Chapter

Simple Approach to Special Polynomials: Laguerre, Hermite, Legendre, Tchebycheff, and Gegenbauer

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Abstract

Special polynomials: Laguerre, Hermite, Legendre, Tchebycheff and Gegenbauer are obtained through well-known linear algebra methods based on Sturm-Liouville theory. A matrix corresponding to the differential operator is found and its eigenvalues are obtained. The elements of the eigenvectors obtained correspond to each mentioned polynomial. This method contrasts in simplicity with standard methods based on solving the differential equation by means of power series, obtaining them through a generating function, using the Rodrigues formula for each polynomial, or by means of a contour integral.

Keywords: special polynomials, special functions, linear algebra, eigenvalues, eigenvectors

1. Introduction

The polynomials covered in this chapter are solutions to an ordinary differential equation (ODE), the hypergeometric equation. In general, the hypergeometric equation may be written as:

$$s(x)F''(x) + t(x)F'(x) + \lambda F(x) = 0, \tag{1}$$

where $F(x)$ is a real function of a real variable $F : U \to \mathbb{R}$, where $U \subset \mathbb{R}$ is an open subset of the real line, and $\lambda \in \mathbb{R}$ a corresponding eigenvalue, and the functions $s(x)$ and $t(x)$ are real polynomials of at most second order and first order, respectively.

There are different cases obtained, depending on the kind of the $s(x)$ function in Eq. (1). When $s(x)$ is a constant, Eq. (1) takes the form $F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0$, and if $\alpha = 1$ one obtains the Hermite polynomials. When $s(x)$ is a polynomial of the first degree, Eq. (1) takes the form $xF''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0$, and when $\alpha = 1$ and $\beta = 0$, one obtains the Laguerre polynomials. There are three different cases when $s(x)$ is a polynomial of the second degree. When the second degree polynomial has two different real roots, Eq. (1) takes the form $(1 - x^2)F''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]F'(x) + \lambda F(x) = 0$; this is the Jacobi equation, and for different values of α and β, one obtains particular cases of polynomials: Gegenbauer polynomials if $\alpha = \beta$, Tchebycheff I and II if $\alpha = \beta = \pm 1/2$, and Legendre polynomials if $\alpha = \beta = 0$. When the second degree polynomial has one double root, Eq. (1) takes the form $x^2 F''(x) + [(\alpha+2)x + \beta]F'(x) + \lambda F(x) = 0$, and when $\alpha = -1$ and $\beta = 0$, one obtains the Bessel polynomials. Finally, when the second degree polynomial has two complex roots, Eq. (1) takes the form $(1+x)^2 F''(x) + (2\beta x + \alpha)F'(x) + \lambda F(x) = 0$, which is the Romanovski equation [1]. These results are summarized in **Table 1**.

The Sturm-Liouville Theory is covered in most advanced physics and engineering courses. In this context, an eigenvalue equation sometimes takes the more general self-adjoint form: $\mathcal{L}u(x) + \lambda w(x)u(x) = 0$, where $\mathcal{L}$ is a differential operator; $\mathcal{L}u(x) = \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x)$, λ an eigenvalue, and $w(x)$ is known as a weight or density function. The analysis of this equation and its solutions is called the Sturm-Liouville theory. Specific forms of $p(x)$, $q(x)$, λ and $w(x)$ are given for Legendre, Laguerre, Hermite and other well-known equations in the given references. There, the close analogy of this theory with linear algebra concepts is also shown. For example, functions here take the role of vectors there, and linear operators here take that of matrices there. Finally, the diagonalization of a real symmetric matrix corresponds to the solution of an ordinary differential equation, defined by a self-adjoint operator $\mathcal{L}$, in terms of its eigenfunctions, which are the "continuous" analog of the eigenvectors [2, 3].

