Chapter

# Pricing Basket Options by Polynomial Approximations

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## Abstract

In this paper, we use polynomial approximations in terms of Taylor, Chebyshev, and cubic splines to compute the price of basket options. The paper extends the use of a similar pricing technique applied under a multivariate Black-Scholes model to a framework where the dynamic of the underlying assets is described by dependent exponential Levy processes generated by a combination of Brownian motions and compound Poisson processes. This model captures some empirical features of the asset dynamics such as common and idiosyncratic random jumps. The approach is implemented in the context of spread options and a multivariate Merton model, i.e., a jump diffusion with Gaussian jumps. Our findings show that, within the range of parameters analyzed, polynomial approximations are comparable in accuracy to a standard Monte Carlo approach with a considerable reduction in computational effort. Among the three expansions, cubic splines show the best performance.

**Keywords:** Taylor approximations, Chebyshev polynomials, cubic splines, basket options, spread options, jump-diffusion model

## 1. Introduction

We study the pricing of basket contracts under a multivariate jump-diffusion process. The paper extends the use of a similar pricing technique applied under a multivariate Black-Scholes model, see [1], to a framework where the dynamic of the underlying assets is described by dependent exponential Levy processes generated by a combination of Brownian motions and compound Poisson processes. This model captures some empirical features of the asset dynamics such as common and idiosyncratic random jumps. The dependence between assets is reflected in both the covariance structure of the Brownian motion and the joint probability law of the common jump sizes.

For such class of models, no pricing closed-form formula is available. In single-asset contracts, well-established numerical methods have proven to be effective, but their extensions to several dimensions reveal important instabilities and a costly computational effort. Our paper introduces a novel approach based on polynomial approximations of the conditional price. It is, in the framework considered, less time demanding than a standard Monte Carlo approach to achieve similar results. Moreover, the use of Chebyshev polynomials and cubic splines improves the convergence over previous attempts based on Taylor expansions.

We consider a pricing methodology consisting in a two-step procedure. First, conditioning on $d-1$ out of the total number of $d$ assets, we find the price of a payoff based on a single asset with a more complex conditional distribution.

Secondly, we consider some expansions of the conditional price, given either in terms of Taylor, Chebyshev, or cubic spline polynomials, allowing to write the corresponding price as a linear combination of mixed exponential-power moments.

This approach is implemented in the context of spread options and a multivariate Merton model, that is, a jump diffusion with Gaussian jumps. Our findings show that, within the range of parameters analyzed, polynomial approximations are comparable in accuracy to a standard Monte Carlo approach with a considerable reduction in computational effort. Among the three expansions, cubic splines show the best performance.

The use of a Taylor expansion to pricing has been considered in the pioneering work of [2] for a vanilla European option and in [3, 4] for spread contracts under a bivariate Black-Scholes model. See also [5]. A Chebyshev expansion has been recently considered in [6]. Applications under a multivariate jump-diffusion model have been less explored. Our paper intends to fill this gap.

Although a comparison with alternative approaches is beyond the scope of this paper, it is worth noticing the existence of pricing methods based on Fourier or Hilbert transforms. For example, for spread contracts under a different class of Levy processes, a Fast Fourier transform method can be found in [7]. See also [8] for expansions in terms of Fourier series and [9] for Hilbert transforms.

The organization of the paper is as follows: in Section 2, we introduce the model and obtain the pricing expressions for basket contracts under the approximations. In Section 3, we specialize the three expansions in the case of spreads contracts. In Section 4, we discuss the implementation of the methods and present our numerical findings. Finally in Section 5, we present conclusions. Proofs are deferred to the appendix.

## 2. Pricing under jump-diffusion models

Let $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered probability space. We define the filtration $\mathcal{F}^{X_t} := \sigma(X_s, 0 \leq s \leq t)$ as the $\sigma$-algebra generated by the random variables $\{X_s, 0 \leq s \leq t\}$ completed in the usual way. Denote by $\mathcal{Q}$ an equivalent martingale measure (EMM), respectively, by $E_{\mathcal{Q}}$, $\varphi_X$, and $M_X$ the expectation, characteristic, and moment-generating functions of a random variable $X$ under $\mathcal{Q}$. The function $f_X$ is its probability density function.

By $r$ we denote the (constant) interest rate, $A \circ B$ is the componentwise product between matrices $A$ and $B$, and $A'$ represents the transpose of matrix $A = \left(a_{ij}\right)_{1\leq i,j\leq d}$, while $\mathrm{diag}(A)$ is a vector with components $(a_{ii})_{1\leq i\leq d}$. The symbol $\delta_{ij}$ is the usual Kronecker's number. The vector $\tilde{Y}$ is created from the vector $Y$ after eliminating the first component. For a function $f$ with domain in $\mathbb{R}^d$ and a vector $L = (l_1, l_2, \ldots, l_d)$ with $l_k \in \mathbb{N}$, the symbol $D^L f$ represents the mixed partial derivative of the function $f$ differentiated $l_k$ times w.r.t. the $k$-th variable.

For vectors $v = (v_1, v_2, \ldots, v_d)$ and $n = (n_1, n_2, \ldots, n_d)$, we set $v! = \prod_{k=1}^{d} v_k$ and $\nu^n = \prod_{k=1}^{d} v_k^{n_k}$.

We introduce the following convenient notations. For a $1 \times (n+1)$ vector $Va$, $b \in \mathbb{R}$, and $n \in \mathbb{N}$

$$\begin{aligned} bin(n, Va, b) &= \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} Va_m b^{n-m} \\ PVa &= \left(1, Va_1, \ldots, Va_n^n\right) \end{aligned}$$

Also, for a differentiable function $f$, we set the vector $DVf = \left(f, D^1 f, \ldots, D^n f\right)$.

The $d$-dimensional process of spot prices is denoted by $(S_t)_{t \geq 0}$, while $(Y_t)_{t \geq 0}$ is the corresponding log-price process. They are related by

$$S_t^{(j)} = S_0^{(j)} \exp\left(Y_t^{(j)}\right), \ j = 1, 2, \ldots, d \tag{1}$$

We analyze European basket options whose payoff at maturity $T$, for a strike price $K$, are given by

$$h(S_T) = \left(\sum_{j=1}^{d} w_j S_T^{(j)} - K\right)_+$$

where $(w_j)_{1 \leq j \leq d}$ are some deterministic weights and $x_+ = max(x, 0)$.

Furthermore, for the log-prices, we assume a multidimensional jump-diffusion dynamics under $\mathcal{Q}$ given by

$$dY_t = \mu dt + \Sigma^{\frac{1}{2}} dB_t + dZ_t \tag{2}$$

where $(B_t)_{t \geq 0}$ is a multivariate Brownian motion with independent components and $\mu = r - \frac{1}{2} diag(\Sigma) - m$. The matrix $\Sigma = \left(\sigma_{jl}\right)_{j,l}$ is symmetric, positive definite, while $\Sigma^{\frac{1}{2}}$ is such that $\Sigma^{\frac{1}{2}}\left(\Sigma^{\frac{1}{2}}\right)' = \Sigma$. The value $m$ is the compensator of a compound Poisson process $m = \log \varphi_{Z_1}(-i)$.

