Chapter

Obtaining Explicit Formulas and Identities for Polynomials Defined by Generating Functions of the Form $F(t)^x \cdot G(t)^\alpha$

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Abstract

In this chapter, we study properties of polynomials defined by generating functions of the form $A(t,x,\alpha) = F(t)^x \cdot G(t)^\alpha$. Based on the Lagrange inversion theorem and the theorem of logarithmic derivative for generating functions, we obtain new properties related to the compositional inverse generating functions of those polynomials. Also we study the composition of generating functions $R(tA(t))$, where $A(t)$ is the generating function of the form $F(t)^x \cdot G(t)^\alpha$. We apply those results for obtaining explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

Keywords: polynomial, identity, generating function, composita, composition, compositional inverse

1. Introduction

Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory, and other fields of mathematics. One of the advantages of generating functions is that an infinite number sequence can be represented in a form of a single expression. Many authors have studied generating functions and their properties and found applications for them (for instance, Comtet [1], Flajolet and Sedgewick [2], Graham et al. [3], Robert [4], Stanley [5], and Wilf [6]).

Generating functions have an important role in the study of polynomials. Vast investigations related to the generating functions for many polynomials can be found in many books and articles (e.g., see [7–17]).

A special place in this area is occupied by research in the field of obtaining new identities for polynomials and special numbers with using their generating functions. Interesting results in the field of obtaining new identities for polynomials can be found in some recent works by Simsek [18–20], Kim et al. [21, 22], and Ryoo [23–25].

Another trend in study of polynomials is getting new representation and explicit formulas for those polynomials. For instance, Qi has recently established explicit formulas for the generalized Motzkin numbers in [26] and the central Delannoy numbers in [27]. One can find interesting results in papers of Srivastava [28, 29], Cenkci [30], and Boyadzhiev [31].

In this chapter, we obtain some interesting properties of polynomials defined by generating functions of the form $F(t)^x \cdot G(t)^\alpha$. As an application, we give some new identities for the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

According to Stanley [32], ordinary generating functions are defined as follows:

Definition 1. *An ordinary generating function of the sequence $(a_n)_{n\geq 0}$ is the formal power series*

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n\geq 0} a_n x^n. \tag{1}$$

Kruchinin et al. [33–35] introduced the mathematical notion of the *composita* of a given generating function, which can be used for calculating the coefficients of a composition of generating functions.

Definition 2. The composita of the generating function $F(x) = \sum_{n>0} f_n x^n$ is the function with two variables

$$F^\Delta(n,k) = \sum_{\pi_k \in C_n} f_{\lambda_1} f_{\lambda_2} \cdots f_{\lambda_k}, \tag{2}$$

where C_n is the set of all compositions of an integer n and π_k is the composition n into k parts such that $\sum_{i=1}^{k} \lambda_i = n$.

Using the expression of the composita of a given generating function $F^\Delta(n,k)$, we can get powers of the generating function $F(x)$:

$$(F(x))^k = \sum_{n\geq k} F^\Delta(n,k) x^n. \tag{3}$$

Compositae also can be used for calculating the coefficients of generating functions obtained by addition, multiplication, composition, reciprocation, and compositional inversion of generating functions (for details see [33–35]).

By the reciprocal generating function we mean the following [6]:

Definition 1. A reciprocal generating function $A(x)$ of a generating function $B(x) = \sum_{n\geq 0} b_n x^n$ is a power series such that satisfies the following condition:

$$A(x)B(x) = 1. \tag{4}$$

By the compositional inverse generating function we mean the following:

Definition 2. A compositional inverse $\overline{F(x)}$ of generating function $F(x) = \sum_{n>0} f_n x^n$ with $f(1) \neq 0$ is a power series such that satisfies the following condition:

$$F\left(\overline{F(x)}\right) = x. \tag{5}$$

Also the compositional inverse can be written as $F^{[-1]}(x)$ or $\overline{F(x)} = RevF$.