$s(x)$	Canonical form and weight function		Example
Constant	$F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0$ $w(x) = e^{-\alpha x^2}$	(2) (3)	When $\alpha = 1$ one obtains the Hermite equation, $F(x) = H(x)$; this produces the Hermite polynomials, denoted as $\{H_n^{(\alpha)}\}$.
First degree	$xF''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0$ $w(x) = x^{\beta} e^{-\alpha x}$	(4) (5)	When $\alpha = 1$ and $\beta = 0$, one obtains the Laguerre equation, $F(x) = L(x)$; this produces the Laguerre polynomials, denoted as $\{L_n^{(\alpha,\beta)}\}$.
Second degree: with two different real roots	$(1-x^2)F''(x) + [\beta - \alpha - (\alpha+\beta+2)x]F'(x) + \lambda F(x) = 0$ $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$	(6) (7)	Eq. (6) is the Jacobi equation, considering $F(x) = P(x)$, and for each pair (α, β), one obtains the Jacobi polynomials, denoted as $\{P_n^{(\alpha,\beta)}\}$. Particular cases: Gegenbauer polynomials if $\alpha = \beta$, Tchebycheff I and II if $\alpha = \beta = \pm\frac{1}{2}$, and Legendre polynomials if $\alpha = \beta = 0$.
Second degree: with one double real root	$x^2 F''(x) + [(\alpha+2)x + \beta]F'(x) + \lambda F(x) = 0$ $w^{(\alpha,\beta)}(x) = x^{\alpha} e^{-\frac{\beta}{x}}$	(8) (9)	When $\alpha = -1$ and $\beta = 0$, one obtains the Bessel equation, $F(x) = B(x)$; this produces the Bessel polynomials, denoted as $\{B_n^{(\alpha,\beta)}\}$.
Second degree: with two complex roots	$(1+x)^2 F''(x) + (2\beta x + \alpha)F'(x) + \lambda F(x) = 0$ $w^{(\alpha,\beta)}(x) = (1+x^2)^{\beta-1} e^{-\alpha \cot^{-1} x}$	(10) (11)	Eq. (10) is the Romanovski equation; considering $F(x) = R(x)$, then one obtains the Romanovski polynomials, denoted as $\{R_n^{(\alpha,\beta)}\}$.

Table 1.
Polynomials obtained depending on the $s(x)$ function of Eq. (1).

The next section shows some of the most important applications of Hermite, Gegenbauer, Tchebycheff, Laguerre and Legendre polynomials in applied Mathematics and Physics. These polynomials are of great importance in mathematical physics, the theory of approximation, the theory of mechanical quadrature, engineering, and so forth.

2. Physical applications

2.1 Laguerre

Laguerre polynomials were named after Edmond Laguerre (1834–1886). Laguerre studied a special case in 1897, and in 1880, Nikolay Yakovlevich Sonin worked on the general case known as Sonine polynomials, but they were anticipated by Robert Murphy (1833).

The Laguerre differential equation and its solutions, that is, Laguerre polynomials, are found in many important physical problems, such as in the description of the transversal profile of Laguerre-Gaussian laser beams [4]. The practical importance of Laguerre polynomials was enhanced by Schrödinger's wave mechanics, where they occur in the radial wave functions of the hydrogen atom [5].

The most important single application of the Laguerre polynomials is in the solution of the Schrödinger wave equation for the hydrogen atom. This equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi, \tag{12}$$

in which $Z = 1$ for hydrogen, 2 for single ionized helium, and so on. Separating variables, we find that the angular dependence of ψ is $Y_L^M(\theta, \varphi)$. The radial part, $R(r)$, satisfies the equation

$$-\frac{\hbar^2}{2\mathrm{m}}\frac{1}{\mathrm{r}^2}\frac{\mathrm{d}}{\mathrm{dr}}\left(\mathrm{r}^2\frac{\mathrm{dR}}{\mathrm{dr}}\right) - \frac{\mathrm{Ze}^2}{\mathrm{r}}\mathrm{R} + \frac{\mathrm{L(L+1)}}{\mathrm{r}^2}\mathrm{R} = \mathrm{ER}. \tag{13}$$

By use of the abbreviations

$$\rho = \alpha r, \quad \text{with } \alpha^2 = -\frac{8\mathrm{mE}}{\hbar^2},\ \mathrm{E} < 0,\ \lambda = \frac{2\mathrm{mZe}^2}{\alpha\hbar^2}, \tag{14}$$

Eq. (14) becomes

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\chi(\rho)}{d\rho}\right) + \left(\frac{\lambda}{\rho} - \frac{1}{4} - \frac{L(L+1)}{\rho^2}\right)\chi(\rho) = 0, \tag{15}$$

where $\chi(\rho) = R(\rho/\alpha)$. Eq. (15) is satisfied by

$$\rho\chi(\rho) = e^{-\frac{\rho}{2}}\rho^{L+1}L_{\lambda-L-1}^{2L+1}(\rho), \tag{16}$$

in which k is replaced by $2L+1$ and n by $\lambda - L - 1$, in order to consider the associated Laguerre polynomials $L_n^k(\rho)$.

These polynomials are also used in problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of propagation of electromagnetic waves along transmission lines, in describing the static Wigner functions of oscillator systems in quantum mechanics in phase space [6], etc.

