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The Generalized Weierstrass System in Three-Dimensional Euclidean Space

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Abstract

In this chapter, some recent advances in the area of generalized Weierstrass representations will be given. This is an approach to the theory of surfaces in Euclidean three space. Weierstrass representations permit the explicit construction of surfaces in the designated space. The discussion proceeds in a novel and introductory manner. The inducing formulas for the coordinates of a surface are derived and important conservation laws are formulated. These lead to the inducing mechanism of a surface in terms of solutions to a system of two-dimensional Dirac equations. A set of fundamental forms as well as expressions for the mean and Gaussian curvatures are derived. The Cartan moving frame picture is also formulated to put everything in a broader perspective. A connection with the nonlinear sigma model is presented, which has important applications in physics. Some relationships are established between integrable systems and geometry by way of conclusion.

Keywords: metric, tensor, manifold, Weierstrass representation, curvature, evolution equation
Mathematics Subject Classification: 35Q51,53A10

1. Introduction

The theory of immersions and deformations of surfaces has been an important area of study as far as classical differential geometry is concerned. An inducing mechanism for describing minimal surfaces imbedded in three-dimensional Euclidean space was first put forward by Enneper and Weierstrass in the nineteenth century [1]. Their basic ideas have been extended and generalized by Konopelchenko and colleagues [2–4]. The connection between certain classes of constant mean curvature surfaces and the trajectories of an infinite-dimensional Hamiltonian system was put forward first by Konopelchenko and Taimanov [2], and has

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proved to be very useful in investigating types of questions related to this and other types of spaces and in higher dimensions [5, 6].

Surfaces and their dynamics play a very crucial and important role in a great number of phenomena which arise in the physical sciences in general. A longer introduction and more examples can be found in [7, 8]. They appear in the study of surface waves, shock waves, deformations of membranes, as well as in many problems in hydrodynamics connected with the motion of boundaries between regions of differing densities and viscosities. At the present time, they are appearing in string theory models [9–11] and in the study of integrable systems in general [12, 13]. A special case is that of surfaces which have zero mean curvature. These surfaces are usually referred to as minimal surfaces. The work of Weierstrass and Enneper originally concerned itself with the construction of minimal surfaces in three-dimensional Euclidean space [14, 15].

It is the intention here to present an introduction to the work of Konopelchenko and referred to presently as the generalized Weierstrass representation. The work presents both mathematical and physical developments in the area which should be relevant to both physicists and mathematicians. The development starts by studying a coupled system of two-dimensional Dirac equations in terms of two complex functions that involves a mass term that depends on two coordinates of the space. This equation can then be decomposed into a system of two simpler equations and their respective complex conjugates. By looking at such things as conservation laws, inducing formulas which specify the coordinates of a surface in Euclidean three space can be deduced, as well as the first and second fundamental forms pertaining to the surface. A remarkable result of this development is that the mass which appears in the Dirac system becomes related to the mean curvature of the surface. One might say this indicates that mass is a consequence of geometry in this type of model. To fit these developments in the larger picture of modern differential geometry, the Cartan moving frame for the system is formulated out of which emerges another remarkable result. Namely, the two-dimensional Dirac equation is a way of writing an affine connection on the surface. Finally, by investigating the Gauss map, it is shown that there is a mathematical way of proceeding from the Dirac system and the nonlinear sigma model in two dimensions [16, 17]. The whole construction leads to a very deep link between nonlinear evolution equations and geometry as a whole [18, 19]. The paper finishes with some interesting examples and outlook for further work.

2. Two-dimensional Dirac equation and construction of surfaces

The process of inducing surfaces in three-dimensional space can be generalized by establishing a system of Dirac equations in terms of a mass parameter and two complex valued functions called ψ_1 and ψ_2. In Euclidean space in two dimensions, the Dirac equation can be written in terms of the set of Pauli matrices $\{\sigma_\mu\}$ as follows:

$$\Psi = i(\sigma_1 \partial_x + \sigma_2 \partial_y)\Psi + m\Psi = 0. \tag{1}$$

In (1), the mass term m has been generalized to be a real function of x and y, which are the Cartesian coordinates of the space. Let us introduce two complex operators defined to be

$$\partial = \frac{1}{2}\left(\partial_x - i\partial_y\right), \qquad \overline{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right). \tag{2}$$

In terms of a complex variable $z = x + iy$, we also define $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$. A, and spinor wavefunction Ψ is specified in terms of two components ψ_1 and ψ_2 as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3}$$

Using (2), the Dirac equation can be developed in terms of the two components of Ψ and their complex conjugates to give the following coupled first-order system of equations:

$$\begin{aligned} \overline{\partial}\psi_1 &= \frac{i}{2}m\psi_2, & \partial\overline{\psi}_1 &= -\frac{i}{2}m\overline{\psi}_2, \\ \partial\psi_2 &= \frac{i}{2}m\psi_1, & \overline{\partial}\,\overline{\psi}_2 &= -\frac{i}{2}m\overline{\psi}_1. \end{aligned} \tag{4}$$

The Dirac equation in the form (4) leads to a variety of differential constraints. The first of which is given by

$$\psi_1\partial\overline{\psi}_1 + \overline{\psi}_1\partial\psi_2 = \psi_1\left(-\frac{i}{2}m\overline{\psi}_2\right) + \overline{\psi}_2\left(\frac{i}{2}m\psi_1\right) = 0,$$

as well as its complex conjugate equation. There is also the expression for a new real variable p

$$\overline{\psi}_1\partial\psi_2 - \psi_2\partial\overline{\psi}_1 = \frac{i}{2}m\left(|\psi_1|^2 + |\psi_2|^2\right) = \frac{i}{2}mp,$$

and its complex conjugate. This also serves to define the real function P

$$p = |\psi_1|^2 + |\psi_2|^2. \tag{5}$$

A system of conservation laws can also be formulated

$$\overline{\psi}_1\partial\overline{\psi}_1 - \overline{\psi}_2\overline{\partial}\,\overline{\psi}_2 = 0, \qquad \overline{\psi}_1\partial\psi_2 + \psi_1\overline{\partial}\,\overline{\psi}_2 = 0, \tag{6}$$

as well as their complex conjugate equations. The complex quantity S is defined as follows:

$$\overline{\psi}_2\partial\psi_1 - \psi_1\partial\overline{\psi}_2 = \frac{i}{2}pS, \quad \overline{\psi}_1\overline{\partial}\psi_2 - \psi_2\overline{\partial}\psi_1 = \frac{i}{2}p\overline{S}. \tag{7}$$

Let Φ be the two-by-two matrix spinor given by

$$\Phi = \begin{pmatrix} \psi_1 & -\overline{\psi}_2 \\ \psi_2 & \overline{\psi}_1 \end{pmatrix}, \tag{8}$$

defining the real variable u so that p in (5) is given by $p = e^u$, and it follows that

$$p = e^u = \det \Phi = |\psi_1|^2 + |\psi_2|^2. \tag{9}$$

Clearly, we have $\Phi\, \Phi^{\dagger} = p\mathbf{I}$ and there follows another differential constraint:

$$p\partial u = \overline{\psi}_1 \partial \psi_1 + \psi_2 \partial \overline{\psi}_2. \tag{10}$$

Differentiating (8) exteriorly, we obtain that

$$d\Phi = \begin{pmatrix} \partial\psi_1 & -\partial\overline{\psi}_2 \\ \partial\psi_2 & \partial\psi_1 \end{pmatrix} dz + \begin{pmatrix} \overline{\partial}\psi_1 & -\overline{\partial}\,\overline{\psi}_2 \\ \overline{\partial}\psi_2 & \overline{\partial}\,\overline{\psi}_1 \end{pmatrix} d\overline{z}. \tag{11}$$