For example, we will use the following formulas:

If we consider the composition $A(x) = R(F(x)) = \sum_{n\geq 0} a_n x^n$ of generating functions $R(x) = \sum_{n\geq 0} r_n x^n$ and $F(x) = \sum_{n>0} f_n x^n$, then we can get the values of the coefficients a_n by using the following formula ([35], Eq. (17)):

$$a_n = \begin{cases} r_0, & \text{for } n = 0; \\ \sum_{k=1}^{n} F^{\Delta}(n,k) r_k, & \text{otherwise}. \end{cases} \tag{6}$$

If we consider the composition $A(x) = R(F(x)) = \sum_{n>0} a_n x^n$ of generating functions $R(x) = \sum_{n>0} r_n x^n$ and $F(x) = \sum_{n>0} f_n x^n$, then we can get the values of the composita $A^{\Delta}(n,k)$ by using the following formula ([35]):

$$A^{\Delta}(n,k) = \sum_{m=k}^{n} F^{\Delta}(n,m) R^{\Delta}(m,k). \tag{7}$$

2. Main results

Let us consider a special case of generating functions that can be presented as the product of the powers of generating functions $F(t)^x \cdot G(t)^{\alpha}$. For such generating functions, we obtain several properties, which are given in the following theorem:

Theorem 1. *If $A(t)$ is a generating function of the following form:*

$$A(t) = F(t)^x \cdot G(t)^{\alpha} = \sum_{n \geq 0} A_n(x,\alpha) t^n, \tag{8}$$

then:

1. *For the composition of generating functions $D(t) = C(B(t)) = \sum_{n \geq 0} B_n t^n$, where $B(t) = tA(t)$ and $C(t) = \sum_{n \geq 0} C_n t^n$, we have*

$$D_n = D_n(x,\alpha) = \sum_{k=1}^{n} A_{n-k}(kx, k\alpha) C_k, \quad D_0 = C_0; \tag{9}$$

2. *For the compositional inverse generating function $\overline{B}(t)$ of $B(t) = tA(t)$, we have*

$$\overline{B}(t) = \sum_{n>0} \frac{1}{n} A_{n-1}(-nx, -n\alpha) t^n; \tag{10}$$

3. *We have the following identities*

$$\sum_{m=k}^{n} A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k} \tag{11}$$

and

$$\sum_{m=k}^{n} \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}, \tag{12}$$

where $\delta_{n,k}$ is the Kronecker delta.

Proof. First we get the k-th power of the generating function $B(t) = tA(t)$

$$\begin{aligned} (B(t))^k = (tA(t))^k = t^k (F(t))^{xk} (G(t))^{\alpha k} = \\ = t^k \sum_{n \geq 0} A_n(kx, k\alpha) t^n = \sum_{n \geq k} A_{n-k}(kx, k\alpha) t^n. \end{aligned}$$

Hence, the composita of $B(t) = tA(t)$ is

$$B^{\Delta}(n,k) = A_{n-k}(kx, k\alpha). \tag{13}$$

Using Eqs. (6) and (13), we get Eq. (9).

According to [36], the composita of the compositional inverse generating function $\overline{A}(t)$ of $A(t) = \sum_{n>0} a_n t^n$ is

$$\overline{A}^{\Delta}(n,k) = \frac{k}{n} R^{\Delta}(2n-k, n), \tag{14}$$

where $R^{\Delta}(n,k)$ is the composita of the generating function $R(t) = \frac{t^2}{A(t)}$.

For getting the composita of the compositional inverse generating function $\overline{B}(t)$ of $B(t) = tA(t)$, we need to know the composita of the generating function

$$R(t) = \frac{t^2}{B(t)} = \frac{t^2}{tA(t)} = \frac{t}{A(t)}. \tag{15}$$

Then we get the k-th power of the generating function $R(t) = \frac{t}{A(t)}$

$$\begin{aligned}(R(t))^k &= \left(\tfrac{t}{A(t)}\right)^k = t^k (F(t))^{-xk} (G(t))^{-\alpha k} = \\ &= t^k \sum_{n \geq 0} A_n(-kx, -k\alpha) t^n = \sum_{n \geq k} A_{n-k}(-kx, -k\alpha) t^n.\end{aligned} \tag{16}$$

Hence, the composita of Eq. (15) is

$$R^{\Delta}(n,k) = A_{n-k}(-kx, -k\alpha). \tag{17}$$

Using Eqs. (14) and (17), we get

$$\overline{B}^{\Delta}(n,k) = \frac{k}{n} R^{\Delta}(2n-k, n) = \frac{k}{n} A_{2n-k-n}(-nx, -n\alpha) = \frac{k}{n} A_{n-k}(-nx, -n\alpha). \tag{18}$$

For $k = 1$, we get Eq. (10).