2.2 Hermite

Hermite polynomials were defined into the theory of probability by Pierre-Simon Laplace in 1810, and Charles Hermite extended them to include several variables and named them in 1864 [7].

Hermite polynomials are used to describe the transversal profile of Hermite-Gaussian laser beams [4], but mainly to analyze the quantum mechanical simple harmonic oscillator [8]. For a potential energy $V = \frac{1}{2}Kz^2 = \frac{1}{2}m\omega^2 z^2$ (force $\boldsymbol{F} = \nabla V = -Kz$), the Schrödinger wave equation is

$$-\frac{\hbar^2}{2\mathrm{m}}\nabla^2\Psi(\mathrm{z}) + \frac{1}{2}\mathrm{Kz}^2\Psi(\mathrm{z}) = E\Psi(\mathrm{z}). \tag{17}$$

The oscillating particle has mass m and total energy E. By use of the abbreviations

$$\mathrm{x} = \alpha z \text{ with } \alpha^4 = \frac{\mathrm{mK}}{\hbar^2} = \frac{\mathrm{m}^2\omega^2}{\hbar^2}, \lambda = \frac{2\mathrm{E}}{\hbar}\left(\frac{\mathrm{m}}{\mathrm{K}}\right)^{1/2} = \frac{2\mathrm{E}}{\hbar\omega}, \tag{18}$$

in which ω is the angular frequency of the corresponding classical oscillator, Eq. (17) becomes

$$\frac{d^2\psi(x)}{dx^2} + \left(\lambda - x^2\right)\psi(x) = 0, \tag{19}$$

where $\psi(x) = \Psi(z) = \Psi(x/\alpha)$. With $\lambda = 2n + 1$, Eq. (19) is satisfied by

$$\psi_n(x) = 2^{-\frac{n}{2}}\pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}H_n(x), \tag{20}$$

where $H_n(x)$ corresponds to Hermite polynomials.

Hermite polynomials also appear in probability as the Edgeworth series, in combinatorics as an example of an Appell sequence, obeying the umbral calculus, in numerical analysis as Gaussian quadrature, etc.

2.3 Legendre

Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre. Spherical harmonics are an important class of special functions that are closely related to these polynomials. They arise, for instance, when Laplace's equation is solved in spherical coordinates. Since continuous solutions of Laplace's equation are *harmonic functions*, these solutions are called *spherical harmonics* [9].

In the separation of variables of Laplace's equation, Helmholtz's or the space-dependence of the classical wave equation, and the Schrödinger wave equation for central force fields,

$$\nabla^2\psi + \mathrm{k}^2 f(r)\psi = 0, \tag{21}$$

the angular dependence, coming entirely from the Laplacian operator, is

$$\frac{\Phi(\phi)}{\sin(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Theta(\theta)}{\sin^2\theta}\frac{d^2\Phi(\phi)}{d\phi^2} + n(n+1)\Theta(\theta)\Phi(\phi) = 0. \tag{22}$$

The separated azimuthal equation is

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -m^2, \tag{23}$$

with an orthogonal and normalized solution,

$$\Phi_m = \frac{1}{\sqrt{2\pi}}e^{im\phi}. \tag{24}$$

Splitting off the azimuthal dependence, the polar angle dependence (θ) leads to the associated Legendre equation, which is satisfied by the associated Legendre functions; that is, $\Theta(\theta) = P_n^m(cos\theta)$. Normalizing the associated Legendre function, one obtains the orthonormal function

$$\wp_n^m(cos\theta) = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta). \tag{25}$$

Taking the product of Eqs. (24) and (25) to define,

$$Y_n^m(\theta,\phi) \equiv (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta)e^{im\phi}. \tag{26}$$

These $Y_n^m(\theta,\phi)$ are the spherical harmonics [10].

Legendre polynomials are frequently encountered in physics and other technical fields. Some examples are the coefficients in the expansion of the Newtonian potential that gives the gravitational potential associated to a point mass or the Coulomb potential associated to a point charge, the gravitational and electrostatic potential inside a spherical shell, steady-state heat conduction problems in spherical problems inside a homogeneous solid sphere, and so forth [11].

2.4 Tchebycheff

Tchebycheff polynomials, named after Pafnuty Tchebycheff (also written as Chebyshev, Tchebyshev or Tschebyschow), are important in approximation theory because the roots of the Tchebycheff polynomials of the first kind, which are also called Tchebycheff nodes, are used as nodes in polynomial interpolation. Approximation theory is concerned with how functions can best be approximated with simpler functions, and through quantitatively characterizing the errors introduced thereby.