Consequently, we find that

$$\begin{aligned} d\Phi \cdot \Phi^{-1} &= \frac{1}{p} \begin{pmatrix} \partial\psi_1 & -\partial\overline{\psi}_2 \\ \partial\psi_2 & \partial\psi_1 \end{pmatrix} \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 \\ -\psi_2 & \psi_1 \end{pmatrix} dz + \frac{1}{p} \begin{pmatrix} \overline{\partial}\psi_1 & -\overline{\partial}\,\overline{\psi}_2 \\ \overline{\partial}\psi_2 & \overline{\partial}\,\overline{\psi}_1 \end{pmatrix} \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 \\ -\psi_2 & \psi_1 \end{pmatrix} d\overline{z} \\ &= \frac{1}{2} \left[\begin{pmatrix} 2\partial u & iS \\ im & 0 \end{pmatrix} dz + \begin{pmatrix} 0 & im \\ i\overline{S} & 2\overline{\partial} u \end{pmatrix} d\overline{z} \right]. \end{aligned} \tag{12}$$

Taking the derivative $\overline{\partial}$ of pS in (7) and substituting system (4), we obtain that

$$\begin{aligned} \overline{\partial}(pS) &= -m\overline{\psi}_1 \partial\psi_1 + \overline{\psi}_2 \partial(m\psi_2) - m\psi_2 \partial\overline{\psi}_2 + \psi_1 \partial(m\overline{\psi}_1) \\ &= -m\overline{\psi}_1 \partial\psi_1 - m\psi_2 \partial\overline{\psi}_2 + p\partial m. \end{aligned}$$

It follows that

$$p^{-1}\partial(pS) = -mp^{-1}\partial p + \partial m = p\partial(p^{-1}m) \ .$$

Let us summarize this as

$$p^{-1}\overline{\partial}(pS) = p\partial(p^{-1}m). \tag{13}$$

Proceeding in a similar fashion, we calculate the following two derivatives:

$$\begin{aligned} \partial(p^{-1}\psi_1) &= \frac{1}{p^2} \left(-|\psi_1|^2 \partial\psi_1 - \psi_1\psi_2 \partial\overline{\psi}_2 + \left(|\psi_1|^2 + |\psi_2|^2 \right) \partial\psi_1 \right) \\ &= \frac{\psi_2}{p^2} \left(-\psi_1 \partial\overline{\psi}_2 + \overline{\psi}_2 \partial\psi_1 \right) = \frac{i}{2} S(p^{-1}\psi_2), \end{aligned} \tag{14}$$

and as well, we have

$$\begin{aligned}\bar{\partial}(p^{-1}\psi_2) &= \frac{1}{p^2}\left(-\psi_1\psi_2\overline{\partial\psi}_1 - |\psi_2|^2\bar{\partial}\psi_2 + |\psi_1|^2\bar{\partial}\psi_2 + |\psi_2|^2\bar{\partial}\psi_2\right);\\ &= \frac{\psi_1}{p}\left(-\psi_2\bar{\partial}\,\overline{\psi}_1 + \overline{\psi}_1\bar{\partial}\psi_2\right) = \frac{i}{2}\overline{S}(p^{-1}\psi_1).\end{aligned} \tag{15}$$

It should be pointed out that the systems (14) and (15) are summarized here

$$\partial(p^{-1}\psi_1) = \frac{i}{2}S(p^{-1}\psi_2), \qquad \bar{\partial}(p^{-1}\psi_2) = \frac{i}{2}\overline{S}(p^{-1}\psi_1). \tag{16}$$

By comparing with (4), it look very much like a Dirac system in their own right if S is thought of as a mass variable. Another quantity, a current, was found in [20] and has the form

$$J = pS.$$

It is possible to construct a vector representation of Φ as well. A matrix such as Φ represents a rotation matrix multiplied by a scaling in $\mathbb{R}^3$ as follows $V = v^i\sigma_i \to V' = \Phi V \Phi^\dagger$. So the matrix Φ can be represented by means of a multiple of an orthogona1 3×3 real matrix. The matrix elements can be found by using the inner product in V, namely $\langle V_1, V_2\rangle = (1/2)$ Tr $[V_1V_2]$, then

$$\varsigma_i^j = \frac{1}{2}Tr\left[\sigma_i\Phi\sigma_j\Phi^\dagger\right]. \tag{17}$$

ς_i^j defines a 3×3 matrix which can be written down by using the usual representation of the Pauli matrices. In particular, the matrix formed out of the following combinations will be very useful:

$$\begin{aligned}\varsigma_+ &= \frac{1}{\sqrt{2}}(\varsigma_1 - i\varsigma_2) = \frac{1}{\sqrt{2}}\left(\psi_1^2 - \overline{\psi}_2^2,\ \ -i\left(\psi_1^2 + \overline{\psi}_2^2\right), 2\psi_1\overline{\psi}_2\right),\\ \varsigma_- &= \frac{1}{\sqrt{2}}(\varsigma_1 + i\varsigma_2) = \frac{1}{\sqrt{2}}\left(\overline{\psi}_1^2 - \psi_2^2,\ i\left(\overline{\psi}_1^2 + \psi_2^2\right), 2\overline{\psi}_1\psi_2\right),\\ \varsigma_3 &= \left(-\psi_1\psi_2 - \overline{\psi}_1\overline{\psi}_2,\ i(\psi_1\psi_2 - \overline{\psi}_1\overline{\psi}_2),\ |\psi_1|^2 - |\psi_2|^2\right).\end{aligned} \tag{18}$$

In terms of matrices, ς and $\varsigma^\dagger$ are represented as:

$$\varsigma = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\psi_1^2 - \overline{\psi}_2^2\right) & -\frac{i}{\sqrt{2}}\left(\psi_1^2 + \overline{\psi}_2^2\right) & 2\psi_1\overline{\psi}_2 \\ \frac{1}{\sqrt{2}}\left(\overline{\psi}_1^2 - \psi_2^2\right) & \frac{i}{\sqrt{2}}\left(\overline{\psi}_1^2 + \psi_2^2\right) & 2\overline{\psi}_1\psi_2 \\ -\psi_1\psi_2 - \overline{\psi}_1\psi_2 & i(\psi_1\psi_2 - \overline{\psi}_1\overline{\psi}_2) & |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}, \tag{19}$$

$$
\varsigma\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\overline{\psi}_1^2 - \overline{\psi}_2^2\right) & \frac{1}{2}\left(\psi_1^2 + \overline{\psi}_2^2\right) & -(\overline{\psi}_1\overline{\psi}_2 + \psi_1\psi_2) \\ \frac{i}{\sqrt{2}}\left(\overline{\psi}_1^2 - \psi_2^2\right) & -\frac{i}{\sqrt{2}}\left(\psi_1^2 + \overline{\psi}_2^2\right) & i(\psi_1\psi_2 - \overline{\psi}_1\overline{\psi}_2) \\ \sqrt{2}\overline{\psi}_1\psi_2 & \sqrt{2}\psi_1\overline{\psi}_2 & |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}
$$

Given this explicit representation, it is now possible to evaluate

$$
p^{-2}d\varsigma\cdot\varsigma^\dagger = p^{-2}\left(\partial\varsigma\cdot\varsigma^\dagger dz + \overline{\partial}\varsigma\cdot\varsigma^\dagger d\overline{z}\right). \tag{20}
$$