Applying Eq. (7) for the composition $C(t) = B(\overline{B}(t)) = t$, we get

$$\begin{aligned}C^{\Delta}(n,k) &= \sum_{m=k}^{n} \overline{B}^{\Delta}(n,m) B^{\Delta}(m,k) = \\ &= \sum_{m=k}^{n} \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}.\end{aligned} \tag{19}$$

Applying Eq. (7) for the composition $D(t) = \overline{B}(B(t)) = x$, we get

$$\begin{aligned}D^{\Delta}(n,k) &= \sum_{m=k}^{n} B^{\Delta}(n,m) \overline{B}^{\Delta}(m,k) = \\ &= \sum_{m=k}^{n} A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k}.\end{aligned} \tag{20}$$

□

As an application of Theorem 1, we present several examples of its usage for such polynomials as the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner.

2.1 Generalized Bernoulli polynomials

The generalized Bernoulli polynomials are defined by the following generating function [37, 38]:

$$B(t,x,\alpha) = e^{xt}\left(\frac{t}{e^t-1}\right)^{\alpha} = \left(e^t\right)^x\left(\frac{t}{e^t-1}\right)^{\alpha} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{21}$$

where

$$B_n^{(\alpha)}(x) = \sum_{i=0}^{n} \frac{n!}{(n+i)!}\binom{n+\alpha}{n-i}\binom{i+\alpha-1}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(x+j)^{n+i}. \tag{22}$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{B_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \tag{23}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x-\alpha}{2} & 1 & & \\
\frac{12x^2-12\alpha x+3\alpha^2-\alpha}{24} & 2x-\alpha & 1 & \\
\frac{8x^3-12\alpha x^2+(6\alpha^2-2\alpha)x-\alpha^3+\alpha^2}{48} & \frac{24x^2-24\alpha x+6a^2-\alpha}{12} & \frac{6x-3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{B_{n-k}^{(-n\alpha)}(-nx)}{(n-k)!}. \tag{24}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{-2x+\alpha}{2} & 1 & & \\
\frac{36x^2-36\alpha x+9a^2+\alpha}{24} & -2x+\alpha & 1 & \\
\frac{-32x^3+48\alpha x^2-(24\alpha^2+2\alpha)x+4\alpha^3+\alpha^2}{12} & \frac{48x^2-48\alpha x+12\alpha^2+\alpha}{12} & \frac{-6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the generalized Bernoulli polynomials:

$$\sum_{m=k}^{n}\frac{m}{n}\frac{B_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!}\frac{B_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \tag{25}$$

and

$$\sum_{m=k}^{n} \frac{B_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{26}$$

2.2 Generalized Euler polynomials

The generalized Euler polynomials are defined by the following generating function [37]:

$$E(t,x,\alpha) = e^{xt}\left(\frac{2}{e^t+1}\right)^{\alpha} = (e^t)^x\left(\frac{2}{e^t+1}\right)^{\alpha} = \sum_{n\geq 0} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{27}$$

where

$$E_n^{(\alpha)}(x) = \sum_{i=0}^{n} \frac{1}{2^i}\binom{i+\alpha-1}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j}(x+j)^n. \tag{28}$$

According to Eq. (13), the composita for the generating function $D(t) = tE(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{E_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \tag{29}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x-\alpha}{2} & 1 & & \\
\frac{4x^2-4\alpha x+\alpha^2-\alpha}{8} & 2x-\alpha & 1 & \\
\frac{8x^3-12\alpha x^2+(6\alpha^2-6\alpha)x-\alpha^3+3\alpha^2}{48} & \frac{8x^2-8\alpha x+2a^2-\alpha}{4} & \frac{6x-3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tE(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{E_{n-k}^{(-n\alpha)}(-nx)}{(n-k)!}. \tag{30}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{-2x+\alpha}{2} & 1 & & \\
\frac{12x^2-12\alpha x+3a^2+\alpha}{8} & -2x+\alpha & 1 & \\
\frac{-32x^3+48\alpha x^2-(24\alpha^2+6\alpha)x+4\alpha^3+3\alpha^2}{12} & \frac{16x^2-16\alpha x+4\alpha^2+\alpha}{4} & \frac{-6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the generalized Euler polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{E_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!}\frac{E_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \tag{31}$$

and

$$\sum_{m=k}^{n} \frac{E_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{E_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{32}$$