One can obtain polynomials very close to the optimal one by expanding the given function in terms of Tchebycheff polynomials, and then cutting off the expansion at the desired degree. This is similar to the Fourier analysis of the function, using the Tchebycheff polynomials instead of the usual trigonometric functions.

If one calculates the coefficients in the Tchebycheff expansion for a function,

$$f(x) \sim \sum_{i=0}^{\infty} c_i T_i(x), \tag{27}$$

and then cuts off the series after the T_N term, one gets an Nth-degree polynomial approximating $f(x)$.

Tchebycheff polynomials are also found in many important physics, mathematics and engineering problems. A capacitor whose plates are two eccentric spheres is an interesting example [12], another one can be found in aircraft aerodynamics [13], etc.

2.5 Gegenbauer

Gegenbauer polynomials, named after Leopold Gegenbauer, and often called ultraspherical polynomials, include Legendre and Tchebycheff polynomials as special or limiting cases, and at the same time, Gegenbauer polynomials are a special case of Jacobi polynomials (see **Table 1**).

Gegenbauer polynomials appear naturally as extensions of Legendre polynomials in the context of potential theory and harmonic analysis. They also appear in the theory of Positive-definite functions [14].

Since Gegenbauer polynomials are a general case of Legendre and Tchebycheff polynomials, more applications are shown in Section 2.3 and 2.4.

The most common methods to obtain the special polynomials are described in the next section.

3. Special polynomials

To obtain the polynomials described in the previous section, one can use different methods, some tougher than others. These polynomials are typically obtained as a result of the solution of each specific differential equation by means of the power series method. Usually, it is also shown that they can be obtained through a generating function and also by using the Rodrigues formula for each special polynomial, or finally, through a contour integral. Most Mathematical Methods courses also include a study of the properties of these polynomials, such as orthogonality, completeness, recursion relations, special values, asymptotic expansions and their relation to other functions, such as polynomials and hypergeometric functions. There is no doubt that this is a challenging and demanding subject that requires a great deal of attention from most students.

3.1 Differential equation

The most common way to solve the special polynomials is solving the associated differential equation through power series and the Frobenius method $y = \sum_{n=0}^{\infty} a_n x^n$. The corresponding polynomials satisfy the following differential equations:

the Laguerre differential equation,

$$xy'' + (1-x)y' + ny = 0, \tag{28}$$

the Hermite differential equation,

$$y' - 2xy' + 2ny = 0, \tag{29}$$

the Legendre differential equation,

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0\,, \tag{30}$$

the Tchebycheff differential equation,

$$\left(1-x^2\right)y'' - xy' + n^2 y = 0, \tag{31}$$

and the Gegenbauer differential equation,

$$\left(1-x^2\right)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0, \tag{32}$$

with $n = 0, 1, 2, 3, \ldots$ in all the previous cases. Note that if $\lambda = \frac{1}{2}$, Eq. (32) reduces to the Legendre differential equation (Eq. (30)), and if $\lambda = 0$, Eq. (32) reduces to the Tchebycheff differential equation (Eq. (31)).

3.2 Rodrigues formula

For polynomials $\psi_n(x)$, with interval I, weight function $w(x)$, and an eigenvalue equation of the form

$$p(x)\psi_n''(x) + q(x)\psi_n'(x) + \lambda_n \psi_n(x) = 0, \tag{33}$$

and with $q(x) = \frac{(p(x)w(x))'}{w(x)}$, the general formula

$$\psi_n(x) = w(x)^{-1} \frac{d^n}{dx^n} [p(x)^n w(x)] \tag{34}$$

is known as the *Rodrigues formula,* useful to obtain the nth-degree polynomial of ψ [15].

3.3 Generating function and contour integral

Let Γ be a curve that encloses $x \in I$ but excludes the endpoints of I. Then, considering the Cauchy integral formula [16] for derivatives of $w(x)p(x)^n$ to derive an integral formula from Eq. (34), one obtains

$$\frac{\psi_n(x)}{n!} = \frac{1}{2\pi i} \int_\Gamma \frac{w(z)}{w(x)} \frac{p(z)^n}{(z-x)^n} \frac{dz}{z-x}. \tag{35}$$

The *generating function* for the orthogonal polynomials $\left\{\frac{\psi_n(x)}{n!}\right\}$ is defined as

$$G(x, s) = \sum_{n=0}^{\infty} \frac{\psi_n(x)}{n!} s^n. \tag{36}$$

In the following section, Laguerre [2], Hermite [17], Legendre, Tchebycheff [18] and Gegenbauer [3] polynomials are obtained through a simple method, using basic linear algebra concepts, such as the eigenvalue and the eigenvector of a matrix.