To obtain an expression for (20), both matrices (19) can be expressed in Maple. Apply the operator $map(\partial)$ to ς, right multiply by $\varsigma^\dagger$ then substitute system (4) of known derivatives to obtain the matrix

$$
\begin{pmatrix} 2(\psi_2\partial\overline{\psi}_2 + \overline{\psi}_1\partial\psi_1)p & 0 & \sqrt{2}(\psi_1\partial\overline{\psi}_2 - \overline{\psi}_2\partial\psi_1)p \\ 0 & 0 & -\frac{i}{\sqrt{2}}mp^2 \\ -\frac{i}{\sqrt{2}}mp^2 & \sqrt{2}(\overline{\psi}_2\partial\overline{\psi}_1 - \psi_1\partial\overline{\psi}_2)p & (\psi_2\partial\overline{\psi}_2 + \overline{\psi}_1\partial\psi_1)p \end{pmatrix} \tag{21}
$$

Similarly, applying $map\left(\overline{\partial}\right)$ to ς then right multiplying by $\varsigma^\dagger$ yields

$$
\begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}}mp^2 \\ 0 & 2(\psi_1\overline{\partial}\psi_1 + \overline{\psi}_2\overline{\partial}\psi_2)p & -\sqrt{2}(\psi_2\overline{\partial}\,\overline{\psi}_1 - \overline{\psi}_1\overline{\partial}\psi_2)p \\ \frac{i}{\sqrt{2}}p^2\overline{S} & \frac{i}{\sqrt{2}}mp^2 & (\psi_1\overline{\partial}\,\overline{\psi}_1 + \psi_2\overline{\partial}\psi_2)p \end{pmatrix} \tag{22}
$$

By (20) and the differential constraints, the vector representation of the Maurer-Cartan form can be expressed as:

$$
p^{-2}d\varsigma\varsigma^\dagger = \begin{pmatrix} 2\partial u & 0 & -\frac{i}{\sqrt{2}}S \\ 0 & 0 & \frac{i}{2}m \\ -\frac{i}{\sqrt{2}}m & \frac{i}{\sqrt{2}}S & \partial u \end{pmatrix} dz + \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}}m \\ 0 & 2\overline{\partial}u & \frac{i}{\sqrt{2}}\overline{S} \\ -\frac{i}{\sqrt{2}}\overline{S} & \frac{i}{\sqrt{2}}m & \overline{\partial}u \end{pmatrix} dz. \tag{23}
$$

According to the properties of the inner product, we can write $E_i = \varsigma_i^j \sigma_j = \Phi^\dagger \sigma_i \Phi$ and calculate that

$$d\varsigma_i \varsigma_j = \langle dE_i, E_j \rangle = \frac{1}{2}\operatorname{Tr}\left[d\left(\Phi^\dagger \sigma_i \Phi\right)\Phi^\dagger \sigma_j \Phi\right] = \frac{1}{2}\operatorname{Tr}\left[\left(d\Phi^\dagger \sigma_i \sigma_j \Phi\right) + \left(d\Phi^\dagger \sigma_i \sigma_j \Phi\right)^\dagger\right]. \tag{24}$$

If a conserved current can be constructed whose components are divergence free, then a differential one-form exists with values in $\mathbb{R}^3$ that will induce a surface upon quadrature. Such a current will be given from the global symmetries of the Lagrangian by means of Noether's theorem. Making the transformations $\psi_1 \to -\overline{\psi}_2$ and $\psi_2 \to \overline{\psi}_1$ in system (4), it is seen to remain invariant. This can be thought of as a charge conjugation. The same solutions are obtained if we put Φ instead of Ψ in the Dirac equation (2). So Φ multiplied on the right by any constant nonsingular matrix is a solution of the equation if Φ is. This implies the full symmetry group is $GL(2\mathbb{C})$. The transformation above is a member of this group, so can be thought of as a continuous transformation. In terms of matrix Φ, the Lagrangian of the Dirac equation can be written as

$$\mathcal{L} = \frac{1}{2}\operatorname{Tr}\left[\Phi^\dagger \Phi\right] = i\overline{\psi}_1 \partial \psi_2 - i\psi_2 \partial \overline{\psi}_1 + i\overline{\psi}_2 \overline{\partial} \psi_1 - i\psi_1 \overline{\partial}\, \overline{\psi}_2 + m\left(|\psi_1|^2 + |\psi_2|^2\right). \tag{25}$$

The currents that correspond to the generators of $SU(2)$ are found to be proportional to the components of ς_+ and ς_-; hence, the required conservation law is

$$\partial \varsigma_- + \overline{\partial} \varsigma_+ = 0. \tag{26}$$

Alternatively, the Dirac equation and its Hermitian conjugate which are given by

$$\Phi^\dagger \left(i\sigma_1 \partial_x + i\sigma_2 \partial_y + m\right)\Phi = 0 \quad \Phi^\dagger \left(i\overleftarrow{\partial}_x \sigma_1 + i\overleftarrow{\partial}_y \sigma_2 - m\right)\Phi = 0, \tag{27}$$

may be added to obtain

$$\partial_x \left(\Phi^\dagger \sigma_1 \Phi\right) + \partial_y \left(\Phi^\dagger \sigma_2 \Phi\right) = 0. \tag{28}$$

Now to describe the surface, define the $\mathbb{R}^3$-valued differential form

$$d\mathrm{r} = \frac{i}{\sqrt{2}} \varsigma_+ \, dz + \frac{i}{\sqrt{2}} \varsigma_- d\overline{z} \tag{29}$$

which is real since $\varsigma_- = \overline{\varsigma}_+$. The differential form (29) is closed under substitution of conservation law (26) since

$$d^2\mathbf{r} = \frac{i}{\sqrt{2}} \overline{\partial} \varsigma_+ \, d\overline{z} \wedge dz - \frac{i}{\sqrt{2}} \partial \varsigma_- dz \wedge d\overline{z} = \frac{i}{\sqrt{2}} \left(\overline{\partial} \varsigma_+ + \partial \varsigma_-\right) dz \wedge d\overline{z} = 0. \tag{30}$$

By Poincare's lemma, the form is exact since every loop in $\mathbb{C}$ can be collapsed to a point. Therefore, the desired expression for a surface will result when the form is integrated along a path Γ^1 in the $(z, \overline{z})$ plane from a fixed point z_0. The components are

$$\begin{aligned} dx_1 &= \frac{i}{2}\left(\psi_1^2 - \overline{\psi}_2^2\right)dz - \frac{i}{2}\left(\overline{\psi}_1^2 - \psi_2^2\right)d\overline{z}, \\ dx_2 &= \frac{1}{2}\left(\psi_1^2 + \overline{\psi}_2^2\right)dz + \frac{1}{2}\left(\overline{\psi}_1^2 + \psi_2^2\right)d\overline{z}, \\ dx_3 &= i\left(\psi_1\overline{\psi}_2 dz - \overline{\psi}_1\psi_2 d\overline{z}\right) \ . \end{aligned} \tag{31}$$

Combining the first two equations in (31) and integrating from z_0, the coordinates of a surface in $\mathbb{R}^3$ are obtained by integrating over any path Γ^1 in the $(z, \overline{z})$ plane

$$\begin{aligned} x_1 + ix_2 &= i\int_\Gamma \left(\psi_1^2 dz' + \psi_2^2 d\overline{z}'\right) \\ x_1 - ix_2 &= -i\int_\Gamma \left(\overline{\psi}_2^2 dz' + \overline{\psi}_1^2 d\overline{z}'\right), \\ x_3 &= i\int_\Gamma \left(\psi_1\overline{\psi}_2 dz' - \overline{\psi}_1\psi_2 d\overline{z}'\right) \ . \end{aligned} \tag{32}$$

In the end, we have set z_0 to be zero, and it may be repeated; the integrals are independent of Γ due to the conservation laws. In (31) and (32), $\mathbf{r}$ is the point of the surface with coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$ and ς_3 is normal to the surface.