We define two sequences of independent and identically distributed $1 \times d$-dimensional random vectors $(X_k)_{k \in \mathbb{N}}$ and $(X_{0,k})_{k \in \mathbb{N}}$. The components of the random vectors in the first sequence are independent.

The process $(Z_t)_{t \geq 0}$ is a d-variate compound Poisson process, independent of $(B_t)_{t \geq 0}$ such that

$$Z_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} X_k^{(j)} + \sum_{k=1}^{N_t^{(0)}} X_{0,k}^{(j)}, \ j = 1, \ldots, d$$

where $(N_t)_{t \geq 0} = \left(N_t^{(0)}, N_t^{(1)}, \ldots, N_t^{(d)}\right)_{t \geq 0}$ is a vector of independent Poisson processes with respective intensities $\lambda_j$.

The processes $\left(N_t^{(j)}\right)_{t \geq 0}$ and $\left(N_t^{(0)}\right)_{t \geq 0}$ correspond, respectively, to idiosyncratic and common jumps of the $j$-th underlying asset on the interval $[0, t]$. Their jump sizes are $X_k^{(j)}$ and $X_{0,k}^{(j)}$.

For the sake of concreteness, we assume Gaussian jumps, i.e., we assume for any $k \in \mathbb{N}$ that $X_k \sim N\left(\mu_J, D_J\right)$, where $D_J$ is a diagonal matrix with components $D_J(j, l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$ and $X_{0,k} \sim N\left(\mu_{0,J}, \Sigma_{0,J}\right)$, with $\Sigma_{0,J}$ a matrix of components $\Sigma_{0,J}(j, l) = \sigma_0^{j,l}$. The compensator across each dimension takes the form

$$m_j = \lambda_j \left( \exp\left( \mu_J^{(j)} + \frac{1}{2}\left(\sigma_J^{(j)}\right)^2 \right) - 1 \right) + \lambda_0 \left( \exp\left( \mu_{0,J}^{(j)} + \frac{1}{2}\left(\sigma_0^{jj}\right)^2 \right) - 1 \right), \quad j = 1, 2, \dots d$$

Let $C_{JD}$ denote the price of a European basket option with payoff $h(S_T)$ under the model given by Eqs. (1) and (2).

First, we write the price of the basket contract in terms of its conditional price when the number of jumps and $d-1$ underlying assets are fixed. Results are given in Theorem 1 below.

Notice that, for any $k \in \mathbb{N}^{d+1}$

$$p_k = P(N_T = k) = \frac{\exp\left(-\sum_{j=0}^{d} \lambda_j T\right) \prod_{j=0}^{d} \lambda_j^{k_j} T^{\sum_{j=0}^{d} k_j}}{k!} \tag{3}$$

We also introduce the vector $\overline{\mu}(k)$ with components

$$\overline{\mu}_j(k) = \mu_j T + k_j \mu_J^{(j)} + k_0 \mu_{0,J}^{(j)} \qquad j = 1, 2, \dots, d.$$

**Theorem 1.** *Let $C_{JD}$ be the price of a European basket contract with maturity $T$, strike price $K$, and payoff $h(Y_T)$, under a model given by Eqs.* (1) *and* (2). *See proof in Appendix A.2.*

*In addition assume $X_k \sim N(\mu_J, D_J)$ and $X_{0,k} \sim N(\mu_{0,J}, \Sigma_{0,J})$ for any $k \in \mathbb{N}$, where $D_J$ is a $d \times d$ diagonal matrix with components $D_J(j,l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$ and $\Sigma_J^{(0)}$ is also a $d \times d$ matrix with components $\Sigma_{0,J}(j,l) = \sigma_0^{j,l}$.*

*Then, we have*

$$C_{JD} = \sum_{k \in \mathbb{N}^{d+1}} C(k) p_k \tag{4}$$

*where for any $k \in \mathbb{N}^{d+1}$*

$$C(k) := w_1 \exp\left(\frac{1}{2}\sigma^2(k) T\right) E_Q\left[\exp\left(\mu(\tilde{Y}_T, N_T)\right) C(\tilde{Y}_T, N_T) / N_T = k\right] \tag{5}$$

$$\begin{aligned} C(y,k) = e^{-rT} E_Q\Bigg[\Bigg(S_0^{(1)} \exp\left(\left(r - \frac{1}{2}\sigma^2(N_T)\right)T\right. \\ \left. + \ \sigma(N_T)\sqrt{T}Z)\right) - K(\tilde{Y}_T, N_T)\Bigg)_+ / N_T = k, \tilde{Y}_T = y\Bigg] \end{aligned} \tag{6}$$

*with $Z$ a standard normal random variable independent of $N_T$ and $\tilde{Y}_T$.*

*Also*

$$K(y,k) = \exp\left(\left(r - \frac{1}{2}\sigma^2(k)\right)T - \mu(y,k)\right)\left[\frac{K}{w_1} - \sum_{j=2}^{d} \frac{w_j}{w_1} S_0^{(j)} \exp\left(y^{(j)}\right)\right], \quad \text{for } y \in \mathbb{R}^{d-1}$$

$$\mu(y,k) = \overline{\mu}_1(k) + \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)(y - \tilde{\mu}(k))'$$

$$\sigma(k) = \frac{1}{T}\left(\sigma_{11}(k) - \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\Sigma_{1\tilde{Y}}'(k)\right)$$

*Here $\sigma_{jl}(k)$ is the $(j,l)$ component of the matrix:*

$$\Sigma_Y(k) = \Sigma T + D_J \circ D_N + k_0 \Sigma_{0,J} \text{ and}$$

$$D_N(j,l) = \delta_{jl} N_T^{(j)}$$

$$\Sigma_{1\tilde{Y}}(k) = (\sigma_{12}(k), \sigma_{13}(k), \dots, \sigma_{1,d-1}(k))'$$

**Remark 2.** *Notice that when $K(y,k)$ is nonnegative, $C(y,k)$ is the well-known Black-Scholes price of a call option with maturity at $T > 0$, volatility $\sqrt{\sigma(k)}$, spot price $S_0^{(1)}$, and strike price $K(y,k)$. A sufficient condition for $K(y,k)$ to be positive is $w_1 \geq 0$ while $w_j \leq 0, 2 \leq j \leq d$. It is the case of spreads and crack spreads. When $K(y,k)$ is negative, it does not have the meaning of a strike price anymore.*

**Remark 3.** *The values $\mu(y,k)$ and $\sigma(k)$ are, respectively, the mean and variance of the first asset after conditioning on a value $y$ of the remaining assets and the certain number of jumps $k$.*

For any fixed $k \in \mathbb{N}^{d+1}$, we approximate the conditional price $C(y,k)$ on the variable $y$ by a suitable polynomial. In particular we consider Taylor, Chebyshev polynomials and cubic splines.

Approximations based on the three expansions are discussed below.

