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Chapter

Cosmological Solutions to Polynomial Affine Gravity in the Torsion-Free Sector

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Abstract

We find possible cosmological models of the polynomial affine gravity described by connections that are either compatible or not with a metric. When possible, we compare them with those of general relativity. We show that the set of cosmological vacuum solutions in general relativity are a subset of the solutions of polynomial affine gravity. In our model, the cosmological constant appears as an integration constant, and, additionally, we show that some forms of matter can be emulated by the affine structure—even in the metric compatible case. In the case of connections not compatible with a metric, we obtain formal families of solutions, which should be constrained by physical arguments. We show that for a certain parametrisation of the connection, the affine Ricci-flat condition yields the cosmological field equations of general relativity coupled with a perfect fluid, pointing towards a geometrical emulation of—what is interpreted in general relativity as—matter effects.

Keywords: affine gravity, exact solutions, cosmological models

1. Introduction

All of the *fundamental* physics is described by four interactions: electromagnetic, weak, strong and gravitational. The former three are bundled into what is known as *standard model* of particle physics, which explains very accurately the physics at very short scales. These three interactions share common grounds, for example, they are modelled by connections with values in a Lie algebra, they have been successfully quantised and renormalised, and the simplest of them—quantum electrodynamics—gives the most accurate results when compared with the experiments.

On the other hand, the model that explains gravitational interaction (general relativity) is a field theory for the metric, which can be thought as a potential for the gravitational connection [1, 2]. Although general relativity is the most successful theory, we have to explain gravity [3–5], it cannot be formulated as a gauge theory (in four dimensions), the standard quantisation methods lead to inconsistencies, and it is non-renormalisable, driving the community to believe it is an effective theory of a yet unknown fundamental one. Within the framework of cosmology, when one wants to conciliate both *standard models*,[1] it was noticed that nearly 95%

[1] Besides the standard model of particles, there is a *standard model of cosmology*.

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of the Universe does not fit into the picture. Therefore, a (huge) piece of the puzzle is missing called the dark sector of the Universe, composed of dark matter and dark energy. In order to solve this problem, one needs to add *new physics*, by either including extra particles (say inspired in beyond standard model physics) or changing the gravitational sector. The latter has inspired plenty of generalisations of general relativity.

Although it cannot be said that the mentioned troubles are due to the fact that the model is described by the metric, given that the *physical* quantity associated with the gravitational interaction—the curvature—is defined for a connection, it is worth to ask ourselves whether a *more* fundamental model of gravitational interactions can be built up using the affine connection as the mediator.

The first affine model of gravity was proposed by Eddington in Ref. [6], where the action was defined by the square root of the determinant of the Ricci tensor:

$$S = \int \sqrt{\det(\mathrm{Ric})}, \tag{1}$$

but in Schrödinger's words [7]:

> *For all that I know, no special solution has yet been found which suggests an application to anything that might interest us...*

However, Eddington's idea serves as a starting point to new proposals [8, 9].

In a series of seminal papers [10–13], Cartan presented a definition of curvature for spaces with torsion and its relevance for general relativity. It is worth mentioning that in pure gravity—described by the Einstein-Hilbert-like action, Cartan's generalisation of gravity yields the condition of vanishing torsion as an equation of motion. Therefore, it was not seriously considered as a generalisation of general relativity, until the inclusion of gravitating fermionic matter [14].

Inspired by Cartan's idea of considering an affine connection into modelling of gravity, a new interesting proposal has been considered. Among the interesting generalisations, we mention a couple: (i) the well-known metric-affine models of gravity [15], in which the metric and connection are not only considered as independent, but the conditions of metricity and vanishing torsion are in general dropped and (ii) the Lovelock-Cartan gravity [16], includes extra terms in the action compatible with the precepts of general relativity, whose variation yields field equations that are second-order differential equations. Nonetheless, the metric plays a very important role in these models.

Modern attempts to describe gravity as a theory for the affine connection have been proposed in Refs. [17–25], and the cosmological implications in an Eddington-inspired affine model were studied in Refs. [26–29].

The recently proposed polynomial affine gravity [24] separates the two roles of the metric field, as in a Palatini formulation of gravity, but does not allow it to participate in the mediation of the interaction, by its exclusion from the action. It turns out that the absence of the metric in the action results in a robust structure that—without the addition of other fields—does not accept deformations. That robustness can be useful if one would like to quantise the theory, because all possible counter-terms should have the form of terms already present in the action.

In this chapter, we focus in finding cosmological solutions in the context of polynomial affine gravity, restricted to torsion-free sector of equi-affine connections, which yields a simple set of field equations generalising those obtained in standard general relativity [25]. This chapter is divided into four sections: In Section 2, we review briefly the polynomial affine model of gravity. In Section 3, we use the

Levì-Civita connection for a Friedman-Robertson-Walker metric, to solve the field equations—obtained in the torsion-free sector—of polynomial affine gravity. Then, in Section 4, we solve for, the case of (affine) Ricci-flat manifold, the field equations for the affine connection. Some remarks and conclusions are presented in Section 5. For completeness, in Appendix A, we include a short exposition of the Lie derivative applied to the connection and show the Killing vectors compatible with the cosmological principle.