4. Simple approach to special polynomials

The general algebraic polynomial of degree n,

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n, \tag{37}$$

with $a_o, a_1, \ldots, a_n \in \Re$, is represented by vector

$$A_n = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}. \tag{38}$$

Taking the first derivative of the above polynomial (x), one obtains the polynomial

$$\frac{d}{dx}\left[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots a_nx^n\right] = a_1 + 2a_2x + 3a_3x^2 + \dots na_nx^{n-1}, \tag{39}$$

which may be written as

$$\frac{dA_n}{dx} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix}. \tag{40}$$

Taking the second derivative of the polynomial (Eq. (37)) one obtains

$$\frac{d^2}{dx^2}\left[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots a_nx^n\right] = 2a_2 + 6a_3x + \dots n(n-1)a_nx^{n-2}, \tag{41}$$

which may be written as

$$\frac{d^2A_n}{dx^2} = \begin{bmatrix} 2a_2 \\ 6a_3 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix}. \tag{42}$$

Using Eq. (40), Eq. (39) may be written as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix}; \tag{43}$$

therefore, the first derivative operator A_n may be written as

$$\frac{d}{dx} \to \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{44}$$

Doing the same for Eq. (41),

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix}, \tag{45}$$

the second derivative operator A_n may be written as

$$\frac{d^2}{dx^2} \rightarrow \begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{46}$$

4.1 Laguerre

The Laguerre differential operator is given by.

$$x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}; \tag{47}$$

substituting Eqs. (41) and (44) into Eq. (47),

$$\begin{aligned} x\left[2a_2 + 6a_3x + \ldots\ + n(n-1)a_nx^{n-2}\right] + (1-x)\left[a_1 + 2a_2x + 3a_3x^2 + \ldots\ + na_nx^{n-1}\right] \\ = a_1 + (4a_2 - a_1)x + (9a_3 - 2a_2)x^2 + (16a_4 + 3a_3)x^3 + \cdots - na_n, \end{aligned} \tag{48}$$

which may be written as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2 & 9 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -3 & 16 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 4a_2 - a_1 \\ 9a_3 - 2a_2 \\ 16a_4 - 3a_3 \\ \vdots \\ -na_n \end{bmatrix}. \tag{49}$$

For simplicity, the Laguerre differential operator, as a $4x4$ matrix, is represented by

$$x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -2 & 9 \\ 0 & 0 & 0 & -3 \end{bmatrix}. \tag{50}$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M - \lambda I) = 0$. However, since the matrix (Eq. (50)) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$, $\lambda_4 = -3$. The corresponding eigenvectors are the solutions of the equation $(M - \lambda_i I) \cdot v = 0$, where the eigenvector $v = [a_0, a_1, a_2, a_3]^T$:

$$\begin{bmatrix} 0-\lambda_i & 1 & 0 & 0 \\ 0 & -1-\lambda_i & 4 & 0 \\ 0 & 0 & -2-\lambda_i & 9 \\ 0 & 0 & 0 & -3-\lambda_i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{51}$$

Substituting eigenvalue $\lambda_1 = 0$ in Eq. (51), we obtain eigenvector v_1:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{52}$$

the elements of this eigenvector correspond to the first Laguerre polynomial, $L_0(x) = 1$.

Substituting eigenvalue $\lambda_2 = -1$ in Eq. (51), we obtain eigenvector v_2:

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}; \tag{53}$$

the elements of this eigenvector correspond to the second Laguerre polynomial, $L_1(x) = 1 - x$.

Substituting eigenvalue $\lambda_3 = -2$ in Eq. (51), we obtain eigenvector v_3:

$$v_3 = \begin{bmatrix} 1 \\ -2 \\ \frac{1}{2} \\ 0 \end{bmatrix}; \tag{54}$$

the elements of this eigenvector correspond to the third Laguerre polynomial, $L_2(x) = 1 - 2x + \frac{1}{2}x^2$.

Substituting eigenvalue $\lambda_4 = -3$ in Eq. (51), we obtain eigenvector v_4:

$$v_4 = \begin{bmatrix} 1 \\ -3 \\ \frac{3}{2} \\ -\frac{1}{6} \end{bmatrix}; \tag{55}$$

the elements of this eigenvector correspond to the fourth Laguerre polynomial, $L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$.