(i) An order $n$ Taylor approximation of $C(y,k)$ around $y^* \in \mathbb{R}^{d-1}$ is described by

$$C^T(y,y^*,k,n) = \sum_{l=0}^{n} \sum_{L \in R_l} \frac{D^L C(y^*,k)}{L!} (y - y*)^L \tag{7}$$

with $L = (l_1, l_2, \dots, l_{d-1})$, where the second sum is taken on the set

$$R_l = \{L \in \mathbb{N}^{d-1} / l_1 + l_2 + \dots + l_{d-1} = l, \quad 0 \leq l_j \leq l\}.$$

Notice the existence of the derivatives of any order in the functions $K(y)$ and $C(y,k)$.

(ii) An approximation based on Chebyshev polynomials is given as follows:

In a region $D \subset \mathbb{R}^{d-1}$, we consider an expansion of order $n = (n_1, n_2, \dots, n_{d-1})$ of the function $C(y,k)$ as

$$\begin{aligned} C^{Ch}(y,k,n) &= \frac{1}{2}\hat{c}_0(k)1_D(y) + \sum_{l \in B_n} \hat{c}_l(k) T_l^D(y) 1_D(y) \\ &= \frac{1}{2}\hat{c}_0(k)1_D(y) + \sum_{l \in B_n} \sum_{m \in C_l} \hat{c}_l(k) b_{m,l} y^{l-2m} 1_D(y) \end{aligned} \tag{8}$$

where the sums are taken over the sets

$$\begin{aligned} B_n &= \{l \in \mathbb{N}^{d-1} / 0 \leq l \leq n_j ; j = 1, 2, \dots, d-1.\} \\ C_l &= \left\{m \in \mathbb{N}^{d-1} / 0 \leq m_j \leq \left[\frac{l_j}{2}\right], j = 1, 2, \dots, d-1\right\} \end{aligned}$$

Here $(T_l^D)_{l \in B_n}$ is a family of $d-1$-dimensional Chebyshev polynomials with degrees $l \in B_n$ defined in the region $D$, while the quantities $\hat{c}_l(k)$ are suitable approximations of the corresponding Chebyshev coefficients $c_l(k)$, computed using the trapezoidal rule.

Notice that, by the orthogonality of the polynomials, the coefficients in the expansion are $c_l(k) = \langle C, T_l^D \rangle_W$, where $\langle f, g \rangle_W$ is the scalar product of functions $f$ and $g$, conveniently weighted by a function $W$. See, for example, [10] for a general account on Chebyshev polynomials.

For convenience, we write the Chebyshev polynomials in terms of powers of their variables, where $b_{m,l}$ are the coefficients of this expansion.

In particular, for a rectangular region $D = [a,b]^{d-1}$ and valued vectors $a = (a_1, a_2, \dots, a_{d-1})$ and $b = (b_1, b_2, \dots, b_{d-1})$, we write

$$T^D(y) := T_l^{a,b}(y) = T_l^{-1,1}\left(-1 + 2\frac{y-a}{b-a}\right)$$

Hence, for $d = 2$

$$\begin{aligned} C^{Ch}(y,k) &= \frac{1}{2}\hat{c}_0(k)1_D(y) \\ &+ \sum_{l=1}^{n}\sum_{m=0}^{\left[\frac{l}{2}\right]}(b-a)^{2m-l}\hat{c}_l(k)b_{m,l}(2y-(a+b))^{l-2m}1_D(y) \end{aligned} \tag{9}$$

See, for example, [11] for specific expressions of $b_{m,l}$ in one dimension.

(iii) Approximation by cubic splines.

On a rectangular region $D = [a,b]^{d-1}$, we consider an approximation based on cubic splines given by

$$C^{spl}(y,k) = \sum_{j=1}^{N}\sum_{l\in B_3}\alpha_{j,l}(k)\left(y-b_{j-1}\right)^l 1_{D_j}(y) \tag{10}$$

where $b_j$ is some point on a (d − 1)-dimensional grid $\{b_0, b_1 \dots, b_N\}$ with $N+1$ points in $D$.

The local coefficients $\alpha_{j,l}(k)$ are determined by imposing the conditions $C\left(y_j,k\right) = z_{jk}, j, k = 1, \dots, N+1$. The family of sets $\left\{D_j,\ j = 0, 1, \dots, N\right\}$ is a partition of $D$. Notice that the coefficients $\alpha_{j,l}(k)$ depend on the particular rectangle in the grid. See [12] for a general account on multivariate splines.

In the case of $d = 2$, splines used to approximate the conditional price become one-dimensional polynomials. Additional conditions on the derivatives to smoothen these curves are imposed, namely, $D_-^l C\left(y_j,k\right) = D_+^l C\left(y_j,k\right)$, $j = 1, 2, \dots, N, l = 1, 2,$ where $D_-^l C\left(y_j,k\right)$ and $D_+^l C\left(y_j,k\right)$ are, respectively, the derivatives from the left and the right of the function $C(y,k)$ at point $y = y_j$. Moreover, for end points in the grid, $D^2\left(y_0,k\right) = D^2\left(y_N,k\right) = 0$.

In order to approximate the prices, we replace the function $C(y,k)$ by its respective expansions. The conditional prices on the event $[N_T = k]$ are estimated by approximating the corresponding conditional expected values. Substituting the approximations of conditional prices into Eq. (4), we obtain, after truncation, estimates of the price of the basket contract, under the jump-diffusion model described by Eqs. (1) and (2). They are denoted, respectively, by $C_{JD}^T(y^*)$, $C_{JD}^{Ch}$, and $C_{JD}^{spl}$.

Notice that these estimates depend on the mixing exponential-power moments of the log-prices. The latter can be computed from its conditional moment-generating function under the selected EMM. Hence, for a vector $X$ and a Borel set $D$, we define

$$\begin{aligned} M_X(u,k) &= E_{\mathcal{Q}}[\exp(uX)/N_T = k] \\ M_X(u,k,D) &= E_{\mathcal{Q}}[\exp(uX)1_D(X)/N_T = k] \end{aligned}$$

In particular when $D = [a,b]^{d-1}$, we write $M_X(u,k,D) = M_X(u,k,a,b)$.

Concrete expressions of these approximations under a two-dimensional Gaussian model are shown in Theorem 4.