2. Polynomial affine gravity

In the standard theory of gravity, general relativity, the fundamental field is the metric, $g_{\mu\nu}$, of the space–time [1, 2]. Nevertheless, the metric has a twofold role in this gravitational model: it measures distances and also defines the notion of parallelism, that is, settles the connection. Palatini, in Ref. [30], considered a somehow separation of these roles, but at the end of the day, the metric was still the sole field of the model. It was understood soon after that the connection, $\Gamma^{\mu}{}_{\rho\sigma}$, does not need to be related with the metric field [10–13, 31], and, therefore, the curvature could be blind to the metric.

In this section, we briefly expose the model proposed in Refs. [24, 25], which is inspired in the aforementioned role separation. The metric is left out the mediation of gravitational interactions by taking it out the action.

The action of the polynomial affine gravity is built up from an affine connection, $\hat{\Gamma}^{\mu}{}_{\rho\sigma}$, which accepts a decomposition on irreducible components as

$$\hat{\Gamma}^{\mu}{}_{\rho\sigma} = \hat{\Gamma}^{\mu}{}_{(\rho\sigma)} + \hat{\Gamma}^{\mu}{}_{[\rho\sigma]} = \Gamma^{\mu}{}_{\rho\sigma} + \varepsilon_{\rho\sigma\lambda\kappa} T^{\mu,\,\lambda\kappa} + A_{[\rho}\delta^{\mu}_{\nu]}, \tag{2}$$

where $\Gamma^{\mu}{}_{\rho\sigma} = \hat{\Gamma}^{\mu}{}_{(\rho\sigma)}$ is symmetric in the lower indices, A_ρ is a vector field corresponding to the trace of torsion, and $T^{\mu,\,\lambda\kappa}$ is a Curtright field [32], which satisfy the properties $T^{\kappa,\,\mu\nu} = -T^{\kappa,\,\nu\mu}$ and $\varepsilon_{\lambda\kappa\mu\nu} T^{\kappa,\,\mu\nu} = 0$. The metric field, which might or might not exist, cannot be used for contracting nor lowering or raising indices. The; relation between the epsilons with lower and upper indices is given by $\epsilon^{\delta\eta\lambda\kappa}\epsilon_{\mu\nu\rho\sigma} = 4!\delta^{\delta}{}_{[\mu}\delta^{\eta}{}_{\nu}\delta^{\lambda}{}_{\rho}\delta^{\kappa}{}_{\sigma]}$.

The most general action preserving diffeomorphism invariance, written in terms of the fields in Eq. (2), is

$$\begin{aligned} S[\Gamma, T, A] = \int \mathrm{d}^4x \Big[& B_1 R_{\mu\nu}{}^{\mu}{}_{\rho} T^{\nu,\,\alpha\beta} T^{\rho,\,\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} + B_2 R_{\mu\nu}{}^{\sigma}{}_{\rho} T^{\beta,\,\mu\nu} T^{\rho,\,\gamma\delta} \epsilon_{\sigma\beta\gamma\delta} + B_3 R_{\mu\nu}{}^{\mu}{}_{\rho} T^{\nu,\,\rho\sigma} A_\sigma + B_4 R_{\mu\nu}{}^{\sigma}{}_{\rho} T^{\rho,\,\mu\nu} A_\sigma \\ & + B_5 R_{\mu\nu}{}^{\rho}{}_{\rho} T^{\sigma,\,\mu\nu} A_\sigma + C_1 R_{\mu\rho}{}^{\mu}{}_{\nu} \nabla_\sigma T^{\nu,\,\rho\sigma} + C_2 R_{\mu\nu}{}^{\rho}{}_{\rho} \nabla_\sigma T^{\sigma,\,\mu\nu} + D_1 T^{\alpha,\,\mu\nu} T^{\beta,\,\rho\sigma} \nabla_\gamma T^{(\lambda,\kappa)\gamma} \epsilon_{\beta\mu\nu\lambda} \epsilon_{\alpha\rho\sigma\kappa} \\ & + D_2 T^{\alpha,\,\mu\nu} T^{\lambda,\,\beta\gamma} \nabla_\lambda T^{\delta,\,\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} + D_3 T^{\mu,\,\alpha\beta} T^{\lambda,\,\nu\gamma} \nabla_\lambda T^{\delta,\,\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} + D_4 T^{\lambda,\,\mu\nu} T^{\kappa,\,\rho\sigma} \nabla_{(\lambda} A_{\kappa)} \epsilon_{\mu\nu\rho\sigma} \\ & + D_5 T^{\lambda,\,\mu\nu} \nabla_{[\lambda} T^{\kappa,\,\rho\sigma} A_{\kappa]} \epsilon_{\mu\nu\rho\sigma} + D_6 T^{\lambda,\,\mu\nu} A_\nu \nabla_{(\lambda} A_{\mu)} + D_7 T^{\lambda,\,\mu\nu} A_\lambda \nabla_{[\mu} A_{\nu]} \\ & + E_1 \nabla_{(\rho} T^{\rho,\,\mu\nu} \nabla_{\sigma)} T^{\sigma,\,\lambda\kappa} \epsilon_{\mu\nu\lambda\kappa} + E_2 \nabla_{(\lambda} T^{\lambda,\,\mu\nu} \nabla_{\mu)} A_\nu + T^{\alpha,\,\beta\gamma} T^{\delta,\,\eta\kappa} T^{\lambda,\,\mu\nu} T^{\rho,\,\sigma\tau} \big(F_1 \epsilon_{\beta\gamma\eta\kappa} \epsilon_{\alpha\rho\mu\nu} \epsilon_{\delta\lambda\sigma\tau} \\ & + F_2 \epsilon_{\beta\lambda\eta\kappa} \epsilon_{\gamma\rho\mu\nu} \epsilon_{\alpha\delta\sigma\tau} \big) + F_3 T^{\rho,\,\alpha\beta} T^{\gamma,\,\mu\nu} T^{\lambda,\,\sigma\tau} A_\tau \epsilon_{\alpha\beta\gamma\lambda} \epsilon_{\mu\nu\rho\sigma} + F_4 T^{\eta,\,\alpha\beta} T^{\kappa,\,\gamma\delta} A_\eta A_\kappa \epsilon_{\alpha\beta\gamma\delta} \Big], \end{aligned} \tag{3}$$

where terms related through partial integration and topological invariant have been dropped.[2] One can prove via a dimensional analysis, the *uniqueness* of the above action (see Ref. [25]).