3. Fundamental forms and Cartan moving frame

The necessary information to write down the traditional data for a surface has been obtained. Since $\varsigma_\pm^2 = 0$ and $\varsigma_+ \cdot \varsigma_- = 0$, the first fundamental form is given by

$$I = d\mathbf{r} \cdot d\mathbf{r} = p^2 dz \otimes d\overline{z} \tag{33}$$

or in a matrix representation,

$$I = \frac{p^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The inverse of (33) is given by

$$I^{-1} = \frac{2}{p^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{34}$$

It is therefore a conformal immersions with isothermal coordinates ζ_1, ζ_2. The second fundamental form of the surface can also be calculated and using $\varsigma_3 \cdot d\mathbf{r} = 0$,

$$II = -d\left(p^{-1}\varsigma_3\right) \cdot \ d\mathbf{r} = -p^{-1}d\varsigma_3 \cdot d\mathbf{r} = -\frac{p}{2}\left(Sdz \otimes dz + 2mdz \otimes d\overline{z} + \overline{S}d\overline{z} \otimes d\overline{z}\right) \tag{35}$$

and in matrix form,

$$II = -\frac{p}{2}\begin{pmatrix} S & m \\ m & \overline{S} \end{pmatrix}.$$

Collecting (34) and (35), we have

$$II \cdot I^{-1} = -\frac{p}{2}\begin{pmatrix} S & m \\ m & \overline{S} \end{pmatrix}\begin{pmatrix} 0 & \frac{2}{p^2} \\ \frac{2}{p^2} & 0 \end{pmatrix} = -\frac{1}{p}\begin{pmatrix} S & m \\ m & \overline{S} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{p}\begin{pmatrix} m & S \\ \overline{S} & m \end{pmatrix}.$$

The usual definitions give the mean curvature H and the Gaussian curvature as well

$$H = \frac{1}{2}\mathrm{Tr}\left(II \cdot I^{-1}\right) = -\frac{m}{p}, \tag{36}$$

$$K = \det\left(II \cdot I^{-1}\right) = \frac{1}{p^2}\left(m^2 - |S|^2\right) = H^2 - \frac{|S|^2}{p^2}. \tag{37}$$

Equation (36) relates the mean curvature H to the mass parameter in the Dirac equation. Konopelchenko obtains the expression

$$K = -4p^{-2}\partial\overline{\partial}u, \tag{38}$$

which is known as the Gauss-Riemann curvature. It has been shown however that it is equivalent to (37) in accord with Gauss' Theorem Egregium.

It is interesting to note that since the difference between the principal curvatures is given as

$$(\Delta\kappa)^2 = 4\left(H^2 - K\right), \tag{39}$$

it also holds that since $H^2 - K = p^{-2}|S|^2$,

$$|\Delta\kappa| = \frac{2}{p}|S|.$$

Thus, the modulus of S is a measure of the local deformation from a spherical surface as m is a measure of the local deformation from the case of a minimal surface, so $\kappa = -p^{-1}(m \pm |S|)$.

A fixed referential frame in $\mathbb{R}^3$ has been implicitly used up to now. By varying the frame with some solution of Dirac system (4), a whole set of surfaces is obtained that may be deduced from each other by means of a rigid motion. Cartan developed a powerful method referred to as the moving frame method to avoid this awkward process.

By introducing differential 1-forms also called Pfaffian forms, we define the system

$$d\mathbf{r} = \omega^j \mathbf{e}_j, \qquad d\mathbf{e}_j = \omega_i^j \mathbf{e}_j, \qquad i, j = +, \, -, \; 3. \tag{40}$$

This is the first system of structure equations introduced by Cartan. The vectors $\mathbf{e}_i$ satisfy orthonormality conditions

$$\mathbf{e}_+^2 = 0, \qquad \mathbf{e}_+ \cdot \mathbf{e}_- = p^2, \qquad \mathbf{e}_\pm \cdot \mathbf{e}_3 = 0, \qquad \mathbf{e}_3^2 = p^2. \tag{41}$$

Differentiating relations (41) and using structure equations (40), the following relations among the differential forms are obtained

$$d\mathbf{e}_+^2 = 0, \qquad 2\mathbf{e}_+ \cdot d\mathbf{e}_+ = 0, \qquad 2\mathbf{e}_+ \left(\omega_+^j \mathbf{e}_j \right) = 0,$$

$$2\mathbf{e}_+ \cdot \omega_+^- \mathbf{e}_- = 0 \quad \omega_+^- = 0. \qquad \square$$
$$d\mathbf{e}_-^2 = 0, \qquad 2\mathbf{e}_- \cdot d\mathbf{e}_- = 0, \qquad 2\mathbf{e}_- \cdot \left(\omega_-^j \mathbf{e}_j \right) = 0,$$
$$2\mathbf{e}_- \; \left(\omega_-^+ \mathbf{e}_+ \right) = 0, \quad \omega_-^+ = 0. \qquad \square$$
$$d\mathbf{e}_+ \cdot \mathbf{e}_- + \mathbf{e}_+ \cdot de_- = 2pdp,$$
$$\omega_+^+ \mathbf{e}_+ \cdot \mathbf{e}_- + \omega_-^- \mathbf{e}_+ \cdot \mathbf{e}_- = 2pdp,$$
$$\omega_-^- + \omega_+^+ = 2\partial u, \qquad \square$$
$$d\mathbf{e}_+ \cdot \mathbf{e}_3 + \mathbf{e}_+ \cdot d\mathbf{e}_3 = 0,$$
$$\omega_j^3 + \omega_3^- = 0. \qquad \square$$
$$d\mathbf{e}_- \cdot \mathbf{e}_3 + \mathbf{e}_- \omega_3^j \mathbf{e}_j = 0,$$
$$\omega_-^3 + \omega_3^+ = 0. \qquad \square$$
$$3\mathbf{e}_3 \cdot d\mathbf{e}_3 = p^2 du \qquad \omega_3^3 = du. \qquad \square$$

This collection of results is summarized all together below

$$\begin{gathered} \omega_-^+ = \omega_+^- = 0, \\ \omega_3^- + \omega_+^3 = \omega_3^+ + \omega_-^3 = 0, \\ \omega_-^- + \omega_+^+ = 2du, \qquad \omega_3^3 = du. \end{gathered} \tag{42}$$

As $\overline{\mathbf{e}}_+ = \mathbf{e}_-$ and $\overline{\mathbf{e}}_3 = \mathbf{e}_3$, it is found that

$$\overline{\omega}_+^3 = \omega_-^3, \qquad \overline{\omega}_+^+ = \omega_-^-. \tag{43}$$

Assuming structure equations (40) are integrable, differentiating and substituting $d\mathbf{e}_i$ where ever possible, compatibility equations are obtained which are referred to as the second system of structure equations, that is first we have

$$d\omega^j \mathbf{e}_j - \omega^j \wedge d\mathbf{e}_j = 0,$$

hence

$$d\omega^s = \omega^j \wedge \omega_j^s,$$

and next

$$d\omega_i^j \mathbf{e}_j - \omega_i^j \wedge d\mathbf{e}_j = 0,$$

hence

$$d\omega_i^j = \omega_i^j \wedge \omega_j^s.$$

Let us summarize these as the pair

$$d\omega^s = \omega^j \wedge \omega_j^s, \qquad d\omega_i^s = \omega_i^j \wedge \omega_j^s. \tag{44}$$