As it is well known, the conditional mixed exponential-power moments of a random vector $X$ are related to the partial derivatives of the corresponding moment-generating function Indeed, for $\nu \in \mathbb{N}^{d-1}$, we have

$$D^{\nu}M_X(u,k,D) = E_{\mathcal{Q}}[\exp(uX)X^{\nu}1_D(X)/N_T = k], \quad u \in \mathbb{R}^{d-1}$$

In order to simplify notations, we introduce the following quantities:

$$A_1(k) = \frac{1}{2}\sigma^2(k)T + \overline{\mu}_1(k) - \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\tilde{\mu}(k)'$$
$$A_2(k) = A_1(k) + \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)y^*$$
$$A_3(k) = A_1(k) + \frac{1}{2}(a+b)\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)$$

and the set

$$\mathbb{N}_M^{d+1} = \{k = (k_0, k_1, \ldots, k_d)/k_j = 0, 1, \ldots, M_j, j = 0, 1, \ldots, d\}$$

**Theorem 4.** *Let $C_{JD}$ be the price of a European basket contract with maturity $T$, strike price $K$, and payoff $h(Y_T)$ under a model given by Eqs. (1) and (2). In addition assume $X_k \sim N(\mu_J, D_J)$ and $X_{0,k} \sim N(\mu_{0,J}, \Sigma_{0,J})$ for any $k \in \mathbb{N}$, where $D_J$ is a $d \times d$ diagonal matrix with components $D_J(j,l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$. Let $\Sigma_J^{(0)}$ be a $d \times d$ matrix with components $\Sigma_{0,J}(j,l) = \sigma_0^{j,l}$.*

*Then, its n-th-order approximation around $y^* \in \mathbb{R}^{d-1}$ in terms of Taylor polynomials is given by*

$$C_{JD}^T(y^*) = w_1 \sum_{k \in \mathbb{N}_M^{d+1}} \sum_{l=0}^{n} \sum_{L \in R_l} \exp(A_2(k)) \frac{D^L C(y^*, k)}{L!} D^L M_{\tilde{Y}_T - y^*}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k\right) p_k \tag{11}$$

*for some truncation vector $M \in \mathbb{N}^{d+1}$.*

*The n-th-order Chebyshev approximation on a region $D = [a,b]^{d-1}$ is*

$$\begin{aligned} C_{JD}^{Ch} = {} & \frac{w_1}{2} \sum_{k \in \mathbb{N}_M^{d+1}} [\hat{c}_0(k) K_1(a,b,k) \\ & + w_1 \sum_{k \in \mathbb{N}_M^{d+1}} \sum_{l \in B_n} \sum_{m \in C_l} \exp(A_3(k)) \hat{c}_l(k) b_{m,l} (b-a)^{2m-l} \\ & D^{l-2m} M_{\tilde{V}_T}(\frac{1}{2}\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, -(b-a), b-a)] p_k \end{aligned} \tag{12}$$

*where $\tilde{V}_T = 2\tilde{Y}_T - (a+b)$ and $K_1(a,b,n) = \exp(A_1(k)) M_{\tilde{Y}_T}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, a, b\right)$.*

*The n-th-order approximation by cubic splines on the region $D = [a,b]^{d-1}$ is given by*

$$\begin{aligned} C_{JD}^{spl} = {} & w_1 \sum_{k \in \mathbb{N}_M^{d+1}} \left[\exp\left(\frac{1}{2}\sigma^2(k)T\right)\right. \\ & \sum_{j=1}^{N} \sum_{l \in B_3} \exp\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k) b_{j-1}\right) \alpha_{j,l}(k) \\ & \left. D^m M_{\tilde{Y} - b_{j-1}}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, D_j\right)\right] p_k \end{aligned} \tag{13}$$

**Remark 5.** *The point $y^*$ around which the Taylor expansion is taken, in general, depends on $k$.*

## 3. Approximating the price of spread contracts

Spread contracts are the most common basket derivatives. In this case the payoff is written as $h(S_T) = \left(S_T^{(1)} - S_T^{(2)} - K\right)_+$.

Hence for $d = 2$, conditionally on $\left[Y_T^{(2)} = y\right] \cap [N_T = k]$, the log-prices of the first asset are normally distributed, i.e., $Y_T^{(1)} \sim N(\mu(y,k), \sigma^2(k))$, with

$$\mu(y,k) = \overline{\mu}_1(k) + \sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(y - \overline{\mu}_2(k))$$
$$\sigma^2(k) = \frac{1}{T}\left(\sigma_{11}(k) - \frac{\sigma_{12}^2(k)}{\sigma_{22}(k)}\right) = \frac{1}{T}\left[1 - (\overline{\rho}(k))^2\right]\sigma_{11}(k)$$

where

$$\overline{\rho}(k) = \frac{\sigma_{12}(k)}{\sqrt{\sigma_{11}(k)}\sqrt{\sigma_{22}(k)}}$$

is the conditional correlation coefficient between the two assets.

A result about the derivatives of the moment-generating function of a constrained standard normal random variable $Z$ on the interval $(-\infty, b)$ is needed. To this end we have

$$\begin{aligned} D^m M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k), k, -\infty, b\right) = \exp\left(\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right) \\ bin\left(m, \mu V\left(-\infty, b - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right), \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right) \end{aligned} \tag{14}$$

where $\mu(m,a,b) = \mu(m,-\infty,b) - \mu(m,-\infty,a) = \int_a^b z^m f_Z(z)\ dz$ is the $m$-th moment of a standard normal random variable constrained to the interval $(a,b)$ and $\mu V(a,b)$ is a vector with components $\mu(j,a,b), j = 0, 1, \ldots, m$.

By integration by parts, the later can be calculated recursively as

$$\begin{aligned} &\mu(0,a,b) = N(b) - N(a) \\ &\mu(1,a,b) = f_Z(a) - f_Z(b) \\ &\mu(m,a,b) = (m-1)\mu(m-2,a,b) + a^{m-1}f_Z(a) - b^{m-1}f_Z(b)), \quad m \geq 2 \end{aligned}$$

For a Taylor expansion, derivatives of the moment-generating function and constrained moment-generating function for the second component of the log-prices are computed as follows:

$$\begin{aligned} D^l M_{Y_T^{(2)} - y*}\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k), k\right) = \exp\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(\overline{\mu}_2(k) - y^*)\right) \\ bin\left(l, PV\left(\sqrt{\sigma_{22}(k)}\right)1 \circ DV\left(M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right)\right), \overline{\mu}_2(k) - y^*\right) \end{aligned}$$

Now, combining the expressions above with Eq. (11), we have

$$\begin{aligned} C_{JD}^T(y^*) = w_1 \sum_{k \in \mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T + \overline{\mu}_1(k)\right) \sum_{l=0}^{n} \binom{l}{m} \left(\frac{D^l C(y^*, k)}{l!}\right) (\overline{\mu}_2(k) - y^*)^{l-m} \\ bin\left(l, PV\left(\sqrt{\sigma_{22}(k)}\right)1 \circ DV\left(M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right)\right), \overline{\mu}_2(k) - y^*\right) p_k \end{aligned} \tag{15}$$

Next, we obtain the Taylor approximations up to third order. By elementary calculation we can compute the derivatives of the function $C(y,k)$ with respect to $y$.

First, notice that, from the Black-Scholes pricing formula:

$$C(y,k) = S_0^{(1)} N\big(d_1(K(y,k)) - K(y,k)e^{-rT} N(d_2(K(y,k))\big)$$

where

$$d_1(K(y,k)) = \frac{\log\left(\dfrac{S_0^{(1)}}{K(y,k)}\right) + \left(r + \dfrac{\sigma(k)}{2}\right)T}{\sigma(k)\sqrt{T}}$$
$$d_2(K(y,k)) = d_1(K(y,k)) - \sigma(k)\sqrt{T}$$

and $N(.)$ is the cumulative distribution function of a standard normal distribution. Hence

$$D^1C(y,k) = -\frac{1}{\sqrt{\sigma(k)T}} T_1(y,k)A(y,k)$$

where

$$\begin{aligned} T_1(y,k) &= \frac{D^1K(y,k)}{K(y,k)} \\ A(y,k) &= S_0^{(1)} f_Z(d_1(K(y,k))) + \sigma(k)\sqrt{T}e^{-rT}K(y,k)N(d_2(K(y,k))) \\ &\quad - e^{-rT}K(y,k)f_Z(d_2(K(y,k))) \end{aligned}$$

Higher derivatives can be calculated recursively.

$$D^nC(y,k) = -\frac{1}{\sqrt{\sigma(k)T}} \sum_{l=0}^{n-1} \binom{n-1}{l} D^l T_1(y,k) D^{n-l-1} A(y,k)$$

Concrete expressions for second- and third-order derivatives are shown in the appendix.