4.2 Hermite

The Hermite differential operator is given by

$$\frac{d^2}{dx^2} - 2x\frac{d}{dx}; \tag{56}$$

substituting Eqs. (41) and (44) into Eq. (56),

$$\begin{aligned}&\left[2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2}\right] - 2x\left[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}\right]\\ &\quad = 2a_2 + (6a_3 - 2a_1)x + (12a_4 - 4a_2)x^2 + (20a_5 - 6a_3)x^3 + \cdots - 2na_n,\end{aligned} \tag{57}$$

which may be written as

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & 6 & 0 & \cdots & 0 \\ 0 & 0 & -4 & 0 & 12 & \cdots & 0 \\ 0 & 0 & 0 & -6 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -2n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 - 2a_1 \\ 12a_4 - 4a_2 \\ 20a_5 - 6a_3 \\ \vdots \\ -2na_n \end{bmatrix}. \tag{58}$$

For simplicity, the Hermite differential operator, as a $4x4$ matrix, is represented by

$$\frac{d^2}{dx^2} - 2x\frac{d}{dx} \to \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}. \tag{59}$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M - \lambda I) = 0$. However, since the matrix (Eq. (59)) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -4$, $\lambda_4 = -6$. The corresponding eigenvectors are the solutions of the equation $(M - \lambda_i I) \cdot v = 0$, where the eigenvector $v = [a_0, a_1, a_2, a_3]^T$:

$$\begin{bmatrix} 0 - \lambda_i & 0 & 2 & 0 \\ 0 & -2 - \lambda_i & 0 & 6 \\ 0 & 0 & -4 - \lambda_i & 0 \\ 0 & 0 & 0 & -6 - \lambda_i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{60}$$

Substituting eigenvalue $\lambda_1 = 0$ in Eq. (60), we obtain eigenvector v_1:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{61}$$

the elements of this eigenvector correspond to the first Hermite polynomial, $H_0(x) = 1$.

Substituting eigenvalue $\lambda_2 = -2$ in Eq. (60), we obtain eigenvector v_2:

$$v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}; \tag{62}$$

the elements of this eigenvector correspond to the second Hermite polynomial, $H_1(x) = 2x$.

Substituting eigenvalue $\lambda_3 = -4$ in Eq. (60), we obtain eigenvector v_3:

$$v_3 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}; \tag{63}$$

the elements of this eigenvector correspond to the third Hermite polynomial, $H_2(x) = 4x^2 - 2$.

Substituting eigenvalue $\lambda_4 = -6$ in Eq. (60), we obtain eigenvector v_4:

$$v_4 = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}; \tag{64}$$

the elements of this eigenvector correspond to the fourth Hermite polynomial, $H_3(x) = 8x^3 - 12x$.

4.3 Legendre

The Legendre differential operator is given by

$$\left(1 - x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx}; \tag{65}$$

substituting Eqs. (41) and (44) into Eq. (65),

$$\begin{aligned}\left(1 - x^2\right)&\left[2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2}\right] - 2x\left[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}\right] \\ &= 2a_2 + (6a_3 - 2a_1)x + (12a_4 - 6a_2)x^2 + (20a_5 - 12a_3)x^3 + \cdots - \left(n^2 + n\right)a_n,\end{aligned} \tag{66}$$

which may be written as

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & 6 & 0 & \cdots & 0 \\ 0 & 0 & -6 & 0 & 12 & \cdots & 0 \\ 0 & 0 & 0 & -12 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -(n^2+n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 - 2a_1 \\ 12a_4 - 6a_2 \\ 20a_5 - 12a_3 \\ \vdots \\ -(n^2+n)a_n \end{bmatrix}. \tag{67}$$

For simplicity, the Legendre differential operator, as a 4*x*4 matrix, is represented by

$$\left(1 - x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} \to \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}. \tag{68}$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M - \lambda I) = 0$. However, since the matrix (Eq. (68)) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -6$, $\lambda_4 = -12$. The corresponding eigenvectors are the solutions of the equation $(M - \lambda_i I) \cdot v = 0$, where the eigenvector $v = [a_0, a_1, a_2, a_3]^T$:

$$\begin{bmatrix} 0-\lambda_i & 0 & 2 & 0 \\ 0 & -2-\lambda_i & 0 & 6 \\ 0 & 0 & -6-\lambda_i & 0 \\ 0 & 0 & 0 & -12-\lambda_i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{69}$$

Substituting eigenvalue $\lambda_1 = 0$ in Eq. (69), we obtain eigenvector v_1:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{70}$$

the elements of this eigenvector correspond to the first Legendre polynomial, $P_0(x) = 1$.