The action in Eq. (3) shows up very interesting features: (i) it is power-counting renormalisable;[3] (ii) all coupling constants are dimensionless which hints the

[2] An example of four-dimensional topological term is the Euler density.

[3] Power-counting renormalisability does not guarantee renormalisability.

conformal invariance of the model [33]; (iii) yields no three-point graviton vertices, which might allow to overcome the *no-go* theorems found in Refs. [34, 35]; (iv) its non-relativistic geodesic deviation agrees with that produced by a Keplerian potential [24]; and (v) the effective equations of motion in the torsion-free limit are a generalisation of Einstein's equations [25]. In the remaining of this section, we will sketch how to find the relativistic limit of this model, when the torsion vanishes.

First, notice that the vanishing torsion condition is equivalent to setting both $T^{\lambda,\mu\nu}$ and A_μ equal to zero. Although this limit is not well defined at the action level, it is well defined at the level of equation of motion.[4] In order to simplify the task of finding the equations of motion to take the limit, we restrict ourselves to the terms in the action which are linear in either $T^{\lambda,\mu\nu}$ or A_μ, since these are the only terms which, after the extremisation, will survive the torsion-free limit. Therefore, after the described considerations, the effective torsion-free action is

$$S_{\text{eff}} = \int \mathrm{d}^4x \left(C_1 R_{\lambda\mu}{}^{\lambda}{}_{\nu}\nabla_\rho + C_2 R_{\mu\rho}{}^{\lambda}{}_{\lambda}\nabla_\nu\right) T^{\nu,\mu\rho}. \tag{4}$$

The nontrivial equations of motion for this action are those for the Curtright field, $T^{\nu,\mu\rho}$:

$$\nabla_{[\rho} R_{\mu]\nu} + \kappa \nabla_\nu R_{\mu\rho}{}^{\lambda}{}_{\lambda} = 0, \tag{5}$$

where κ is a constant related with the original couplings of the model.

In the Riemannian formulation of differential geometry, since the curvature tensor is antisymmetric in the last couple of indices, the second term in Eq. (5) vanishes identically. However, for non-Riemannian connections, such term still vanishes if the connection is compatible with a volume form. These connections are known as equi-affine connections [36, 37]. In addition, the Ricci tensor for equi-affine connections is symmetric. For these connections, the gravitational equations are simply

$$\nabla_{[\rho} R_{\mu]\nu} = 0. \tag{6}$$

Eq. (6) is a generalisation of the parallel Ricci curvature condition, $\nabla_\rho R_{\mu\nu} = 0$, which is a known extension of Einstein's equations [38, 39]. Moreover, these field equations are also obtained as part of a Ã la Palatini approach to a Yang-Mills formulation of gravity, known as the Stephenson-Kilmister-Yang (or SKY) model, proposed in Refs. [40–42]. Such Yang-Mills-like gravity is described by the action

$$S_{sky} = \int \mathrm{d}^4x \sqrt{g} g^{\mu\nu} g^{\sigma\tau} R_{\mu\sigma}{}^{\lambda}{}_{\rho} R_{\nu\tau}{}^{\rho}{}_{\lambda}, \tag{7}$$

which can be written using the curvature two-form as

$$S_{sky} = \int \mathrm{Tr}(\boldsymbol{\mathcal{R}} \star \boldsymbol{\mathcal{R}}) = \int \left(\boldsymbol{\mathcal{R}}^a_b \star \boldsymbol{\mathcal{R}}^b_a\right). \tag{8}$$

Although the field equations of the connection obtained from Eq. (7) are the harmonic curvature condition [43],

[4] The field equations can be consistently truncated under the requirement of vanishing torsion. It is worth noticing that this condition does not yield the Riemannian theory, since we are not yet asking for a metricity condition.

$$\nabla_\lambda R_{\mu\nu}{}^{\lambda}{}_{\rho} = 0, \tag{9}$$

these are equivalent to Eq. (6) through the second Bianchi identity [39, 44].

The Stephenson-Kilmister-Yang model is a field theory for the metric—not for the connection, and thus there is an extra field equation for the metric. The field equation for the metric is very restrictive, and it does not accept Schwarzschild-like solutions [45]. However, in the polynomial affine gravity, since the metric does not participate in the mediation of gravitational interaction, that problem is solved trivially. Meanwhile, the physical field associated with the gravitational interaction is the connection. This difference makes a huge distinction in the phenomenological interpretation of these models.

In the following sections, we shall present solutions to the field Eqs. (6), in the cases where the connection is metric or not. To this end, in Appendix A we show how to propose an ansatz compatible with the desired symmetries. Moreover, Eq. (6) can be solved in three ways, yielding to a sub-classification of the solutions: (i) Ricci flat solutions, $R_{\mu\nu} = 0$; (ii) parallel Ricci solutions, $\nabla_\lambda R_{\mu\nu} = 0$; and (iii) harmonic Riemann solutions, $\nabla_\lambda R_{\mu\nu}{}^{\lambda}{}_{\rho} = 0$.