Regarding the approximation based on Chebyshev polynomials, we first compute the moment-generating function of the random variables $Y_T^{(2)}$ and $V_T^{(2)}$ constrained to the interval $(a,b)$. To this end we denote

$$\tilde{b} = \frac{b - \overline{\mu}_2(k)}{\sqrt{\sigma_{22}(k)}}, \quad \tilde{a} = \frac{a - \overline{\mu}_2(k)}{\sqrt{\sigma_{22}(k)}} \tag{16}$$

Notice that, taking into account Eq. (14),

$$\begin{aligned} &D^m M_Z\left(\sqrt{\sigma_{11}}\overline{\rho}(k), \tilde{a}, \tilde{b}\right) = \exp\left(\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right) \\ &bin\left(m, \mu\left(m_1, \tilde{a} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k), \tilde{b} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right), \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right) \end{aligned} \tag{17}$$

Moreover

$$\begin{aligned} M_{Y_T^{(2)}}\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k), k, -\infty, b\right) &= \exp\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)\overline{\mu}_2(k) + \frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right) \\ &\quad N\left(\tilde{b} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right) \end{aligned}$$

Hence

$$D^{\nu}M_{V_T^{(2)}}\left(\frac{1}{2}\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k),k,-(b-a),b-a\right)$$
$$= \exp\left(\frac{1}{2}\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(2\overline{\mu}_2(k)-a-b)\right)G(\nu,k)$$

where

$$G(\nu,k) = bin\left(\nu, M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k),k,\tilde{a},\tilde{b}\right)\circ PV\left(2\sigma_{22}^{\frac{1}{2}}(k)1\right),2\overline{\mu}_2(k)-a-b\right)$$

Then, combining Eq. (12) with the results above, we get

$$C^{Ch}(k,n) = \frac{w_1}{2}\hat{c}_0(k)K_1(a,b,k)$$
$$+ w_1 \exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)\right)\sum_{l=1}^{n}\sum_{m=0}^{\left[\frac{l}{2}\right]}\hat{c}_l(k)b_{m,l}K(a,b,l,m)G(l-2m,k)$$

Finally, the n-th-order Chebyshev approximation is given by

$$C_{JD}^{Ch} = \sum_{k\in\mathbb{N}_M^3} C^{Ch}(k,n)p_k$$

Similarly for a cubic spline approximation, we specialize Eq. (13) with $D=(a,b), D_j=(b_{j-1},b_j), b_0=a, b_{N+1}=b$. Therefore, we have

$$C_{JD}^{spl} = w_1\sum_{k\in\mathbb{N}_M^{d+1}}\left[\exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)\right)\right.$$
$$\left.\sum_{j=1}^{N}\sum_{l=0}^{3}\alpha_{j,l}(k)(\sigma_{22}(k))^{\frac{l}{2}}bin(l,DVM_Z(\sqrt{\sigma_{11}(k)}\overline{\rho}(k),\tilde{b}_{j-1},\tilde{b}_j),-\tilde{b}_{j-1})\right]p_k \tag{18}$$

where $\tilde{b}_j$ is defined as $\tilde{b}$ in Eq. (16) but replacing $b$ by $b_j$.

## 4. Numerical results

We implement the results from the previous section to price spread contracts and show that the approximations considered above produce accurate price values when compared with a standard Monte Carlo approach, at a lesser computational effort.

To this end we consider the following benchmark set of parameters:

The contract specifications consist a strike price of $K=\$1$, maturity $T=1$ year, spot prices $S_0^{(1)}=\$100$, $S_0^{(2)}=\$96$, and a fix interest rate of 3%.

Volatilities corresponding to the diffusion part of both assets are $\sigma_1=10\%$ and $\sigma_2=30\%$, while the correlation coefficient between the two Brownian noises is $\rho=0.3$. Regarding the jump part, we consider an average intensity of the common jumps equal to $\lambda_0=3$ jumps per year and idiosyncratic intensities $\lambda_1=\lambda_2=2$ jumps per year for the respective assets, while jump sizes have means equal to zero; volatilities of common jump sizes are $\sigma_{0,1}=1\%$, $\sigma_{0,2}=5\%$, with a linear correlation $\rho_J=0.5$. Volatilities of the idiosyncratic jumps are taken as $\sigma_{J,1}=10\%$ and $\sigma_{J,2}=20\%$.

| | MC | Taylor (f.o) | Taylor (s.o.) | Spl. |
|---|---|---|---|---|
| Price | 14.7784 | 10.2980 | 14.29068 | 14.8842 |
| Interval | (14.7683, 14.7885) | — | — | — |
| Run time | 624.312 | 1.68806 | 1.68806 | 54.1720 |

*In row three the average computer time (in seconds) for different pricing methods is shown.*

**Table 1.**
*Prices obtained using the benchmark parameter set and Monte Carlo, first- and second-order Taylor, and cubic spline approximations.*

Although these values are somehow arbitrary, they have been selected to produce *reasonable* asset prices in connection with contracts based on crude oil prices. It is worth noting that there is not a general agreement about the range of the parameters in a jump-diffusion model. Indeed they may depend on the market into consideration.

In **Table 1** prices of spread contracts under different methods are shown. Prices are obtained using Taylor and cubic splines approximations and contrasted with a Monte Carlo approach. For the latter we carry $10^7$ repetitions to achieve stable results, with a relative average error of 0.1%. In addition, 95% Monte Carlo confidence intervals and running times are provided. Implementation is done on a Surface Pro 4 i7 computer, using MATLAB language.

The efficiency of the Monte Carlo method can be improved by considering only the simulation of a single asset with the corresponding conditional probability and then computing the discounted average of the conditional Black-Scholes price. It reduces the computational time by half, still considerably higher than those based on polynomial expansions. Chebyshev polynomial approximation is discussed in [1].

The expansions also require repetitive evaluations of conditional prices, which turn out to be given by simple Black-Scholes closed formulas.

For a Taylor approach of order $n$, evaluations in the order of $nM^3$ are needed, where $M$ is the maximum truncation level in the number of jumps. In a Chebyshev approach of the same order about $n^2NM^3$, evaluations of the conditional price should be performed, when a grid of $N$ points is used in a trapezoidal approximation of the corresponding integrals. In a cubic splines approximation $3NM^3$. Here $N$ is also the number of points in the grid where the polynomial coefficients are adjusted.