2.3 Frobenius-Euler polynomials

The Frobenius-Euler polynomials are defined by the following generating function [39]:

$$H(t,x,\alpha,\lambda) = e^{xt}\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} = (e^t)^x\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} = \sum_{n\ge 0} H_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}, \tag{33}$$

where

$$H_n^{(\alpha)}(x,\lambda) = \sum_{i=0}^{n} \frac{1}{(1-\lambda)^i}\binom{i+\alpha-1}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(x+j)^n. \tag{34}$$

According to Eq. (13), the composita for the generating function $D(t) = tH(t,x,\alpha,\lambda)$ is

$$D^{\Delta}(n,k) = \frac{H_{n-k}^{(k\alpha)}(kx,\lambda)}{(n-k)!}. \tag{35}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
\dfrac{(\lambda-1)x+\alpha}{\lambda-1} & 1 & \\
\dfrac{(\lambda^2-2\lambda+1)x^2+(2\lambda-2)\alpha x+\alpha^2+\lambda\alpha}{2\lambda^2-4\lambda+2} & \dfrac{(2\lambda-2)x+2\alpha}{\lambda-1} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tH(t,x,\alpha,\lambda)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{H_{n-k}^{(-n\alpha)}(-nx,\lambda)}{(n-k)!}. \tag{36}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
-\dfrac{(\lambda-1)x+\alpha}{\lambda-1} & 1 & \\
\dfrac{(3\lambda^2-6\lambda+3)x^2+(6\lambda-6)\alpha x+3\alpha^2-\lambda\alpha}{2\lambda^2-4\lambda+2} & -\dfrac{(2\lambda-2)x+2\alpha}{\lambda-1} & 1
\end{array}$$

Also we can get the following new identities for the Frobenius-Euler polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{H_{n-m}^{(-n\alpha)}(-nx,\lambda)}{(n-m)!}\frac{H_{m-k}^{(k\alpha)}(kx,\lambda)}{(m-k)!} = \delta_{n,k} \tag{37}$$

and

$$\sum_{m=k}^{n} \frac{H_{n-m}^{(m\alpha)}(mx,\lambda)}{(n-m)!} \frac{k}{m} \frac{H_{m-k}^{(-m\alpha)}(-mx,\lambda)}{(m-k)!} = \delta_{n,k}. \tag{38}$$

2.4 Generalized Sylvester polynomials

The generalized Sylvester polynomials are defined by the following generating function [40]:

$$F(t,x,\alpha) = (1-t)^{-x} e^{\alpha x t} = \left(\frac{e^{\alpha t}}{1-t}\right)^{x} = \sum_{n\geq 0} F_n(x,\alpha) t^n, \tag{39}$$

where

$$F_n(x,\alpha) = \sum_{i=0}^{n} \frac{(\alpha x)^{n-i}}{(n-i)!} \binom{i+x-1}{i}. \tag{40}$$

According to Eq. (13), the composita for the generating function $D(t) = tF(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = F_{n-k}(kx,\alpha). \tag{41}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
(\alpha+1)x & 1 & & \\
\dfrac{(\alpha^2+2\alpha+1)x^2+x}{2} & (2\alpha+2)x & 1 & \\
\dfrac{(\alpha^3+3\alpha^2+3\alpha+1)x^3+(3\alpha+3)x^2+2x}{6} & (2\alpha^2+4\alpha+2)x^2+x & (3\alpha+3)x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tF(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} F_{n-k}(-nx,\alpha). \tag{42}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-(\alpha+1)x & 1 & & \\
\dfrac{(3\alpha^2+6\alpha+3)x^2-x}{2} & -(2\alpha+2)x & 1 & \\
-\dfrac{(8\alpha^3+24\alpha^2+24\alpha+8)x^3-(6\alpha+6)x^2+x}{3} & (4\alpha^2+8\alpha+4)x^2-x & -(3\alpha+3)x & 1
\end{array}$$

Also we can get the following new identities for the generalized Sylvester polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} F_{n-m}(-nx,\alpha) F_{m-k}(kx,\alpha) = \delta_{n,k} \tag{43}$$

and

$$\sum_{m=k}^{n} F_{n-m}(mx,\alpha)\frac{k}{m}F_{m-k}(-mx,\alpha)=\delta_{n,k}. \tag{44}$$

2.5 Generalized Laguerre polynomials

The generalized Laguerre polynomials are defined by the following generating function [8]:

$$L(t,x,\alpha)=(1-t)^{-\alpha-1}e^{\frac{xt}{t-1}}=\left(e^{\frac{t}{t-1}}\right)^{x}\left(\frac{1}{1-t}\right)^{\alpha+1}=\sum_{n\geq 0}L_n^{(\alpha)}(x)t^n, \tag{45}$$

where

$$L_n^{(\alpha)}(x)=\sum_{i=0}^{n}\frac{(-x)^i}{i!}\binom{n+\alpha}{n-i}. \tag{46}$$