3. Cosmological metric solutions

The conditions of isotropy and homogeneity are very stringent, when imposed on a symmetric rank-two tensor, and the possible ansatz is just the Friedmann-Robertson-Walker metric:

$$g = G_{00}(t)\mathrm{d}t \otimes \mathrm{d}t + G_{11}(t)\left(\frac{1}{1-\kappa r^2}\mathrm{d}r \otimes \mathrm{d}r + r^2\mathrm{d}\theta \otimes \mathrm{d}\theta + r^2\sin(\theta)^2\mathrm{d}\phi \otimes \mathrm{d}\phi\right). \tag{10}$$

In the remaining of this section, we shall use the standard parametrisation of a Friedmann-Robertson-Walker metric, that is

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\left(\frac{1}{1-\kappa r^2}\mathrm{d}r \otimes \mathrm{d}r + r^2\mathrm{d}\theta \otimes \mathrm{d}\theta + r^2\sin(\theta)^2\mathrm{d}\phi \otimes \mathrm{d}\phi\right). \tag{11}$$

The nonvanishing components of the Levi-Cività connection for the metric in Eq. (11) are

$$\begin{aligned}
&\Gamma^t_{rr} = -\frac{a\dot{a}}{kr^2-1} && \Gamma^t_{\theta\theta} = r^2 a\dot{a} && \Gamma^t_{\phi\phi} = r^2 a\sin^2(\theta)\dot{a}\\
&\Gamma^r_{tr} = \frac{\dot{a}}{a} && \Gamma^r_{rt} = \frac{\dot{a}}{a} && \Gamma^r_{rr} = -\frac{kr}{kr^2-1}\\
&\Gamma^r_{\theta\theta} = kr^3 - r && \Gamma^r_{\phi\phi} = \left(kr^3 - r\right)\sin^2(\theta) && \Gamma^\theta_{t\theta} = \frac{\dot{a}}{a}\\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^\theta_{\theta t} = \frac{\dot{a}}{a} && \Gamma^\theta_{\theta r} = \frac{1}{r}\\
&\Gamma^\theta_{\phi\phi} = -\cos(\theta)\sin(\theta) && \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a} && \Gamma^\phi_{r\phi} = \frac{1}{r}\\
&\Gamma^\phi_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)} && \Gamma^\phi_{\phi t} = \frac{\dot{a}}{a} && \Gamma^\phi_{\phi r} = \frac{1}{r}\\
& && \Gamma^\phi_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)} &&
\end{aligned} \tag{12}$$

3.1 ... with vanishing Ricci

This particular case is a metric model of gravity, whose field equations are vanishing Ricci. It is expected to obtain the cosmological vacuum solution of general relativity (without cosmological constant), that is, Minkowski space–time.

From the connection in Eq. (12), it is straightforward to calculate the Ricci tensor, and the field equations are then

$$R_{tt} = -\frac{3\ddot{a}}{a} = 0, \qquad R_{ii} = f_i(r,\theta)\left(\dot{a}^2 + a\ddot{a} + 2\kappa\right) = 0, \tag{13}$$

where the functions f_i are $f_r = (1-\kappa r^2)^{-1}$, $f_\theta = r^2$ and $f_\varphi = r^2\sin^2(\theta)$.

The solutions to Eq. (13) are shown in **Table 1** and (as expected) are two parametrisations of Minkowski space–time (see, for example, Ref. [46]).

3.2 ... with parallel Ricci

Secondly, we shall analyse the possible solutions to the parallel Ricci equations:

$$\nabla_\lambda R_{\mu\nu} = 0. \tag{14}$$

Notice that in the case of Riemannian geometry, there is a *natural* parallel symmetric $\binom{0}{2}$-type tensor, that is, the metric. Therefore, a simple solution to Eq. (14) is that the Ricci is proportional to the metric—the space–time is an Einstein manifold, and the proportionality factor is related with the cosmological constant.

The independent components of Eq. (14) for the ansätze in Eq. (11) are

$$\nabla_t R_{tt} \simeq \dot{a}\ddot{a} - a\ddot{a} = 0, \tag{15}$$

$$\nabla_i R_{ti} \simeq \left(\dot{a}^2 - a\ddot{a} + \kappa\right)\dot{a} = 0. \tag{16}$$

Additionally, Eq. (15) can be rewritten as

$$\frac{d}{dt}\left(\frac{\ddot{a}}{a}\right) = 0 \quad \Rightarrow \quad \ddot{a} + Ca = 0. \tag{17}$$

According to the value of the integration constant C, we parametrise it as

$$C = \begin{cases} \omega^2 & \text{for } C > 0 \\ \omega = 0 & \text{for } C = 0 \\ v & \text{for } C < 0 \end{cases}$$

Using Eq. (17) to eliminate the $\ddot{a}$ dependence from Eq. (16) yields

$$\dot{a}^2 + Ca^2 + \kappa = 0. \tag{18}$$

Scale factor for the metric vanishing Ricci case		
$\kappa = -1$	$\kappa = 0$	$\kappa = 1$
$\sqrt{2}t + B$	$B \in \mathbb{R}^+$	$\nexists$

Table 1.
Scale factor solving the vanishing Ricci condition, for a cosmological metric connection.