The second equality is always true as long as the frames are given, and the first is the equivalent, expressed in the formalism of a moving frame, of the requirement that the form $d\mathbf{r}$ be exact. Writing $d\mathbf{r}$ as

$$d\mathbf{r} = \frac{i}{\sqrt{2}}\mathbf{e}_+ dz - \frac{i}{\sqrt{2}}\mathbf{e}_- d\overline{z} = \omega^j \mathbf{e}_j. \tag{45}$$

Let us identify the forms

$$\omega^+ = \frac{i}{\sqrt{2}}dz, \qquad \omega^- = -\frac{i}{\sqrt{2}}d\overline{z}, \qquad \omega^3 = 0, \;\; \overline{\omega}^+ = \omega^-. \tag{46}$$

The equations for the remaining one-forms can be represented by writing the structure equation in the form

$$d\mathbf{e} = \Omega\mathbf{e}. \tag{47}$$

In (47), Ω is represented by the 3×3 matrix of forms

$$\Omega = \begin{pmatrix} \omega_+^+ & \omega_+^- & \omega_+^3 \\ \omega_-^+ & \omega_-^- & \omega_-^3 \\ \omega_3^+ & \omega_3^- & \omega_3^3 \end{pmatrix} \tag{48}$$

Since $\mathbf{e} \cdot \mathbf{e}^\dagger = p^2 \mathbf{I}$, (47) can be right multiplied by $\mathbf{e}^\dagger$ to obtain

$$\Omega = p^{-2} \;\; d\mathbf{e} \cdot \mathbf{e}^\dagger. \tag{49}$$

This implies that Ω can be identified with the Maurer-Cartan form given in (22). Introduce the vector of differential forms Θ as

$$\Theta = (\omega^+, \;\; \omega^-, \;\; 0), \qquad d\Theta = 0. \tag{50}$$

In terms of Θ the compatibility equations take the form

$$d\Theta = \ominus \wedge \Omega, \qquad d\Omega = \Omega \wedge \Omega. \tag{51}$$

It is clear from the Maurer-Cartan form that it can be decomposed in the following manner

$$\Omega = M_1 dz + M_2 d\bar{z}, \tag{52}$$

where M_1 and M_2 are defined to be the matrices

$$M_1 \begin{pmatrix} 2\partial u & 0 & -\frac{i}{\sqrt{2}} S \\ 0 & 0 & \frac{i}{\sqrt{2}} m \\ -\frac{i}{\sqrt{2}} m & \frac{i}{\sqrt{2}} S & \partial u \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}} m \\ 0 & 2\bar{\partial} u & \frac{i}{\sqrt{2}} \bar{S} \\ -\frac{\bar{S}}{\sqrt{2}} & i\frac{m}{\sqrt{2}} & \bar{\partial} u \end{pmatrix}. \tag{53}$$

The first structure equation in (33) is then

$$\partial e = M_1 e, \qquad \bar{\partial} e = M_2 e. \tag{54}$$

This corresponds to the Gauss-Weingarten equation and the second compatibility equation

$$\bar{\partial} M_1 - \partial M_2 + [M_1, M_2] = 0, \tag{55}$$

is also known as the Gauss-Codazzi-Mainardi equations. All of these have been seen here before in (37) and (45). It has been shown that many nonlinear partial differential equations can be expressed within this formalism. In a spinor representation, the corresponding representation in the form of matrices can be obtained out of the Maurer-Cartan form

$$Z_1 = \frac{1}{2} \begin{pmatrix} 2\partial u & iS \\ im & 0 \end{pmatrix}, \qquad Z_2 = \frac{1}{2} \begin{pmatrix} 0 & im \\ i\bar{S} & 2\bar{\partial} u \end{pmatrix}. \tag{56}$$

In terms of these matrices, the linear system is

$$\partial \Phi = Z_1 \Phi, \qquad \bar{\partial} \Phi = Z_2 \Phi,$$
$$\bar{\partial} Z_1 - \partial Z_2 + [Z_1, \ Z_2] = 0. \tag{57}$$

The differential form Ω is a connection and actually an affine connection on $\mathbb{R}^3$. It is flat on the surface. This is the meaning of the second system of structure equations. This means that the two-dimensional Dirac equation can be regarded as a way of expressing an affine connection.

Two make further progress, Ω can be used in the following way. As $\omega^3 = 0$, from the compatibility equation for $d\omega^3$, we have

$$\omega^+ \wedge \omega^3_+ + \omega^- \wedge \omega^3_- = 0.$$

On account of Cartan's lemma, both ω^3_+ and ω^3_- are equal to a linear combination of ω^+ and ω^-

$$\omega^3_+ = h_{++}\omega^+ + h_{+-}\omega^-, \qquad \omega^3_- = h_{-+}\omega^+ + h_{--}\omega^-, \tag{58}$$

where $h_{-+} = h_{+-}$, as can be seen by substituting ω_+^3 and ω_-^3 into the constraint above. Since

$$\Omega = \begin{pmatrix} 2\partial u & 0 & -\frac{i}{\sqrt{2}}S \\ 0 & 0 & \frac{i}{\sqrt{2}}m \\ -\frac{i}{\sqrt{2}}m & \frac{i}{\sqrt{2}}S & \partial u \end{pmatrix} dz + \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}}m \\ 0 & \bar{\partial} u & \frac{i}{\sqrt{2}}\overline{S} \\ -\frac{i}{\sqrt{2}}\overline{S} & \frac{i}{\sqrt{2}}m & \bar{\partial} u \end{pmatrix} d\bar{z}. \tag{59}$$

Using $\omega^+, \ \ \omega^-$ and ω^3 from (46), we have

$$\omega_+^3 = h_{++}\left(\frac{i}{\sqrt{2}}\right)dz + h_{+-}\left(-\frac{i}{\sqrt{2}}\right)d\bar{z} = -\frac{i}{\sqrt{2}}Sdz - \frac{i}{\sqrt{2}}md\bar{z}. \tag{60}$$

This relation implies that

$$h_{++} = -S, \qquad h_{+-} = -m, \tag{61}$$

and moreover, it follows that

$$\omega_-^3 = h_{-+}\omega^+ + h_{--}\omega^- = h_{-+}\left(\frac{i}{\sqrt{2}}\right)dz + h_{--}\left(-\frac{i}{\sqrt{2}}\right)d\bar{z} = \frac{i}{\sqrt{2}}mdz + \frac{i}{\sqrt{2}}\overline{S}d\bar{z}.$$

This implies that

$$h_{-+} = m, \qquad h_{--} = -\overline{S}. \tag{62}$$

It is important to note that these coefficients can be used together with the structure equations to express the fundamental forms of the surface in terms of Pfaffian forms. The first fundamental form is given as

$$I = 2p^2\omega^+ \otimes \omega^- = 2p^2\left(\frac{1}{2}\right)dz \otimes d\bar{z}, \tag{63}$$

and the second fundamental form can be written as

$$\begin{aligned} II &= -p\left(\omega^+ \otimes \omega_+^3 + \omega^- \otimes \omega_-^3\right) \\ &= -p(h_{++}\omega^+ \otimes \omega^+ + (h_{+-} + h_{-+})\omega^+ \otimes \omega^- + h_{--}\omega^- \otimes \omega^-). \end{aligned} \tag{64}$$

The element of surface is given by

$$d\mathcal{S} = ip^2\omega^+ \wedge \omega^-, \tag{65}$$

and the corresponding surface element on the Gauss map is

$$d\sigma = i\omega_-^3 \wedge \omega_+^3 = i(h_{+-}h_{-+} - h_{++}h_{--})\omega^+ \wedge \omega^-. \tag{66}$$