According to Eq. (13), the composita for the generating function $D(t)=tL(t,x,\alpha)$ is

$$D^{\Delta}(n,k)=L_{n-k}^{(k\alpha+k-1)}(kx). \tag{47}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ -x+\alpha+1 & 1 & \\ \dfrac{x^2-(2\alpha+4)x+\alpha^2+3\alpha+2}{2} & -2x+2\alpha+2 & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t)=tL(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k)=\frac{k}{n}L_{n-k}^{(-n\alpha-n-1)}(-nx). \tag{48}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ x-\alpha-1 & 1 & \\ \dfrac{3x^2-(6\alpha+4)x+3\alpha^2+5\alpha+2}{2} & 2x-2\alpha-2 & 1 \end{array}$$

Also we can get the following new identities for the generalized Laguerre polynomials:

$$\sum_{m=k}^{n}\frac{m}{n}L_{n-m}^{(-n\alpha-n-1)}(-nx)L_{m-k}^{(k\alpha+k-1)}(kx)=\delta_{n,k} \tag{49}$$

and

$$\sum_{m=k}^{n}L_{n-m}^{(m\alpha+m-1)}(mx)\frac{k}{m}L_{m-k}^{(-m\alpha-m-1)}(-mx)=\delta_{n,k}. \tag{50}$$

2.6 Abel polynomials

The Abel polynomials are defined by the following generating function [8, 41]:

$$A(t,x,\alpha) = e^{\frac{W(\alpha t)x}{\alpha}} = \left(e^{\frac{W(\alpha t)}{\alpha}}\right)^x = \sum_{n\geq 0} A_n(x,\alpha)\frac{t^n}{n!}, \tag{51}$$

where $W(t)$ is the Lambert W function and

$$A_n(x,\alpha) = x(x-\alpha n)^{n-1}. \tag{52}$$

According to Eq. (13), the composita for the generating function $D(t) = tA(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{A_{n-k}(kx,\alpha)}{(n-k)!}. \tag{53}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
x & 1 & & & \\
\frac{x^2-2\alpha x}{2} & 2x & 1 & & \\
\frac{x^3-6\alpha x^2+9\alpha^2 x}{6} & 2x^2-2\alpha x & 3x & 1 & \\
\frac{x^4-12\alpha x^3+48\alpha^2x^2-64\alpha^3 x}{24} & \frac{4x^3-12\alpha x^2+9\alpha^2 x}{3} & \frac{9x^2-6\alpha x}{2} & 4x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tA(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{A_{n-k}(-nx,\alpha)}{(n-k)!}. \tag{54}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
-x & 1 & & & \\
\frac{3x^2+2\alpha x}{2} & -2x & 1 & & \\
-\frac{16x^3+24\alpha x^2+9\alpha^2 x}{6} & 4x^2+2\alpha x & -3x & 1 & \\
\frac{125x^4+300\alpha x^3+240\alpha^2x^2+64\alpha^3 x}{24} & -\frac{25x^3+30\alpha x^2+9\alpha^2 x}{3} & \frac{15x^2+6\alpha x}{2} & -4x & 1
\end{array}$$

Also we can get the following new identities for the Abel polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{A_{n-m}(-nx,\alpha)}{(n-m)!}\frac{A_{m-k}(kx,\alpha)}{(m-k)!} = \delta_{n,k} \tag{55}$$

and

$$\sum_{m=k}^{n} \frac{A_{n-m}(mx,\alpha)}{(n-m)!} \frac{k}{m} \frac{A_{m-k}(-mx,\alpha)}{(m-k)!} = \delta_{n,k}. \tag{56}$$