For a theoretical analysis of the error using Taylor and Chebyshev expansions, although in different contexts, see [13] for Taylor and [6] for Chebyshev cases.

In **Figure 1a**, a graph of conditional prices in function of log-price values of the first asset (blue line) with average number of jumps equal to $k_0 = 3$ and $k_1 = k_2 = 2$

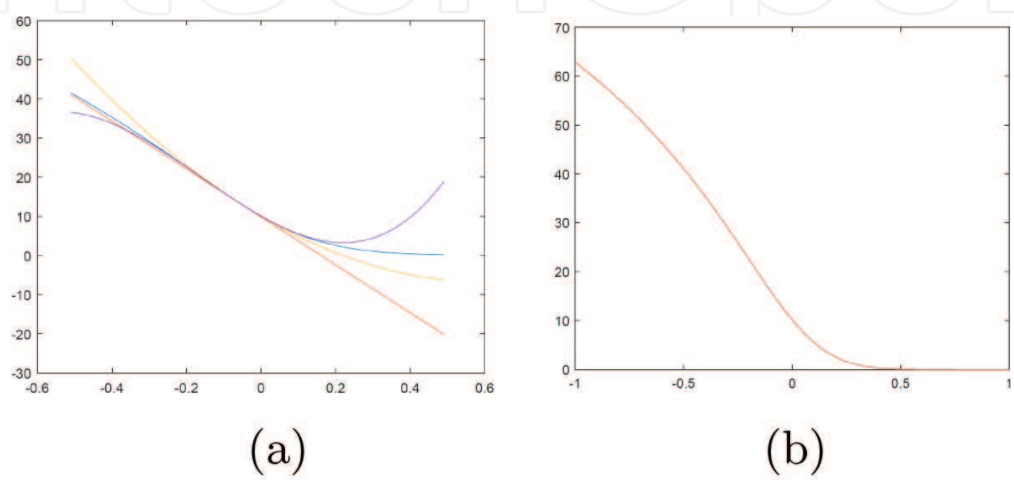


**Figure 1.**
*(a) Conditional price (blue curve) as function of log-price values and its Taylor approximations up to third order around the average. (b) Conditional price vs. its cubic spline approximation.*

is shown. The remaining three curves represent the first-order (green), second-order (red), and third-order (magenta) Taylor polynomials around the average value $y^* = E_{\mathbb{Q}}\left(Y_T^{(2)}\right)$. In **Figure 1b**, conditional prices and its cubic spline approximation are shown. At this scale both are indistinguishable. Notice that, although the Taylor approximation is excellent in a neighborhood of the expansion point, there are significant deviations for values far from the mean. These deviations, under the assumption of normality of the jump sizes, result to be infrequent; therefore, they do not impact the global error, but might be significant when other probability distributions, in particular heavy-tailed ones, or even normal jumps with higher volatilities, are taken into account. Instead of local approximations, as the case of Taylor polynomial expansion, uniform approximations on a given interval may reduce the error. Expansions based on orthogonal basis, e.g., Chebyshev or varying coefficients as in the case of cubic splines, are suggested. Notice that the function $C(y, k)$ is continuous in $y$ for any value of $k$; therefore, Weierstrass' theorem of uniform convergence applies. Curiously, the convergence of Bernstein polynomials, applied in the original proof of the theorem, is remarkably slow.

**Figure 2** shows the differences between the conditional price and the cubic spline for different values of the underlying price. Truncation values were selected as $a = -1$ and $b = 1$. Generally speaking the choice of these values depends on the probability distribution of the underlying asset. In practice it requires an exploratory study of the available data. On the other hand, the larger the interval, the more accurate is the approximation but also is the computational effort. Moreover, we have found that the results are sensible to this choice, though rather robust to the number of splines and the truncation values.

Truncation values for the number of jumps, denoted in the paper by $M_0$, $M_1$ and $M_2$, should cover most of the jump probability distribution $\{p_k, k \in \mathbb{N}^3\}$. An efficient way of choosing these values consists in starting to evaluate the sum at a point close to where the maximum value of the $p_k$'s is attained, namely,

$$k = (k_0, k_1, k_2) = \left([\lambda_0 T - 1]_+, [\lambda_1 T - 1]_+, [\lambda_2 T - 1]_+\right)$$

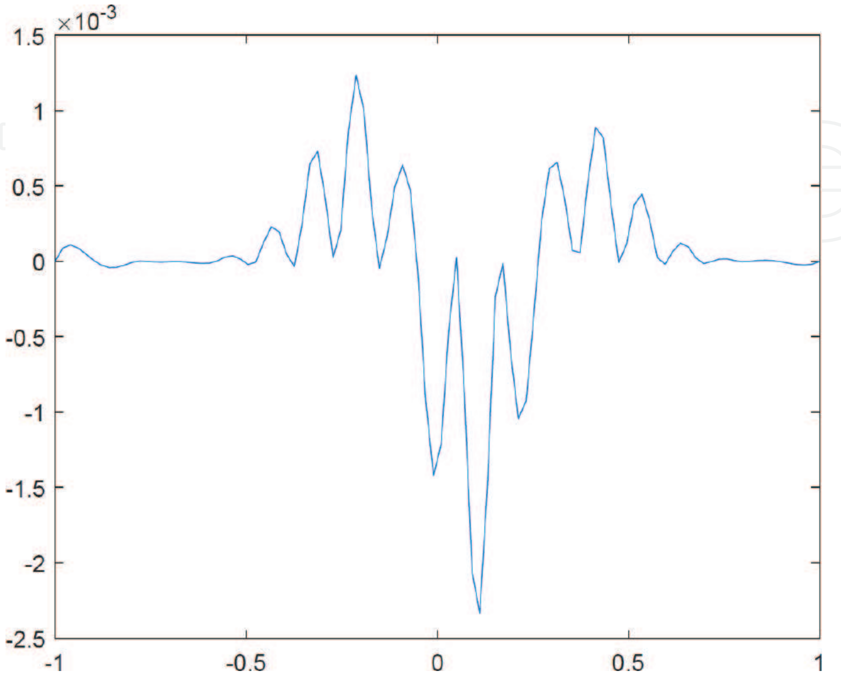


**Figure 2.**
*Curve representing the difference between conditional price and cubic spline approximation for the benchmark parameters.*