The total curvature would be the ratio of the former to the latter,

$$K = p^{-2}(h_{+-}h_{-+} - h_{++}h_{--}). \tag{67}$$

Finally, the mean curvature is given as

$$H = -\frac{1}{2p}(h_{+-} + h_{-+}). \tag{68}$$

4. The Gauss map and nonlinear Sigma model

Under the condition that a given moving frame is integrable, the surface is defined up to a translation. Conversely, given the three vectors which constitute the frame, only one is determined uniquely by the surface, and that is the normal vector. For this reason, it is often referred to as the Gauss or spherical map, as it maps the parameter plane to the sphere of radius one in two dimensions. The map in this instance is given as

$$\phi = \frac{\mathbf{e}_3}{p}, \tag{69}$$

so the north pole corresponds to $\psi_2 = 0$, while the south pole to $\psi_1 = 0$. If the first column of $\Phi^\dagger$ is considered as well as the associated fundamental field

$$\rho = -\frac{\psi_2}{\psi_1}, \tag{70}$$

then dividing the numerator and denominator by $|\psi_1|^2$ in (69), we obtain

$$\phi = \frac{1}{1+|\rho|^2}\left(\rho + \overline{\rho},\ i(\overline{\rho} - \rho), 1 - |\rho|^2\right). \tag{71}$$

This quantity is a function of only ρ. It may be thought that ρ plays the role of stereographic projection of the Gauss map from the south pole. Moreover, for a minimal surface where $m = 0$, it is readily shown that $\overline{\rho}$ is an analytic function of z.

Using the differential constraints, the derivatives of ρ are found to be

$$\partial\rho = -ip\frac{m}{2\overline{\psi}_1^2}, \qquad \partial\overline{\rho} = ip\frac{S}{2\psi_1^2}. \tag{72}$$

By using derivatives (72), the following three relations can be worked out

$$4\frac{\partial\rho\overline{\partial}\overline{\rho}}{\left(1+|\rho|^2\right)^2} = m^2, \qquad 4\frac{\partial\rho\partial\overline{\rho}}{\left(1+|\rho|^2\right)^2} = mS, \qquad 4\frac{\overline{\partial}\rho\partial\overline{\rho}}{\left(1+|\rho|^2\right)^2} = |S|^2. \tag{73}$$

Thus, the quantities m and S can be written as a function of only ρ. It may be asked, can the component of the Maurer-Cartan form ∂u be written in a similar way? Starting with the differential constraint for ∂u,

$$\partial u = \frac{1}{p}\left(\overline{\psi}_1 \partial \psi_1 + \psi_2 \partial \overline{\psi}_2\right) = e^{-u/2} \partial \left(\psi_1, -\overline{\psi}_2\right) e^{-u/2} \begin{pmatrix} \overline{\psi}_1 \\ \psi_2 \end{pmatrix}. \tag{74}$$

Since the spinor product $\left(\psi_1, -\overline{\psi}_2\right)\left(\psi_1, -\overline{\psi}_2\right)^\dagger = e^u$, we have

$$\partial\left[e^{-u/2}\left(\psi_1, -\overline{\psi}_2\right)\right] e^{-u/2} \begin{pmatrix} \overline{\psi}_1 \\ -\psi_2 \end{pmatrix} = -\frac{1}{2} \partial u \left(\psi_1, -\overline{\psi}_2\right) e^{-u} \begin{pmatrix} \overline{\psi}_1 \\ -\psi_2 \end{pmatrix} + e^{-u/2} \partial \left(\psi_1, -\overline{\psi}_2\right) e^{-u/2} \begin{pmatrix} \overline{\psi}_1 \\ -\psi_2 \end{pmatrix}.$$

Combining these last two results, we obtain

$$\frac{1}{2} \partial u = \partial\left[e^{-u/2}\left(\psi_1, -\overline{\psi}_2\right)\right] e^{-u/2} \begin{pmatrix} \overline{\psi}_1 \\ -\psi_2 \end{pmatrix}. \tag{75}$$

If we define the spinor $\alpha = e^{-u/2}\left(\psi_1, -\overline{\psi}_2\right)$ which satisfies $\alpha \ \alpha^\dagger = \mathbf{1}$, (75) becomes

$$\frac{1}{2} \partial u = \partial \alpha \alpha^\dagger. \tag{76}$$

Let us show that α can be expressed as a function of ρ. Using the definition of ρ, a parameterization for α exists as

$$\alpha = \frac{1}{\sqrt{1 + |\rho|^2}} (1, \overline{\rho}) \left(\frac{\psi_1}{\overline{\psi}_2}\right)^{1/2}. \tag{77}$$

To obtain an expression for ψ_1, use differential constraint (2) its conjugate and (70) to arrive at

$$\overline{\psi}_1^2 \partial \rho = -\frac{i}{2} pm. \tag{78}$$

Dividing this by its complex conjugate gives α as a function of ρ as

$$\alpha = \frac{1}{\sqrt{1 + |\rho|^2}} (1, \ \rho) \left(\frac{\partial \rho}{\overline{\partial} \overline{\rho}}\right)^{1/4} e^{i(2n+1)\pi/4}, \qquad n \in \mathrm{Z}. \tag{79}$$

Inserting α into the expression for ∂u provides expressions for ∂u and $\overline{\partial} u$

$$\partial \left(\frac{\partial \rho}{\overline{\partial} \overline{\rho}}\right) = 2 \log \left(\frac{\partial \rho}{\overline{\partial} \overline{\rho}}\right) \ \cdot \frac{\partial \rho}{\overline{\partial} \overline{\rho}}. \tag{80}$$

Differentiating the components of α with respect to z, we find that

$$\begin{aligned}\partial\alpha\alpha^{\dagger} &= -\frac{(\overline{\rho}\partial\rho + \rho\partial\overline{\rho})}{2\left(1+|\rho|^2\right)^2} + \frac{1}{4\left(1+|\rho|^2\right)}\partial\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) \\ &\quad + \frac{-\rho\overline{\rho}^2\partial\rho - \rho^2\overline{\rho}\,\partial\overline{\rho} + 2\rho\left(1+|\rho|^2\right)\partial\overline{\rho}}{2\left(1+|\rho|^2\right)^2} + \frac{|\rho|^2}{4\left(1+|\rho|^2\right)}\partial\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) \\ &= \frac{-\overline{\rho}\,\partial\rho + \rho\partial\overline{\rho}}{2\left(1+|\rho|^2\right)} + \frac{1}{4}\partial\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right).\end{aligned}$$

Returning to the expression for $(1/2)\partial u$, we can now write

$$\frac{1}{2}\partial u = \frac{1}{4}\partial\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + \frac{1}{2}\frac{\rho\partial\overline{\rho} - \overline{\rho}\partial\rho}{1+|\rho|^2}, \tag{81}$$

$$\frac{1}{2}\overline{\partial}u = -\frac{1}{4}\overline{\partial}\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + \frac{1}{2}\frac{\overline{\rho}\overline{\partial}\rho - \rho\overline{\partial}\overline{\rho}}{1+|\rho|^2} \tag{82}$$

There is no simple integral of the second term in general. It may be stated that $(1/2)\partial u$ has the form of a potential with a fixed gauge, because ρ is given as a function of z and $\overline{z}$, so the directions of the axes $\mathbf{e}_+$ and $\mathbf{e}_-$ have been fixed so that a gauge transformation is a rotation of them.