2.7 Bessel polynomials

The Bessel polynomials are defined by the following generating function [8]:

$$B(t,x) = e^{x\left(1-\sqrt{1-2t}\right)} = \left(e^{1-\sqrt{1-2t}}\right)^{x} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}, \tag{57}$$

where

$$B_n(x) = \begin{cases} 1, & n = 0; \\ \sum_{k=1}^{n} \frac{(2n-k-1)!}{(n-k)!(k-1)!}\frac{x^k}{2^{n-k}}, & n > 0. \end{cases} \tag{58}$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t,x)$ is

$$D^{\Delta}(n,k) = \frac{B_{n-k}(kx)}{(n-k)!}. \tag{59}$$

The triangular form of this composita is

$$\begin{array}{lllll} 1 & & & & \\ 2 & 1 & & & \\ \frac{x^2+x}{2} & 2x & 1 & & \\ \frac{x^3+3x^2+3x}{6} & 2x^2+x & 3x & 1 & \\ \frac{x^4+6x^3+15x^2+15x}{24} & \frac{4x^3+6x^2+3x}{3} & \frac{9x^2+3x}{2} & 4x & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t,x)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{B_{n-k}(-nx)}{(n-k)!}. \tag{60}$$

The triangular form of this composita is

$$\begin{array}{lllll} 1 & & & & \\ -x & 1 & & & \\ \frac{3x^2-x}{2} & -2x & 1 & & \\ -\frac{16x^3-12x^2+3x}{6} & 4x^2-x & -3x & 1 & \\ \frac{125x^4-150x^3+75x^2-15x}{24} & -\frac{25x^3-15x^2+3x}{3} & \frac{15x^2-3x}{2} & -4x & 1 \end{array}$$

Also we can get the following new identities for the Bessel polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{B_{n-m}(-nx)}{(n-m)!} \frac{B_{m-k}(kx)}{(m-k)!} = \delta_{n,k} \tag{61}$$

and

$$\sum_{m=k}^{n} \frac{B_{n-m}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{62}$$

2.8 Stirling polynomials

The Stirling polynomials are defined by the following generating function [8, 42]:

$$S(t,x) = \left(\frac{t}{1-e^{-t}}\right)^x = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!}, \tag{63}$$

where

$$S_n(x) = \sum_{i=0}^{n} \binom{x+i}{i} \sum_{j=0}^{i} \frac{j!}{(n+j)!}(-1)^{n+j}\binom{i}{j}\left\{ {n+j \atop j} \right\}. \tag{64}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t,x)$ is

$$D^{\Delta}(n,k) = S_{n-k}(kx+k-1). \tag{65}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
\frac{x+1}{2} & 1 & & & \\
\frac{3x^2+5x+2}{24} & x+1 & 1 & & \\
\frac{x^3+2x^2+x}{48} & \frac{6x^2+11x+5}{12} & \frac{3x+3}{2} & 1 & \\
\frac{15x^4+30x^3+5x^2-18x-8}{5760} & \frac{2x^3+5x^2+4x+1}{12} & \frac{9x^2+17x+8}{8} & 2x+2 & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t,x)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} S_{n-k}(-nx-n-1). \tag{66}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
-\frac{x+2}{2} & 1 & & & \\
\frac{9x^2+19x+10}{24} & -x-1 & 1 & & \\
-\frac{4x^3+13x^2+14x+5}{12} & \frac{12x^2+25x+13}{12} & -\frac{3x+3}{2} & 1 & \\
\frac{1875x^4+8250x^3+13525x^2+9798x+2648}{5760} & -\frac{25x^3+80x^2+85x+30}{24} & \frac{15x^2+31x+16}{8} & -2x-2 & 1
\end{array}$$

Also we can get the following new identities for the Stirling polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} S_{n-m}(-nx-n-1) S_{m-k}(kx+k-1) = \delta_{n,k} \tag{67}$$

and

$$\sum_{m=k}^{n} S_{n-m}(mx+m-1) \frac{k}{m} S_{m-k}(-mx-m-1) = \delta_{n,k}. \tag{68}$$

2.9 Narumi polynomials

The Narumi polynomials are defined by the following generating function [8]:

$$S(t,x,\alpha) = \left(\frac{t}{\ln(1+t)}\right)^{\alpha} (1+t)^x = (1+t)^x \left(\frac{t}{\ln(1+t)}\right)^{\alpha} = \sum_{n\ge 0} S_n(x,\alpha) \frac{t^n}{n!}, \tag{69}$$