Substituting eigenvalue $\lambda_2 = -2$ in Eq. (69), we obtain eigenvector v_2:

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \tag{71}$$

the elements of this eigenvector correspond to the second Legendre polynomial, $P_1(x) = x$.

Substituting eigenvalue $\lambda_3 = -6$ in Eq. (69), we obtain eigenvector v_3:

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}; \tag{72}$$

the elements of this eigenvector correspond to the third Legendre polynomial, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$.

Substituting eigenvalue $\lambda_4 = -12$ in Eq. (69), we obtain eigenvector v_4:

$$v_4 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -5 \end{bmatrix}; \tag{73}$$

the elements of this eigenvector correspond to the fourth Legendre polynomial, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$.

4.4 Tchebycheff

The Tchebycheff differential operator is given by

$$\left(1-x^2\right)\frac{d^2}{dx^2}-x\frac{d}{dx}; \tag{74}$$

substituting Eqs. (41) and (44) into Eq. (74),

$$\begin{aligned}\left(1-x^2\right)&\left[2a_2+6a_3x+\dots\ +n(n-1)a_nx^{n-2}\right]-x\left[a_1+2a_2x+3a_3x^2\right.\\&\left.+\dots\ +na_nx^{n-1}\right]=2a_2+(6a_3-a_1)x+(12a_4-4a_2)x^2\\&+(20a_5-9a_3)x^3+\cdots-n^2a_n,\end{aligned} \tag{75}$$

which may be written as

$$\begin{bmatrix}0&0&2&0&0&\cdots&0\\0&-1&0&6&0&\cdots&0\\0&0&-4&0&12&\cdots&0\\0&0&0&-9&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&0&\cdots&-n^2\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\\vdots\\a_n\end{bmatrix}=\begin{bmatrix}2a_2\\6a_3-a_1\\12a_4-4a_2\\20a_5-9a_3\\\vdots\\-n^2a_n\end{bmatrix}. \tag{76}$$

For simplicity, the Tchebycheff differential operator, as a $4x4$ matrix, is represented by

$$\left(1-x^2\right)\frac{d^2}{dx^2}-x\frac{d}{dx}\to\begin{bmatrix}0&0&2&0\\0&-1&0&6\\0&0&-4&0\\0&0&0&-9\end{bmatrix}. \tag{77}$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M-\lambda I)=0$. However, since the matrix (Eq. (77)) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1=0$, $\lambda_2=-1$, $\lambda_3=-4$, $\lambda_4=-9$. The corresponding eigenvectors are the solutions of the equation $(M-\lambda_i I)\cdot v=0$, where the eigenvector $v=[a_0,a_1,a_2,a_3]^T$;

$$\begin{bmatrix}0-\lambda_i&0&2&0\\0&-1-\lambda_i&0&6\\0&0&-4-\lambda_i&0\\0&0&0&-9-\lambda_i\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{78}$$

Substituting eigenvalue $\lambda_1=0$ in Eq. (78), we obtain eigenvector v_1:

$$v_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}; \tag{79}$$

the elements of this eigenvector correspond to the first Tchebycheff polynomial, $T_0(x)=1$.

Substituting eigenvalue $\lambda_2=-1$ in Eq. (78), we obtain eigenvector v_2:

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \tag{80}$$

the elements of this eigenvector correspond to the second Tchebycheff polynomial, $T_1(x) = x$.

Substituting eigenvalue $\lambda_3 = -4$ in Eq. (78), we obtain eigenvector v_3:

$$v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}; \tag{81}$$

the elements of this eigenvector correspond to the third Tchebycheff polynomial, $T_2(x) = 2x^2 - 1$.

Substituting eigenvalue $\lambda_4 = -9$ in Eq. (78), we obtain eigenvector v_4:

$$v_4 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}. \tag{82}$$

the elements of this eigenvector correspond to the fourth Tchebycheff polynomial, $T_3(x) = 4x^3 - 3x$.