The solutions to Eq. (17) are presented in **Table 2**, and they are known from general relativity (see, for example, Ref. [46]). Interestingly, our integration constant, C, could be identified as $C = -\frac{\Lambda}{3}$ from the vacuum Friedmann's equations. However, our equations are compatible with Friedmann's equations, interacting with a vacuum energy perfect fluid, if the integration constant is identified with

$$C = \frac{4\pi G_N}{3}(\rho + 3p) - \frac{\Lambda}{3}. \tag{19}$$

3.3 ... with harmonic Riemann

Now that we showed that the solutions of the parallel Ricci equations are equivalent to those of general relativity, we turn our attention to Eq. (6). For the metric ansatz in Eq. (11), interestingly, only an independent equation is obtained:

$$\frac{2\dot{a}^3 - a\dot{a}\ddot{a} - a^2\dddot{a} + 2\kappa\dot{a}}{a} = 0,$$

that should determine the scale factor. It can be rewritten as

$$-\frac{d}{dt}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2}\right) = 0, \tag{20}$$

that is,

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} = -C. \tag{21}$$

After a change of variable, $f = a^2$, Eq. (21) becomes

$$\ddot{f} + 2Cf + 2\kappa = 0, \tag{22}$$

whose solutions are

$$f(t) = \begin{cases} -\kappa t^2 + At + B & C = 0 \\ A\sin(\omega t) + B\cos(\omega t) - \dfrac{2\kappa}{\omega^2} & 2C = \omega^2 > 0 \\ A\sinh(\omega t) + B\cosh(\omega t) + \dfrac{2\kappa}{\omega^2} & 2C = -\omega^2 < 0 \end{cases} \tag{23}$$

Therefore, the scale factors are those presented in **Table 3**. Notice, however, that in this case we are not separating the cases according to the value of κ, but the

Scale factor for the metric parallel Ricci case			
	$\kappa = -1$	$\kappa = 0$	$\kappa = 1$
$C = -\omega^2 < 0$	$\pm\frac{\sinh(\omega t)}{2\omega}$	$A\exp(\pm\omega t)$	$\pm\frac{\cosh(\omega t)}{2\omega}$
$C = \omega = 0$	$\pm t + B$	$B \in {}^{+}$	$\nexists$
$C = \omega^2 > 0$	$\frac{\sin(\omega t + \varphi)}{\omega}$	$\nexists$	$\nexists$

Table 2.
Scale factor solving the parallel Ricci condition, for a cosmological metric connection.

Scale factor for the metric harmonic curvature case	
	$\kappa = -1, 0, 1$
$C = -\omega^2 < 0$	$\sqrt{A\sinh(\omega t) + B\cosh(\omega t) + \frac{2\kappa}{\omega^2}}$
$C = \omega = 0$	$\sqrt{-\kappa t^2 + At + B}$
$C = \omega^2 > 0$	$\sqrt{\frac{\sin(\omega t+\varphi)}{\omega}}$

Table 3.
Scale factor solving the harmonic curvature condition, for a cosmological metric connection.

existence of a solution for a given κ is determined by the domain of time and also by the values of the integration constants A and B.

4. Cosmological nonmetric solutions

In order to solve the set of coupled, non-linear, partial differential equations for the connection, one proceeds—just as in general relativity—by giving an ansatz compatible with the symmetries of the problem. Using the Lie derivative, we have found the most general torsion-free connection compatible with the cosmological principle [47].[5] The nonvanishing components of the connection are

$$
\begin{array}{lll}
\Gamma^{t}_{\ tt} = f(t) & \Gamma^{t}_{\ rr} = \dfrac{g(t)}{1-\kappa r^2} & \Gamma^{t}_{\ \theta\theta} = r^2 g(t) \\
\Gamma^{t}_{\ \phi\phi} = r^2 g(t)\sin^2(\theta) & \Gamma^{r}_{\ tr} = h(t) & \Gamma^{r}_{\ rt} = h(t) \\
\Gamma^{r}_{\ rr} = \dfrac{\kappa r}{1-\kappa r^2} & \Gamma^{r}_{\ \theta\theta} = \kappa r^3 - r & \Gamma^{r}_{\ \phi\phi} = (\kappa r^3 - r)\sin^2(\theta) \\
\Gamma^{\theta}_{\ t\theta} = h(t) & \Gamma^{\theta}_{\ r\theta} = \dfrac{1}{r} & \Gamma^{\theta}_{\ \theta t} = h(t) \\
\Gamma^{\theta}_{\ \theta r} = \dfrac{1}{r} & \Gamma^{\theta}_{\ \phi\phi} = -\cos(\theta)\sin(\theta) & \Gamma^{\phi}_{\ t\phi} = h(t) \\
\Gamma^{\phi}_{\ r\phi} = \dfrac{1}{r} & \Gamma^{\phi}_{\ \theta\phi} = \dfrac{\cos(\theta)}{\sin(\theta)} & \Gamma^{\phi}_{\ \phi t} = h(t) \\
\Gamma^{\phi}_{\ \phi r} = \dfrac{1}{r} & \Gamma^{\phi}_{\ \phi\theta} = \dfrac{\cos(\theta)}{\sin(\theta)} &
\end{array}
\tag{24}
$$

with f, g and h the unknown functions of time to be determined. The Levi-Cività connection compatible with the Friedman-Robertson-Walker metric is obtained from Eq. (24) by setting $f = 0, g = a\dot{a}$ and $h = \frac{\dot{a}}{a}$—compare with Eq. (12).