where

$$S_n(x,\alpha) = n! \sum_{i=0}^{n} \binom{x}{n-i} \sum_{j=0}^{i} \binom{j+\alpha-1}{j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{l!}{(l+i)!} \left[{l+i \atop l} \right]. \tag{70}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{S_{n-k}(kx,k\alpha)}{(n-k)!}. \tag{71}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x+\alpha}{2} & 1 & & \\
\frac{12x^2+(12\alpha-12)x+3\alpha^2-5\alpha}{24} & 2x+\alpha & 1 & \\
\frac{8x^3+(12\alpha-24)x^2+(6\alpha^2-22\alpha+16)x+\alpha^3-5\alpha^2+6\alpha}{48} & \frac{24x^2+(24\alpha-12)x+6\alpha^2-5\alpha}{12} & \frac{6x+3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} \frac{S_{n-k}(-nx,-n\alpha)}{(n-k)!}. \tag{72}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-\frac{2x+\alpha}{2} & 1 & & \\
\frac{36x^2+(36\alpha+12)x+9\alpha^2-5\alpha}{24} & -2x-\alpha & 1 & \\
-\frac{64x^3(96\alpha+48)x^2+(48\alpha^2+44\alpha+8)x+8\alpha^3+10\alpha^2+3\alpha}{24} & \frac{48x^2+(48\alpha+12)x+12\alpha^2+5\alpha}{12} & -\frac{6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the Narumi polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{S_{n-m}(-nx,-n\alpha)}{(n-m)!} \frac{S_{m-k}(kx,k\alpha)}{(m-k)!} = \delta_{n,k} \tag{73}$$

and

$$\sum_{m=k}^{n} \frac{S_{n-m}(mx, m\alpha)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\alpha)}{(m-k)!} = \delta_{n,k}. \tag{74}$$

2.10 Peters polynomials

The Peters polynomials are defined by the following generating function [8]:

$$S(t, x, \mu, \lambda) = \left(1 + (1+t)^{\lambda}\right)^{-\mu} (1+t)^{x} = (1+t)^{x} \left(\frac{1}{1 + (1+t)^{\lambda}}\right)^{\mu} = \sum_{n \geq 0} S_n(x, \mu, \lambda) \frac{t^n}{n!}, \tag{75}$$

where

$$S_n(x, \mu, \lambda) = n! \sum_{i=0}^{n} \binom{x}{n-i} \sum_{j=0}^{i} \frac{1}{2^{j+\mu}} \binom{j+\mu-1}{j} \sum_{l=0}^{j} (-1)^{l} \binom{j}{l} \binom{l\lambda}{i}. \tag{76}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t, x, \mu, \lambda)$ is

$$D^{\Delta}(n, k) = \frac{S_{n-k}(kx, k\mu, \lambda)}{(n-k)!}. \tag{77}$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^{-\mu} & & \\ 2^{-\mu-1}(2x - \lambda\mu) & 2^{-2\mu} & \\ 2^{-\mu-3}\left(4x^2 - (4\lambda\mu + 4)x + \lambda^2\mu^2 - \lambda^2\mu + 2\lambda\mu\right) & 2^{-2\mu}(2x - \lambda\mu) & 2^{-3\mu} \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t, x, \mu, \lambda)$ is

$$\overline{D}^{\Delta}(n, k) = \frac{k}{n} \frac{S_{n-k}(-nx, -n\mu, \lambda)}{(n-k)!}. \tag{78}$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^{\mu} & & \\ 2^{2\mu-1}(-2x + \lambda\mu) & 2^{2\mu} & \\ 2^{3\mu-3}\left(12x^2 + (4 - 12\lambda\mu)x + 3\lambda^2\mu^2 + \lambda^2\mu - 2\lambda\mu\right) & 2^{3\mu}(\lambda\mu - 2x) & 2^{3\mu} \end{array}$$

Also we can get the following new identities for the Peters polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{S_{n-m}(-nx, -n\mu, \lambda)}{(n-m)!} \frac{S_{m-k}(kx, k\mu, \lambda)}{(m-k)!} = \delta_{n,k} \tag{79}$$

and

$$\sum_{m=k}^{n} \frac{S_{n-m}(mx, m\mu, \lambda)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\mu, \lambda)}{(m-k)!} = \delta_{n,k}. \tag{80}$$

2.11 Gegenbauer polynomials

The Gegenbauer polynomials are defined by the following generating function [43]:

$$C(t,x,\alpha)=\left(1-2xt+t^2\right)^{-\alpha}=\left(\frac{1}{1-2xt+t^2}\right)^{\alpha}=\sum_{n\geq 0}C_n^{(\alpha)}(x)t^n, \tag{81}$$

where

$$C_n^{(\alpha)}(x)=\sum_{i=0}^{n}(-1)^{n-i}\binom{i}{n-i}\binom{i+\alpha-1}{i}(2x)^{2i-n}. \tag{82}$$