Suppose it is asked under what condition a given complex function $\rho(z,\overline{z})$ is the Gauss map of some surface. A necessary condition can be obtained by working out the compatibility condition for the linear system (81) and (82), that is, first

$$\begin{aligned}\partial\overline{\partial}u &= -\frac{1}{2}\partial\overline{\partial}\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + \frac{\partial\overline{\rho}\,\overline{\partial}\rho + \overline{\rho}\partial\overline{\partial}\rho - \partial\rho\,\overline{\partial}\,\overline{\rho} - \rho\partial\,\overline{\partial}\overline{\rho}}{1+|\rho|^2} - \frac{\overline{\rho}\overline{\partial}\rho - \rho\overline{\partial}\,\overline{\rho}}{\left(1+|\rho|^2\right)^2}(\rho\partial\,\overline{\rho} + \overline{\rho}\partial\rho) \\ &= -\frac{1}{2}\partial\overline{\partial}\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + \frac{\overline{\partial}\rho\partial\overline{\rho} - \overline{\partial}\overline{\rho}\partial\rho - \overline{\partial}\rho\partial\rho\overline{\rho}^2 + \partial\overline{\rho}\overline{\partial}\overline{\rho}\rho^2 - \left(1+|\rho|^2\right)(\rho\partial\overline{\partial}\overline{\rho} - \overline{\rho}\partial\overline{\partial}\rho)}{\left(1+|\rho|^2\right)^2},\end{aligned}$$

and the result for the other mixed derivative is

$$\overline{\partial}\partial u = \frac{1}{2}\overline{\partial}\partial\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + \frac{\partial\overline{\rho}\overline{\partial}\rho - \overline{\partial}\overline{\rho}\partial\rho + \left(1+|\rho|^2\right)(\rho\partial\overline{\partial}\overline{\rho} - \overline{\rho}\partial\overline{\partial}\rho) + \overline{\rho}^2\partial\rho\overline{\partial}\rho - \rho^2\partial\overline{\rho}\overline{\partial}\overline{\rho}}{\left(1+|\rho|^2\right)^2}.$$

Equating these mixed partial derivatives, the necessary condition takes the form

$$\partial\overline{\partial}\log\left(\frac{\partial\rho}{\overline{\partial}\overline{\rho}}\right) + 2\frac{\left(1+|\rho|^2\right)(\rho\partial\overline{\partial}\overline{\rho} - \overline{\rho}\partial\overline{\partial}\rho) + \overline{\rho}^2\partial\rho\overline{\partial}\rho - \rho^2\partial\overline{\rho}\overline{\partial}\overline{\rho}}{\left(1+|\rho|^2\right)^2}. \tag{83}$$

If it is satisfied, it has the implication that

$$\partial\bar{\partial}u = \frac{\bar{\partial}\rho\partial\bar{\rho} - \partial\rho\bar{\partial}\bar{\rho}}{\left(1+|\rho|^2\right)^2}. \tag{84}$$

Using the previous expressions (73) for the derivatives of ρ, this can be put into the form of the Gauss equation. Consequently, one of the integrability conditions is fulfilled. Since

$$\frac{\bar{\partial}\rho\partial\bar{\rho} - \partial\rho\bar{\partial}\bar{\rho}}{\left(1+|\rho|^2\right)^2} = \partial\left(\frac{1}{2}\frac{\bar{\rho}\bar{\partial}\rho - \rho\bar{\partial}\bar{\rho}}{1+|\rho|^2}\right) + \bar{\partial}\left(\frac{1}{2}\frac{\rho\partial\bar{\rho} - \bar{\rho}\partial\rho}{1+|\rho|^2}\right), \tag{85}$$

it follows that

$$\partial\bar{\partial}u = \partial\left(\frac{1}{2}\frac{\bar{\rho}\bar{\partial}\rho - \rho\bar{\partial}\bar{\rho}}{1+|\rho|^2}\right) + \bar{\partial}\left(\frac{1}{2}\frac{\rho\partial\bar{\rho} - \bar{\rho}\partial\rho}{1+|\rho|^2}\right) = \frac{\bar{\partial}\rho\partial\bar{\rho} - \partial\rho\bar{\partial}\bar{\rho}}{\left(1+|\rho|^2\right)^2} = \frac{1}{4}p^2K. \tag{86}$$

Due to cancelations, some shorthand expressions might be quoted

$$\partial\bar{\partial}u = \partial\left(\frac{\bar{\rho}\bar{\partial}\rho}{1+|\rho|^2}\right) + \bar{\partial}\left(-\frac{\bar{\rho}\,\partial\rho}{1+|\rho|^2}\right) = \partial\left(-\frac{\rho\bar{\partial}\bar{\rho}}{1+|\rho|^2}\right) + \bar{\partial}\left(\frac{\rho\partial\bar{\rho}}{1+|\rho|^2}\right) = \frac{\bar{\partial}\rho\partial\bar{\rho} - \partial\rho\bar{\partial}\bar{\rho}}{\left(1+|\rho|^2\right)^2}. \tag{87}$$

The integrability condition can be expressed in the form of a zero curvature condition

$$\partial\left(\frac{\partial\bar{\partial}\rho}{\partial\rho} - 2\frac{\bar{\rho}\bar{\partial}\rho}{1+|\rho|^2}\right) - \bar{\partial}\left(\frac{\partial\bar{\partial}\bar{\rho}}{\bar{\partial}\bar{\rho}} - 2\frac{\rho\partial\bar{\rho}}{1+|\rho|^2}\right) = 0. \tag{88}$$

It is clear that provided we have

$$B(\rho,\bar{\rho}) = \partial\bar{\partial}\rho - \frac{2\bar{\rho}}{1+|\rho|^2}\partial\rho\bar{\partial}\rho = 0, \tag{89}$$

the condition is satisfied automatically. This may be recognized as the equation describing the nonlinear sigma model. As well it is the equation which is satisfied by the Gauss map of a constant mean curvature surface which is harmonic.

It is well known that for a given Gauss map ρ such that $\partial\rho = 0$, there is a one parameter family of surfaces called the associated family which is obtained through the transformation

$$\psi_1 \to q^{1/2}\psi_1, \qquad \psi_2 \to \bar{q}^{1/2}\psi_2. \tag{90}$$

This keeps p and ρ invariant if q is a complex constant of modulus one. If $m \neq 0$, it is not possible since m would not stay real. In the latter case, the only allowed values are $q = e^{in\pi}$. To construct the surface, take α and replace the phase factor by $q^{1/2}$, so $p = 1$, and we obtain

$$\psi_1 = \frac{q^{1/2}}{\sqrt{1+|\rho|^2}}, \qquad \psi_2 = -\frac{q^{1/2}\rho}{\sqrt{1+|\rho|^2}}. \tag{91}$$

Substituting (91) into the inducing formulae (30), the Weierstrass representation for $q = 1$ can be observed.