The Ricci tensor calculated for the connection in Eq. (24) has only two independent components:

$$R_{tt} = 3fh - 3h^2 - 3\frac{\partial h}{\partial t}, \tag{25}$$

$$R_{ii} \simeq fg + gh + 2\kappa + \frac{\partial g}{\partial t}. \tag{26}$$

[5] See Appendix A for a brief comment about the Lie derivative of a connection.

We now proceed to find solutions to Eq. (6). As in the previous section, we present the three possibilities of solutions, but we will restrict ourselves to finding solutions to the (affine) Ricci-flat case.

4.1 ... with vanishing Ricci

The first kind of solutions can be found by solving the system of equations determined by vanishing Ricci. However, this strategy requires the fixing of one of the unknown functions. The equations to solve are written as

$$\dot{h} - (f - h)h = 0, \tag{27}$$

$$\dot{g} + (f + h)g + 2\kappa = 0. \tag{28}$$

Noticing that in the above equations f is not a dynamical function, from Eq. (27) we can solve h as a function of f:

$$h(t) = \frac{\exp(F(t))}{C_h + \int dt \exp(F)}, \tag{29}$$

where we have defined $F = \int dt f(t)$ and C_h is an integration constant. Then, Eq. (28) can be solved for g:

$$g(t) = \exp(-\Sigma(t))\left(C_g - 2\kappa\left(\int dt \exp(\Sigma(t))\right)\right), \tag{30}$$

where $\Sigma(t) = \int dt\,(f(t) + h(t))$ and C_g is another integration constant.

A particular solution inspired in the components of the connection for Friedmann-Robertson-Walker, in whose case $f = 0$, gives

$$f(t) = 0, \;\; g(t) = \frac{1}{t + C_h}\left(C_g - \kappa(t + C_h)^2\right), \;\; h(t) = \frac{1}{t + C_h}, \tag{31}$$

which for $C_h = C_g = 0$ and $\kappa = -1$ yields the expected solution from **Table 1**.[6] However, in Eq. (30) there are Ricci-flat solutions which cannot be associated with the sole existence of a metric, that is, non-Riemannian manifolds, as, for example, solutions with $\kappa > 0$.

There are special solutions that cannot be obtained from Eqs. (29) and (30), since they represent degenerated point in the moduli space.

Case $f = h$: In these particular subspaces on the moduli, the first equation is linear, and therefore the solution above is not valid. However, the solutions to Eqs. (27) and (28) are given by

$$f = C_h \qquad h = C_h \qquad g = C_g \exp(-2Ct) - \frac{\kappa}{C_h}. \tag{32}$$

Case $h = -f$: In this case again, Eq. (27) decouples from Eq. (28), and there solutions are given by

[6] The standard parametrisation of Minkowski space–time is achieved by the trivial solution of Eqs. (27) and (28), i.e. $f = g = h = \kappa = 0$.

$$f = -\frac{1}{2t + C_h} \qquad h = \frac{1}{2t + C_h} \qquad g = C_g - 2\kappa t \tag{33}$$

Case $h = 0$ and f given: In this case, Eq. (28) becomes an identity, and g can still be solved for a given function f as

$$g(t) = \exp\left(-F(t)\right)\left(C_g - 2\kappa\left(\int dt \exp\left(F(t)\right)\right)\right). \tag{34}$$

Case $g = 0$ and f given: In this case, Eq. (28) requires $\kappa = 0$, and h can still be solved for a given function f as in Eq. (29).

At this point, we have shown that a space–time described by a Ricci flat, torsion-free, equi-affine connection with the form presented in Eq. (24) reproduces the cosmological Ricci-flat solutions to general relativity, presented in **Table 1**, and there exist generalisations to these solutions which are not possibly obtained in the Riemannian case. However, one can go even further and ask oneself whether the affine Ricci-flat condition yields more—real life—useful solutions, such as those solutions of general relativity presented in **Table 2**.

Therefore, we would like to obtain the Einstein equations from the affine Ricci-flat equation, that is,

$$R^{\text{Aff}}_{\mu\nu} = M^{\text{GR}}_{\mu\nu} = 0, \tag{35}$$

where

$$M^{\text{GR}}_{\mu\nu} = R^{\text{GR}}_{\mu\nu} - \Lambda g_{\mu\nu} - 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\, T g_{\mu\nu}\right).$$

In the following, we are considering that the stress-energy tensor describes a perfect fluid, that is,

$$T_{\mu\nu} = \text{diag}\left(\rho \quad \frac{pa^2}{1-\kappa r^2} \quad pa^2r^2 \quad pa^2r^2\sin^2\theta\right).$$

In general relativity, the Einstein equations in the form of Ricci, for the cosmological ansatz, yield

$$M^{\text{GR}}_{tt} \simeq 3\ddot{a} - \Lambda a + 4\pi G a(\rho + 3p), \qquad M^{\text{GR}}_{tt} \simeq a\ddot{a} + 2\dot{a}^2 - \Lambda a^2 - 4\pi G a^2(\rho - p) + 2\kappa, \tag{36}$$