According to Eq. (13), the composita for the generating function $D(t)=tC(t,x,\alpha)$ is

$$D^{\Delta}(n,k)=C_{n-k}^{(k\alpha)}(x). \tag{83}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
2\alpha x & 1 & & \\
(2\alpha^2+2\alpha)x^2-\alpha & 4\alpha x & 1 & \\
\dfrac{(4\alpha^3+12\alpha^2+8\alpha)x^3-(6\alpha^2+6\alpha)x}{3} & (8\alpha^2+4\alpha)x^2-2\alpha & 6\alpha x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t)=tC(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k)=\frac{k}{n}C_{n-k}^{(-n\alpha)}(x). \tag{84}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-2\alpha x & 1 & & \\
(6\alpha^2-2\alpha)x^2+\alpha & -4\alpha x & 1 & \\
\dfrac{(64\alpha^3-48\alpha^2+8\alpha)x^3+(24\alpha^2-6\alpha)x}{3} & (16\alpha^2-4\alpha)x^2+2\alpha & -6\alpha x & 1
\end{array}$$

Also we can get the following new identities for the Gegenbauer polynomials:

$$\sum_{m=k}^{n}\frac{m}{n}C_{n-m}^{(-n\alpha)}(x)C_{m-k}^{(k\alpha)}(x)=\delta_{n,k} \tag{85}$$

and

$$\sum_{m=k}^{n}C_{n-m}^{(m\alpha)}(x)\frac{k}{m}C_{m-k}^{(-m\alpha)}(x)=\delta_{n,k}. \tag{86}$$

2.12 Meixner polynomials of the first kind

The Meixner polynomials of the first kind are defined by the following generating function [8, 44]:

$$M(t,x,\beta,c)=\left(1-\frac{t}{c}\right)^{x}(1-t)^{-x-\beta}=\left(\frac{c-t}{c(1-t)}\right)^{x}\left(\frac{1}{1-t}\right)^{\beta}=\sum_{n\geq 0}M_n(x,\beta,c)\frac{t^n}{n!}, \tag{87}$$

where

$$M_n(x,\beta,c)=(-1)^n n!\sum_{i=0}^{n}\binom{x}{i}\binom{-x-\beta}{n-i}c^{-i}. \tag{88}$$

According to Eq. (13), the composita for the generating function $D(t)=tM(t,x,\beta,c)$ is

$$D^{\Delta}(n,k)=\frac{M_{n-k}(kx,k\beta,c)}{(n-k)!}. \tag{89}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ \frac{(c-1)x+\beta c}{c} & 1 & \\ \frac{(c^2-2c+1)x^2+((2\beta+1)c^2-2\beta c-1)x+\left(\beta^2+\beta\right)c^2}{2c^2} & \frac{(2c-2)x+2\beta c}{c} & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t)=tM(t,x,\beta,c)$ is

$$\overline{D}^{\Delta}(n,k)=\frac{k}{n}\frac{M_{n-k}(-nx,-n\beta,c)}{(n-k)!}. \tag{90}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ \frac{(1-c)x-\beta c}{c} & 1 & \\ \frac{(3c^2-6c+3)x^2+((6\beta-1)c^2-6\beta c+1)x+\left(3\beta^2-\beta\right)c^2}{2c^2} & \frac{(2-2c)x-2\beta c}{c} & 1 \end{array}$$

Also we can get the following new identities for the Meixner polynomials of the first kind:

$$\sum_{m=k}^{n}\frac{m}{n}\frac{M_{n-m}(-nx,-n\beta,c)}{(n-m)!}\frac{M_{m-k}(kx,k\beta,c)}{(m-k)!}=\delta_{n,k} \tag{91}$$

and

$$\sum_{m=k}^{n}\frac{M_{n-m}(mx,m\beta,c)}{(n-m)!}\frac{k}{m}\frac{M_{m-k}(-mx,-m\beta,c)}{(m-k)!}=\delta_{n,k}. \tag{92}$$

3. Conclusions and future developments

In this chapter, we find new explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized

Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials that are defined by generating functions of the form $A(t, x, \alpha) = F(t)^x \cdot G(t)^\alpha$.

A lot of studies have recently showed that polynomials are a solution for practical problems related to modeling, quantum mechanics, and other areas. So a study of obtaining explicit formulas and representations of polynomials will be important and influential. Also the further research can be conducted to find practical means of obtained properties.

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