4.5 Gegenbauer

The Gegenbauer differential operator is given by

$$\left(1 - x^2\right)\frac{d^2}{dx^2} - (2\lambda + 1)x\frac{d}{dx}; \tag{83}$$

substituting (41) and (44) into (83),

$$\begin{aligned} \left(1 - x^2\right)\left[2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2}\right] - (2\lambda + 1)x[a_1 \\ + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}\right] = 2a_2 + [6a_3 - (2\lambda + 1)a_1]x \\ + [12a_4 - 4(\lambda + 1)a_2]x^2 + [20a_5 - 3(2\lambda + 3)a_3]x^3 \\ + \dots - \left[n^2 + 2\lambda n\right]a_n, \end{aligned} \tag{84}$$

which may be written as

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & -(2\lambda+1) & 0 & 6 & 0 & \cdots & 0 \\ 0 & 0 & -4(\lambda+1) & 0 & 12 & \cdots & 0 \\ 0 & 0 & 0 & -3(2\lambda+3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -n^2 - 2\lambda n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 - (2\lambda+1)a_1 \\ 12a_4 - 4(\lambda+1)a_2 \\ 20a_5 - 3(2\lambda+3)a_3 \\ \vdots \\ -(n^2 + 2\lambda n)a_n \end{bmatrix}. \tag{85}$$

For simplicity, the Gegenbauer differential operator, as a $4x4$ matrix, is represented by

$$\left(1-x^2\right)\frac{d^2}{dx^2}-(2\lambda+1)x\frac{d}{dx}\rightarrow\begin{bmatrix}0 & 0 & 2 & 0\\ 0 & -(2\lambda+1) & 0 & 6\\ 0 & 0 & -4(\lambda+1) & 0\\ 0 & 0 & 0 & -3(2\lambda+3)\end{bmatrix}. \quad (86)$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M-\lambda' I)=0$. However, since the matrix (Eq. (86)) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1'=0$, $\lambda_2'=-(2\lambda+1)$, $\lambda_3'=-4(\lambda+1)$, $\lambda_4'=-3(2\lambda+3)$. The corresponding eigenvectors are the solutions of the equation $\left(M-\lambda_i' I\right)\cdot v=0$, where the eigenvector $v=[a_0,a_1,a_2,a_3]^T$;

$$\begin{bmatrix}0-\lambda_i' & 0 & 2 & 0\\ 0 & -(2\lambda+1)-\lambda_i' & 0 & 6\\ 0 & 0 & -4(\lambda+1)-\lambda_i' & 0\\ 0 & 0 & 0 & -3(2\lambda+3)-\lambda_i'\end{bmatrix}\cdot\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}. \quad (87)$$

Substituting eigenvalue $\lambda_1'=0$ in Eq. (87), we obtain eigenvector v_1:

$$v_1=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}; \quad (88)$$

the elements of this eigenvector correspond to the first Gegenbauer polynomial, $C_0^\lambda(x)=1$.

Substituting eigenvalue $\lambda_2'=-(2\lambda+1)$ in Eq. (87), we obtain eigenvector v_2:

$$v_2=\begin{bmatrix}0\\ 2\lambda\\ 0\\ 0\end{bmatrix}; \quad (89)$$

the elements of this eigenvector correspond to the second Gegenbauer polynomial, $C_1^\lambda(x)=2\lambda x$.

Substituting eigenvalue $\lambda_3'=-4(\lambda+1)$ in Eq. (87), we obtain eigenvector v_3:

$$v_3=\begin{bmatrix}-\lambda\\ 0\\ 2\lambda(1+\lambda)\\ 0\end{bmatrix}; \quad (90)$$

the elements of this eigenvector correspond to the third Gegenbauer polynomial, $C_2^\lambda(x)=-\lambda+2\lambda(1+\lambda)x^2$.

Substituting eigenvalue $\lambda_4'=-3(2\lambda+3)$ in Eq. (87), we obtain eigenvector v_4:

$$v_4 = \begin{bmatrix} 0 \\ -2\lambda(1+\lambda) \\ 0 \\ \frac{4}{3}\lambda(1+\lambda)(2+\lambda) \end{bmatrix}; \tag{91}$$

the elements of this eigenvector correspond to the fourth Gegenbauer polynomial, $C_3^{\lambda}(x) = 2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3$.

5. Conclusions

Laguerre, Hermite, Legendre, Tchebycheff and Gegenbauer polynomials are obtained in a simple and straightforward way using basic linear algebra concepts, such as the eigenvalue and the eigenvector of a matrix. Once the matrix of the corresponding differential operator is obtained, the eigenvalues of this matrix are found, and the elements of its eigenvectors correspond to the coefficients of each kind of polynomials. Using a larger matrix, higher order polynomials may be found; however, the general case for an nxn matrix was not obtained since it seems that in this general case, standard methods would be easier to use. The main advantage of this method lies in its easiness, since it relies on simple linear algebra concepts. This method contrasts in simplicity with standard methods based on solving the differential equation using power series, using the generating function, using the Rodrigues formula, or using a contour integral.

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