Now, comparing Eq. (36) with Eqs. (27) and (28), a parametrisation for f, g and h can be found such that once one computes the Ricci tensor for the affine connection, the compatibility in Eq. (35) is satisfied. The parametrisation is given by

$$h = \dot{a} + x, \qquad f = x, \qquad g = a\dot{a} + y, \tag{37}$$

where the functions x and y satisfy the equations

$$\dot{x} + x\dot{a} - F_1 = 0, \tag{38}$$

$$\dot{y} + yF_2 - F_3 = 0, \tag{39}$$

with

$$F_1 = \frac{a}{3}(4\pi G(3p + \rho) - \Lambda) - (\dot{a})^2, \qquad F_2 = \dot{a} + 2x, \qquad F_3 = a^2(4\pi G(p - \rho) - \Lambda) - 2a\dot{a}x - a(\dot{a})^2 + (\dot{a})^2. \tag{40}$$

Eqs. (38) and (39) can be formally integrated in terms of functions a, ρ and p, yielding

$$x = e^{-a}\left(C_x + \int dtF_1 e^a\right), \tag{41}$$

$$y = e^{-\int dtF_2}\left(C_y + \int dtF_3 e^{\int dtF_2}\right). \tag{42}$$

Therefore, a subspace of the possible solutions of the affine Ricci-flat geometries describes the cosmological scenarios from general relativity coupled with perfect fluids. However, the explicit expressions for Eqs. (41) and (42) for obtaining specific solutions to Friedmann-Lemaître-Robertson-Walker models are very complicated.

4.2 ... with parallel Ricci

The second class of solutions can be found by solving the parallel Ricci equation, $\nabla_\lambda R_{\mu\nu} = 0$, which yield three independent field equations:

$$\nabla_t R_{tt} \simeq \ddot{h} - (3f - 2h)\dot{h} - h\dot{f} + 2fh(f - h), \tag{43}$$

$$\nabla_i R_{ti} = \nabla_i R_{it} \simeq 3g\dot{h} - h\dot{g} - 2gh(2f - h) - 2\kappa h, \tag{44}$$

$$\nabla_t R_{ii} \simeq \ddot{g} + g\left(\dot{f} + \dot{h}\right) + (f - h)\dot{g} - gh(f + h) - 4\kappa h. \tag{45}$$

However, the system of equations is complicated enough to avoid an analytic solution.

Despite the complication, we can try a couple of assumptions that simplify the system of equations, for example, if one considers the parametrisation inspired in the Friedmann-Robertson-Walker results, that is, setting $f = 0$, and can solve h from Eq. (43), which is a total derivative in this particular case. Nonetheless, despite the value of the first integration constant, the system of equations imposes that both κ and g vanish.

4.3 ... with harmonic curvature

Finally, the third class of solutions are those of Eq. (6). The set of equations degenerate and yield a single independent field equation:

$$\nabla_\lambda R_{ti}{}^{\lambda}{}_{i} \simeq \ddot{g} + g\dot{f} + f\dot{g} - 2g\dot{h} + 2gh(f - 2h) - 2\kappa h \tag{46}$$

Therefore, we need to set two out of the three unknown functions to be able to solve for the connection.

5. Conclusions and remarks

In this chapter, we have shortly reviewed the recently proposed model of polynomial affine gravity, which is an alternative model for gravitational interactions described solely by the connection, that is, the metric does not play any role in the mediation of the interaction. Among the features of the model, one encounters that despite the numerous possible terms in the action (see Eq. (3)), the absence of a metric tensor gives a sort of rigidity to the action, in the sense that only a very restricted set of terms can be added. Such rigidity suggests that if one attempts to quantise the model, it could be renormalisable. Additionally, all of the coupling

constants, in the pure gravity regime, are dimensionless, pointing to a possible conformal invariance of the (pure) gravitational interaction.[7]

Restricting ourselves to equi-affine, torsion-free connections, the field equations are a generalisation of those from general relativity (Eq. (6)). We solved the field equations for an isotropic and homogeneous connection, either compatible with a metric or not. These solutions are classified under three conditions: Ricci flat, parallel Ricci and harmonic curvature.

When the affine connection is the Levi-Cività connection for a Friedman-Robertson-Walker metric, we show that the sole solution for a Ricci-flat space–time is described by the connection of Minkowski's space (see **Table 1**). In the parallel Ricci case, we show that—as intuitively expected—one recovers the vacuum cosmological models from general relativity (see **Table 2**), where the cosmological constant enters as an integration constant, but such constant could be interpreted as (partially) coming from the stress-energy tensor of a vacuum energy perfect fluid, as mentioned—in the context of general relativity—in Ref. [48]. Finally, the (formal) solutions to the harmonic curvature are presented in **Table 3**, but yet some work remains to be done to extract the phenomenology from these solutions.

In the case of the cosmological affine connection, we found that the Ricci-flat condition yields only two independent equations, which are not enough to find the three unknown functions that parametrise the homogeneous and isotropic connection. Nonetheless, since f is not a dynamical function, it serves as a *parametric* function to solve the remaining two, that is, g and h. Interestingly, the three functions can be chosen in a way that Ricci-flat condition for the affine connection yields the Friedmann-Lemaître equations from general relativity coupled with a perfect fluid. In this sense, the pure polynomial affine gravity supersizes general relativity, since geometrically it can mimic effects that are usually interpreted as *matter* effects. However, among the possible solutions for the Ricci-flat condition, there are countless (yet) nonphysical solutions, and what is more, there is nothing that favours the specific choice in Eq. (37) over others. Such landscape drives us to think that another type of condition should be used to restrict even further the possible solutions for the affine connection.

The conditions of affine parallel Ricci could be the cornerstone in solving the aforementioned degeneracy, since these conditions raise three independent equations that would serve to determine the three unknown functions. However, at the moment we have not yet achieved any interesting result in pursuing this goal.

