descriptive, clear, and easy to use and computerize and thus could be helpful in many application areas including the areas of industry, biology, medicine, etc. It can be used for proper parameterization of robust drive controllers, for example, in the area of railway traffic control, in particular for the cases of tackling the problems of breaking and skidding.

Acknowledgements

The author acknowledges the support of this work by the Belarusian Republican Foundation for Fundamental Research.

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