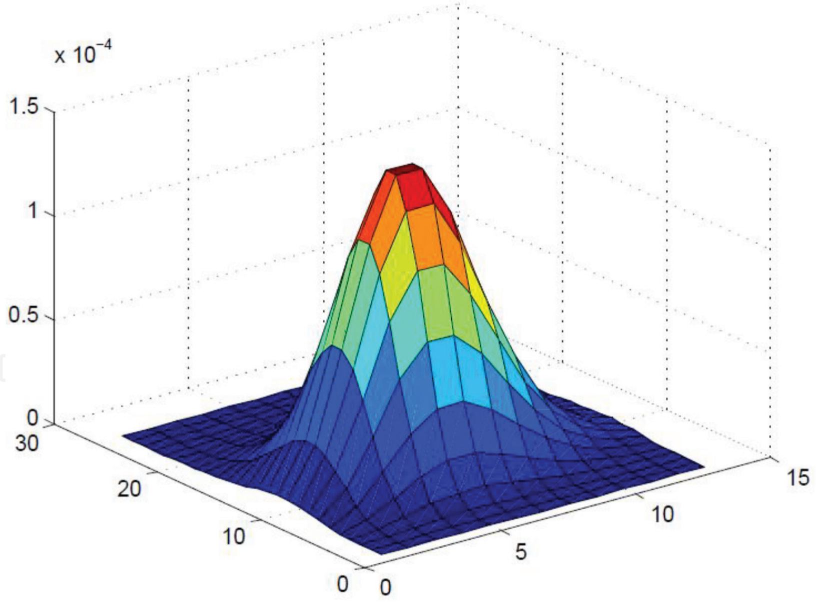


**Figure 3.**
*Probabilities $p_k$ to observe $k = (k_0, k_1)$ jumps when $k_2 = 5$. Truncation values $M_0 = 15, M_1 = 10, M_2 = 10$ capture 99.67% of the probability distribution in the number of jumps.*

where $[x]_+$ represents the maximum of the integer part of $x$ and zero, then adding expression (18) for points $j = (j_0, j_1, j_2)$ over the set

$$N_M(k) = \left\{ (k_0 + j_0, k_1 + j_1, k_2 + j_2) \in \mathbb{N}^3 / \left( k_l - \frac{M_l}{2} \right)_+ \leq j_l \leq k_l + \frac{M_l}{2}, l = 0, 1, 2 \right\}$$

until $\sum_k p_k \geq \delta$, where $\delta$ is a predetermined value close to one.

In **Figure 3** we show the probability distribution $\{p_k, k \in \mathbb{N}^3\}$, for $k_2 = 5$ varying $k_0$ and $k_1$. We observe probabilities become negligible after certain values of $(k_0, k_1)$ with a peak around the center of the distribution. For the benchmark parameter set truncation values $M_0 = 15, M_1 = 10, M_2 = 10$ capture 99.67% of the probability mass.

## 5. Conclusions and future developments

The paper establishes a methodology over the use of polynomial approximations based on Taylor, Chebyshev, and cubic splines to the price of basket contracts. This approach produces accurate results at a lesser computational effort than a standard Monte Carlo technique. The claim is supported by numerical evidence in the case of spread options, under a bivariate jump-diffusion model with a complex Gaussian jump structure that allows to capture the dependence between assets.

The study needs to be extended to different parameter values to corroborate the results in a wider scope. Moreover, optimal choices in the numerical implementation, for example, the order of the polynomials, the number of points in the grid, and truncation levels, require a further study.

Sensitivities with respect to the parameters in the model and the contract, i.e., maturity, strike, interest rate, correlation, etc., can be easily calculated with a straightforward adaptation of the current method. It is enough to approximate the corresponding derivatives instead.

A natural question is how to adapt our method when a non-Gaussian joint distribution of the jump sizes is considered. In this setting, the conditional probability distribution is generally unknown; nonetheless, the use of a copula approach to capture the dependence may provide some insight.

## Acknowledgements

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# A. Appendix

## A.1 Taylor implementation up to third order

After computing the second and third derivatives of $C(y,k)$ and the corresponding derivatives of the moment-generating function of $Z$, we can compute Taylor approximations up to third order around the point $y^*$ as

$$C_{JD}^T(y^*,1) = w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T + \overline{\mu}_1(k) + \frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)$$

$$\left[C(y^*,k) + D^{(1)}C(y^*,k)((\overline{\mu}_2(k) - y^*) + \sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k))\right]p_k$$

$$C_{JD}^T(y^*,2) = C_{JD}^T(y^*,1) + w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T + \overline{\mu}_1(k) + \frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)D^{(2)}$$

$$C(y^*,k)\left[\frac{1}{2}(\overline{\mu}_2(k) - y^*)^2 + (\overline{\mu}_2(k) - y^*)\sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k)\right.$$

$$\left.+\frac{1}{2}\left(\sigma_{22}(k)(1+\sigma_{11}(k))(\overline{\rho}(k))^2\right)\right]p_k$$

$$C_{JD}^T(y^*,3) = C_{JD}^T(y^*,2) + \ w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T + \overline{\mu}_1(k) + \frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)D^{(3)}C(y^*,k)$$

$$\left[\frac{1}{6}(\overline{\mu}_2(k) - y^*)^3 + \frac{1}{2}(\overline{\mu}_2(k) - y^*)^2\sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k)\right.$$

$$+\frac{1}{2}(\overline{\mu}_2(k) - y^*)\sigma_{22}(k)\left(1 + \sigma_{11}(k)(\overline{\rho}(k))^2\right)$$

$$\left.+\frac{1}{6}\sigma_{22}(k)^{\frac{3}{2}}\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\left(\sigma_{11}(k)(\overline{\rho}(k))^2 + 3\right)\right]p_k$$

## A.2 Proof of Theorem 1

From Eq. (2) written in its integral form

$$Y_T = \mu T + \Sigma^{\frac{1}{2}}B_T + Z_T$$

it is easy to see that

$$E_Q(Y_T/N_T) = \mu T + \tilde{N}_T \circ \mu_J + N_T^{(0)}\mu_{0,J} = \overline{\mu}_j\left(\tilde{N}_T\right)$$

$$\Sigma_Y(N_T) := Var(Y_T/N_T) = \Sigma^{\frac{1}{2}} Var(B_T)\Sigma^{\frac{1}{2}} + Var(Z_T/N_T)$$

$$= \Sigma T + Var(Z_T/N_T)$$

From the expression above, in the case of $j \neq l$, we have

$$\mathrm{cov}\left(\left(Z_T^{(j)}, Z_T^{(l)}\right)/N_T\right) = E_Q\left(\sum_{k=1}^{N_T^{(0)}}\sum_{k'=1}^{N_T^{(0)}}\left(X_{0,k}^{(j)} - E_Q X_{0,k}^{(j)}\right)\left(\tilde{X}_{k'}^{(l)} - E_Q X_{0,k}^{(l)}\right)/N_T\right)$$

$$= N_T^{(0)}\mathrm{cov}\left(X_{0,k}^{(j)}, X_{0,k}^{(l)}/N_T\right) = N_T^{(0)}\sigma_0^{j,l}$$

Similarly, for $j = l$

$$\mathrm{cov}\left(Z_T^{(j)}, Z_T^{(l)}/N_T\right) \quad = \quad N_T^{(j)}\left(\sigma_J^{(j)}\right)^2 + N_T^{(0)}\sigma_0^{j,j}$$

Then, conditionally on $N_T$, we have

$$Y_T \sim N\left(\mu T + N_T \circ \mu_J + N_T^{(0)}\mu_{0,J}, \Sigma_Y(N_T)\right) \tag{19}$$