On the other hand, the harmonic curvature condition yields a sole (independent) field equation, and therefore the solutions are even more degenerated than those from the Ricci-flat condition, leaving even more space for nonphysical solutions.

Our research has stressed the importance of considering the connection as the mediator of the gravitational interactions. We have confirmed that in the framework of polynomial affine gravity, the cosmological solutions described by a connection compatible with a Friedmann-Robertson-Walker metric are compatible with those of general relativity, with the possible exception of the case of harmonic curvature. The impact of our contribution lies in the fact that for a generic affine connection, even the simplest condition—Ricci flatness—allows solutions which are (dynamically) equivalent to the system of Friedmann-Lamaître-Robertson-Walker for a perfect fluid (in general relativity), despite the absence of matter in the affine model.

We would like to finish our discussion highlighting that the geometric emulation of matter content can serve as a starting point to a change of paradigm related with the interpretation of the matter content of the Universe, in particular the *dark*

[7] At least at classical level.

sector. We think that our findings might be useful for providing a gravitational origin interpretation of the dark matter and/or energy, driven by the inclusion of extra degrees of freedom coming from the nonmetricity of the connection. Further studies, which take the observations reported in Refs. [49, 50] into account, will need to be performed, to be able of discern between the possibilities of, for example, dark matter that has been originated as a gravitational versus matter effect. Similar analysis should be carried with the dark energy [51, 52].

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A. Lie derivative and killing vectors

The usual procedure for solving Einstein's equation is to propose an ansatz for the metric. That ansatz must be compatible with the symmetries we would like to respect in the problem. A first application is seen in Schwarzschild's metric [60], which is the *most general* symmetric rank-two tensor compatible with the rotation group in three dimensions, an thus is *spherically symmetric*.

The formal study of the symmetries of the fields is accomplished via the Lie derivative (for reviews, see Refs. [61–64]). Below, we briefly explain the use of the Lie derivative for obtaining ansatzes for either the metric or the connection.

The Lie derivative of a connection possesses an inhomogeneous part, in comparison with the one of a rank-three tensor. This can be written schematically as

$$\mathcal{L}_\xi \Gamma^a{}_{bc} = \mathcal{L}_\xi T^a{}_{bc} + \frac{\partial^2 \xi^a}{\partial x^b \partial x^c},$$

or explicitly

$$\mathcal{L}_\xi \Gamma^a{}_{bc} = \xi^m \partial_m \Gamma^a{}_{bc} - \Gamma^m{}_{bc} \partial_m \xi^a + \Gamma^a{}_{mc} \partial_b \xi^m + \Gamma^a{}_{bm} \partial_c \xi^m + \frac{\partial^2 \xi^a}{\partial x^b \partial x^c}, \tag{47}$$

where ξ is the vector defining the symmetry flow.

In particular, for cosmological applications, one asks for *isotropy* and *homogeneity*, which in four dimensions restricts the isometry group to either $SO(4)$, $SO(3,1)$ or $ISO(3)$. The algebra of these groups can be obtained from the algebra $\mathfrak{so}(4)$ through a $3+1$ decomposition, i.e. $J_{AB} = \{J_{ab}, J_{a*}\}$, where the extra dimension has been denoted by an asterisk. In terms of these new generators, the algebra reads

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} + \delta_{ad} J_{bc} - \delta_{bd} J_{ac}, \\ [J_{ab}, J_{c*}] &= \delta_{bc} J_{a*} - \delta_{ac} J_{b*}, \\ [J_{a*}, J_{c*}] &= -\kappa J_{ac}, \end{aligned} \tag{48}$$

with[8]

$$\kappa = \begin{cases} 1 & SO(4) \\ 0 & ISO(3) \\ -1 & SO(3,1) \end{cases}. \tag{49}$$

The six Killing vectors of these algebras, expressed in spherical coordinates, are,

$$\begin{aligned}
J_1 &= J_{23} = (\,0 \quad 0 \quad -\cos(\varphi) \quad \cot(\theta)\sin(\varphi)\,), \\
J_2 &= J_{31} = (\,0 \quad 0 \quad \sin(\varphi) \quad \cos(\varphi)\cot(\theta)\,), \\
J_3 &= J_{12} = (\,0 \quad 0 \quad 0 \quad 1\,), \\
P_1 &= J_{1*} = \sqrt{1-\kappa r^2}\left(\,0 \quad \cos(\varphi)\sin(\theta) \quad \frac{\cos(\varphi)\cos(\theta)}{r} \quad -\frac{\sin(\varphi)}{t\sin(\theta)}\,\right), \\
P_2 &= J_{2*} = \sqrt{1-\kappa r^2}\left(\,0 \quad \sin(\varphi)\sin(\theta) \quad \frac{\cos(\theta)\sin(\varphi)}{r} \quad \frac{\cos(\varphi)}{t\sin(\theta)}\,\right), \\
P_3 &= J_{3*} = \sqrt{1-\kappa r^2}\left(\,0 \quad \cos(\theta) \quad -\frac{\sin(\theta)}{r} \quad 0\,\right).
\end{aligned} \tag{50}$$

Using Eq. (47), for the above Killing vectors, the most general connection compatible with the desired symmetries can be obtained [47], giving the components structure shown in Eq. (24).

[8] The inhomogeneous algebra of $ISO(n)$ can be obtained from those of $SO(n+1)$ or $SO(n,1)$ through the Inönü-Wigner contraction [65].

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