Finally, using (36) and recalling that

$$\partial H = -p^{-2}\overline{\partial}(pS), \qquad \psi_1^2 \partial\overline{\rho} = \frac{i}{2} pS \tag{92}$$

the second equation in (92) is differentiated with respect to $\overline{\partial}$ to obtain,

$$im\psi_1\psi_2\partial\overline{\rho} + \psi_1^2\overline{\partial}\partial\overline{\rho} = \frac{i}{2}\overline{\partial}(pS). \tag{93}$$

Taking the conjugate of the first expression in (72) then solving for ψ_1^2 and substituting into (93) we have

$$-\left(2\frac{\psi_1\psi_2}{p}\partial\overline{\rho} + \frac{\partial\overline{\partial}\rho}{\overline{\partial}\overline{\rho}}\right) = -m^{-1}p^{-1}\overline{\partial}(pS). \tag{94}$$

Using (70) and the relation $|\psi_1|^2/p = 1/\left(1+|\rho|^2\right)$, we obtain the desired result

$$-H^{-1}\partial H = \frac{\partial\overline{\partial}\rho}{\overline{\partial}\overline{\rho}} - 2\frac{\rho\partial\overline{\rho}}{1+|\rho|^2}. \tag{95}$$

Differentiating $J = pS$ with respect to $\overline{\partial}$ then multiplying by $(pS)^{-1}$, we obtain

$$(pS)^{-1}\overline{\partial}(pS) = m\left(2\frac{\psi_1\psi_2}{pS}\partial\overline{\rho} + \frac{1}{S}\frac{\overline{\partial}\partial\overline{\rho}}{\overline{\partial}\partial\overline{\rho}}\partial\overline{\rho}\right) = m\left(\frac{\partial\overline{\rho}}{\overline{\partial}\overline{\rho}}\right)\left(2\frac{\psi_1\psi_2}{pS}\overline{\partial}\overline{\rho} + \frac{1}{S}\frac{\overline{\partial}\partial\overline{\rho}}{\partial\overline{\rho}}\right)$$

$$= \frac{m}{S}\frac{\partial\overline{\rho}}{\overline{\partial}\overline{\rho}}\left(\frac{\overline{\partial}\partial\overline{\rho}}{\partial\overline{\rho}} - 2\frac{\rho\overline{\partial}\overline{\rho}}{1+|\rho|^2}\right) = \frac{\overline{\partial}\partial\overline{\rho}}{\partial\overline{\rho}} - 2\rho\frac{\partial\overline{\rho}}{1+|\rho|^2}.$$

To obtain this, the first two derivatives in (73) have been used to write $\partial\overline{\rho}/\overline{\partial}\overline{\rho} = S/m$. Summarizing these calculations, the following relations have been proved:

$$-H^{-1}\partial H = \frac{\partial\overline{\partial}\rho}{\overline{\partial}\overline{\rho}} - 2\frac{\rho\partial\overline{\rho}}{1+|\rho|^2} = \frac{\overline{B}}{\overline{\partial}\overline{\rho}}, \qquad J^{-1}\partial J = \frac{\partial\overline{\partial}\overline{\rho}}{\partial\overline{\rho}} - 2\rho\frac{\partial\overline{\rho}}{1+|\rho|^2} = \frac{\overline{B}}{\partial\overline{\rho}}. \tag{96}$$

Thus, for the parameters that are proportional to a power of p, the logarithmic derivatives can still be computed. For a constant mean curvature surface $\partial H = 0$ and so $B(\rho, \overline{\rho}) = 0$ hence

$$\bar{\partial} J = \partial \bar{J} = 0, \tag{97}$$

and the current is conserved, of J is a holomorphic function.

5. Summary and conclusions

It should be said that this work has deep implications for the study of manifolds and their relationship with integrable systems in general [21–24]. It would be worth illustrating this more clearly as a way to conclude. As a particular example, consider the case of a spherical surface for which $S = 0$ so that

$$K = H^2 = \frac{m^2}{p}, \tag{98}$$

where K is now a constant and the Gauss equation simplifies to

$$\partial \bar{\partial} u + m^2 = 0. \tag{99}$$

If we choose $K = 1$, this implies that $m = p$; hence $m = e^u$ and (99) is then the nonlinear Liouville equation

$$\partial \bar{\partial} u + e^{2u} = 0$$

is obtained in terms of the only remaining variable u. This procedure has resulted in a nonlinear equation with a link to surfaces. Since $p^{-1} m = 1$, the Codazzi-Mainardi equation is trivially satisfied.

Due to the spinor representation of the Maurer-Cartan form, from which Z_1 and Z_2 are deduced, for any nonsingular matrix τ, there is a gauge transformation given by [19]

$$\begin{aligned} \Phi &\to \tau \Phi, \\ Z_1 &\to \tau Z_1 \tau^{-1} + \partial \tau \cdot \tau^{-1}, \\ Z_2 &\to \tau Z_2 \tau^{-1} + \bar{\partial} \tau \cdot \tau^{-1}, \end{aligned} \tag{100}$$

for which the nonlinear zero curvature equation still holds. For example, suppose we take

$$\tau = \begin{pmatrix} \bar{\lambda}^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix} \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} e^{-u/2}. \tag{101}$$

In (101), λ can be thought of as a complex spectral parameter that satisfies $|\lambda|^2 = 1$. Starting with (8), we find that

$$\Phi'_\lambda = \tau\Phi = \begin{pmatrix} \overline{\lambda}^{1/2}\psi_1 & -\overline{\lambda}^{1/2}\overline{\psi}_2 \\ \lambda^{1/2}e^{-u}\psi_2 & \lambda^{1/2}e^{-u}\overline{\psi}_1 \end{pmatrix}, \qquad \det \Phi'_\lambda = 1. \tag{102}$$

It is straightforward to calculate that

$$\tau Z_1 \tau^{-1} = \frac{1}{2}\begin{pmatrix} 2\partial u & 0 \\ i\lambda & 0 \end{pmatrix}, \qquad \partial\tau = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda^{1/2}e^{-u}\partial u \end{pmatrix}, \qquad \partial\tau\cdot\tau^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -\partial u \end{pmatrix},$$

Therefore, we get

$$\tau Z_1 \tau^{-1} + \partial\tau\tau^{-1} = \frac{1}{2}\begin{pmatrix} 2\partial u & 0 \\ i\lambda & -2\partial u \end{pmatrix},$$

and proceeding in a similar fashion, one finds

$$\tau Z_2 \tau^{-1} + \overline{\partial}\tau\tau^{-1} = \frac{1}{2}\begin{pmatrix} 0 & i\overline{\lambda}e^{2u} \\ 0 & 0 \end{pmatrix}.$$

The linear system for the case in which $S = 0$ and $m = p$ is given by

$$\partial\Phi'_\lambda = \frac{1}{2}\begin{pmatrix} 2\partial u & 0 \\ i\lambda & -2\partial u \end{pmatrix}\Phi'_\lambda, \qquad \overline{\partial}\Phi'_\lambda = \frac{1}{2}\begin{pmatrix} 0 & i\overline{\lambda}e^{2u} \\ 0 & 0 \end{pmatrix}\Phi'_\lambda, \tag{103}$$

where Φ'_λ is given by (102). Other choices for the gauge function τ will lead to other systems: for example, taking

$$\tau = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\lambda}^{1/2} & 0 \\ 0 & \lambda^{1/2}e^{-u} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \overline{\lambda}^{1/2} & -\lambda^{1/2}e^{-u} \\ \overline{\lambda}^{1/2} & \lambda^{1/2}e^{-u} \end{pmatrix} \tag{104}$$

an AKNS type system is obtained

$$\partial\tilde{\Phi} = \frac{1}{4}\begin{pmatrix} -i\lambda & 4\partial u - i\lambda \\ 4\partial u + i\lambda & i\lambda \end{pmatrix}\tilde{\Phi}_\lambda, \qquad \overline{\partial}\tilde{\Phi} = i\frac{\lambda}{4}\begin{pmatrix} -e^{2u} & e^{2u} \\ -e^{2u} & e^{2u} \end{pmatrix}\tilde{\Phi}_\lambda. \tag{105}$$

Therefore, hierarchies may be generated and this linear system which is derived from the Dirac equation and used to create surfaces provides the link between nonlinear evolution equations and geometry [25].

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