Hence, the price is expressed as

$$C_{JD} = e^{-rT}\sum_{k \in \mathbb{N}^{d+1}} E_Q(h(S_T)/N_T = k)p_k = \sum_{k \in \mathbb{N}^{d+1}} C(k)p_k \tag{20}$$

where $C(k) = e^{-rT}E_Q[h(S_T)/N_T = k]$.
On the other hand, conditioning on $[N_T = k] \cap \tilde{Y}_T$:

$$C(k) := e^{-rT}$$
$$E_Q[h(S_T)/N_T = k] = e^{-rT}E_Q\left[E_Q\left(h(S_T)/N_T = k, \tilde{Y}T\right)/N_T = k\right]$$
$$= w_1 e^{-rT}$$
$$E_Q\left[E_Q\left[\left(S_0^{(1)}\exp\left(Y_T^{(1)}\right) - \left(\frac{K}{w_1} - \sum_{j=2}^{d}\frac{w_j}{w_1}S_0^{(j)}\exp\left(Y_T^{(j)}\right)\right)\right)_+ /N_T = k, \tilde{Y}_T\right]/N_T = k\right]$$
$$= w_1 e^{-rT}E_Q\left[E_Q\left[\left(S_0^{(1)}\exp\left(Y_T^{(1)}\right) - K_1\left(\tilde{Y}_T\right)\right)_+ /N_T = k, \tilde{Y}_T\right]/N_T = k\right] \tag{21}$$

where $K_1(y) = \frac{K}{w_1} - \sum_{j=2}^{d}\frac{w_j}{w_1}S_0^{(j)}e^{y^{(j)}}$.

Taking into account Eq. (19), again conditioning on the events $\tilde{Y}_T = y$ and $N_T = k$, it is well known that $Y_T^{(1)}$ has a univariate normal distribution with mean and variance given, respectively, by $\mu(y,k)$ and $\sigma^2(k)T$. See, for example, [14].

Hence, we can write, on the set $\left[\tilde{Y} = y \cap N_T = k\right]$:

$$Y_T^{(1)} = \mu(y,k) + \sigma(k)\sqrt{T}Z$$

Then, replacing the expression above in Eq. (21), we have

$$\begin{aligned}C(k) &= w_1 e^{-rT} E_{\mathcal{Q}}\Big[E_{\mathcal{Q}}\Big[\Big(S_0^{(1)}\exp\big(\mu(\tilde{Y}_T,N_T)+\sigma(N_T)\sqrt{T}Z\big)-K_1(\tilde{Y}_T)\big)_+/\mathcal{F}^{\tilde{Y}_T},N_T=k\Big]/N_T=k\Big]\\ &= w_1 e^{-rT} E_{\mathcal{Q}}\Big[\exp\Big(-\Big(r-\frac{1}{2}\sigma(k)\Big)T+\mu(\tilde{Y}_T,k)\Big)E_{\mathcal{Q}}\Big[\Big(S_0^{(1)}\exp\Big(\Big(r-\frac{1}{2}\sigma^2(N_T)\Big)T+\sigma(N_T)\sqrt{T}Z\Big)\\ &\quad -\exp\big(\big(r-\tfrac{1}{2}\sigma^2(N_T)\big)T-\mu(\tilde{Y}_T,N_T)\big)K_1(\tilde{Y}_T)\big)_+/\mathcal{F}^{\tilde{Y}_T},N_T\Big]/N_T=k\Big]\\ &= w_1\exp\Big(\frac{1}{2}\sigma^2(k)T\Big)E_{\mathcal{Q}}\big[\exp\big(\mu(\tilde{Y}_T,N_T)\big)C(\tilde{Y}_T,N_T)/N_T=k\big]\end{aligned} \tag{22}$$

Eq. (4) easily follows after replacing Eq. (22) into Eq. (20).

## A.3 Proof of Theorem 4

In Eq. (6) we replace the function $C(y,k)$ by its Taylor expansion given in Eq. (7).

Then, the Taylor approximation of $C(k)$ is

$$\begin{aligned}C^T(y^*,k) &= w_1\exp\Big(\frac{1}{2}\sigma^2(k)T\Big)E_{\mathcal{Q}}\big[\exp\big(\mu(\tilde{Y}_T,N_T)\big)C^T(\tilde{Y}_T,y^*,N_T)/N_T=k\big]\\ &= w_1\sum_{l=0}^{n}\sum_{L\in R_l}\frac{D^L C(y^*,k)}{L!}\,exp\Big(\frac{1}{2}\sigma^2(k)T+\overline{\mu}_1(k)-\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\tilde{\mu}(k)'\Big)\\ &\quad E_{\mathcal{Q}}\Big[\exp\Big(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\Big)\big(\tilde{Y}_T-y^*\big)^L/N_T=k\Big]\\ &= w_1\exp\big(A_2(k)\big)\sum_{l=0}^{n}\sum_{R_l}\frac{D^L C(y^*,k)}{L!}D^L M_{\tilde{Y}_T-y^*}\Big(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k),k\Big)\end{aligned}$$

Eq. (11) follows after replacing $C(k)$ in Eq. (20) by the expression above and truncating at point $M$.

After replacing Eq. (9) into Eq. (22), we have

$$\begin{aligned}C^{Ch}(k) &= \frac{w_1}{2}\hat{c}_0(k)\exp\big(A_1(k)\big)E_{\mathcal{Q}}\Big[\exp\Big(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\Big)1_D\big(\tilde{Y}_T\big)/N_T=k\Big]\\ &\quad +w_1\exp\big(A_1(k)\big)\sum_{l\in B_n}\hat{c}_l(k)E_{\mathcal{Q}}\Big[\exp\Big(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\Big)T_l^D\big(\tilde{Y}_T\big)/N_T=k\Big]\\ &= \frac{w_1}{2}\hat{c}_0(k)K_1(k,a,b)+w_1\exp\big(A_1(k)\big)\sum_{l\in B_n}\sum_{m\in C_l}\hat{c}_l(k)b_{m,l}\\ &\quad E_{\mathcal{Q}}\left[\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)\left(-1+2\frac{\tilde{Y}_T-a}{b-a}\right)^{l-2m}1_D\big(\tilde{Y}_T\big)\right]\end{aligned}$$

Eq. (12) easily follows.

Finally, by similar arguments,

$$\begin{aligned}C_{JD}^{spl}(k) &= w_1\exp\Big(\frac{1}{2}\sigma^2(k)T\Big)\\ &\quad \sum_{j=1}^{N}\sum_{l\in B_3}\alpha_{j,l}(k)E_{\mathcal{Q}}\Big(\exp\Big(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\Big)\big(\tilde{Y}_T-c_j\big)^l 1_{D_j}\big(\tilde{Y}_T\big)/N_T=k\Big)\end{aligned}$$

$$= w_1 \exp\left(\frac{1}{2}\sigma^2(k)T\right)$$
$$\sum_{j=1}^{N}\sum_{l\in B_3} \exp\left(\Sigma_{1\tilde{Y}}(k)\Sigma\tilde{Y}^{-1}(k)c_j\right)\alpha_{j,l}(k)D^l M_{\tilde{Y}-c_j}\left(\Sigma_{1\tilde{Y}}(k)\Sigma\tilde{Y}^{-1}(k),k,D_j\right)$$

from which (13) follows.

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