Chapter

A Probabilistic Interpretation of Nonlinear Integral Equations

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Abstract

We study a probabilistic interpretation of solutions to a class of nonlinear integral equations. By considering a branching model and defining a star-product, we construct a tree-based star-product functional as a probabilistic solution of the integral equation. Although the original integral equation has nothing to do with a stochastic world, some probabilistic technique enables us not only to relate the deterministic world with the stochastic one but also to interpret the equation as a random quantity. By studying mathematical structure of the constructed functional, we prove that the function given by expectation of the functional with respect to the law of a branching process satisfies the original integral equation.

Keywords: nonlinear integral equation, branching model, tree structure, star-product, probabilistic solution

AMS classification: Primary 45G10; Secondary 60 J80, 60 J85, 60 J57

1. Introduction

This chapter treats a topic on probabilistic representations of solutions to a certain class of deterministic nonlinear integral equations. Indeed, this is a short review article to introduce the star-product functional and a probabilistic construction of solutions to nonlinear integral equations treated in [1]. The principal parts for the existence and uniqueness of solutions are taken from [1] with slight modification. Since the nonlinear integral equations which we handle are deterministic, they have nothing to do with random world. Hence, we assume that an integral formula may hold, which plays an essential role in connecting a deterministic world with a random one. Once this relationship has been established, we begin with constructing a branching model and we are able to construct a star-product functional based upon the model. At the end we prove that the function provided by the expectation of the functional with respect to the law of a branching process in question solves the original integral equations (see also [2–4]).

More precisely, in this chapter we consider the deterministic nonlinear integral equation of the type:

$$\begin{aligned} e^{\lambda t|x|^2}u(t,x) = u_0(x) + \frac{\lambda}{2}\int_0^t ds\, e^{\lambda s|x|^2}\int p(s,x,y;u)n(x,y)\,dy \\ + \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2} f(s,x)\,ds. \end{aligned} \tag{1}$$

One of the reasons why we are interested in this kind of integral equations consists in its importance in applicatory fields, especially in mathematical physics. For instance, in quantum physics or applied mathematics, a variety of differential equations have been dealt with by many researchers (e.g., [5, 6]), and in most cases, their integral forms have been discussed more than their differential forms on a practical basis. There can be found plenty of integral equations similar to Eq. (1) appearing in mathematical physics.

The purpose of this article is to provide with a quite general method of giving a probabilistic interpretation to deterministic equations. Any deterministic representation of the solutions to Eq. (1) has not been known yet in analysis. The main contents of the study consist in derivation of the probabilistic representation of the solutions to Eq. (1). Our mathematical model is a kind of generalization of the integral equations that were treated in [7], and our kernel appearing in Eq. (1) is given in a more abstract setting. We are aiming at establishment of new probabilistic representations of the solutions.

This paper is organized as follows: In Section 2 we introduce notations which are used in what follows. In Section 3 principal results are stated, where we refer the probabilistic representation of the solutions to a class of deterministic nonlinear integral equations in question. Section 4 deals with branching model and its treelike structure. Section 5 treats construction of star-product functional based upon those tree structures of branching model described in the previous section. The proof of the main theorem will be stated in Sections 6 and 7. Section 6 provides with the proof of existence of the probabilistic solutions to the integral equations. We also consider *-product functional, which is a sister functional of the star-product functional. This newly presented functionals play an essential role in governing the behaviors of star-product functionals via control inequality. Section 7 deals with the proof of uniqueness for the constructed solutions, in terms of the martingale theory [8].

We think that it would not be enough to derive simply explicit representations of probabilistic solutions to the equations, but it is extremely important to make use of the formulae practically in the problem of computations. We hope that our result shall be a trigger to further development on the study in this direction.

2. Notations

Let $D_0 := \mathbb{R}^3\ \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, the symbol $\alpha \cdot \beta$ means the inner product, and we define $e_x := x/|x|$ for every $x \in D_0$. We consider the following deterministic nonlinear integral equation:

$$
\begin{aligned}
e^{\lambda t|x|^2} u(t,x) = u_0(x) &+ \frac{\lambda}{2}\int_0^t \mathrm{d}s\, e^{\lambda s|x|^2} \int p(s,x,y;u) n(x,y) \mathrm{d}y \\
&+ \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2} f(s,x) s, \quad \text{for} \quad \forall (t,x) \in \mathbb{R}_+ \times D_0.
\end{aligned}
\tag{2}
$$

Here, $u \equiv u(t,x)$ is an unknown function: $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \to \mathbb{C}^3$ are the initial data such that $u(t,x)|_{t=0} = u_0(x)$. Moreover, $f(t,x)$: $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$ for each $x \in D_0$. The integrand p in Eq. (2) is actually given by

$$
p(t,x,y;u) = u(t,y) \cdot e_x\{u(t,x-y) - e_x(u(t,x-y) \cdot e_x)\}. \tag{3}
$$

Suppose that the integral kernel $n(x,y)$ is bounded and measurable with respect to $x \times y$. On the other hand, we consider a Markov kernel $K: D_0 \to D_0 \times D_0$. Namely, for every $z \in D_0$, $K_z(x,y)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel k is given by $k(x,y) = i|x|^{-2}n(x,y)$, then we define K_z as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$\iint h(x,y)K_z(x,y) = \int h(x,z-x)k(x,z)\mathrm{d}x. \tag{4}$$

Moreover, we assume that for every measurable functions $f, g > 0$ on $\mathbb{R}^+$,

$$\int h(|z|)\nu(z) \int g(|x|)K_z(x,y) = \int g(|z|)\nu(z) \int h(|y|)K_z(\mathrm{d}x, \mathrm{d}y) \tag{5}$$

holds, where the measure ν is given by $\nu(dz) = |z|^{-3}dz$.

The equality (Eq. (4)) is not only a simple integral transform formula. In fact, in the analytical point of view, it merely says that the double integral with respect to K_z is changed into a single integral with respect to x just after the execution of iterative integration of $h(x,y)$ with respect to the second parameter y. However, our point here consists in establishing a great bridge between a deterministic world and a stochastic world. The validity of the assumed equality (Eq. (5)) means that a sort of symmetry in a wide sense may be posed on our kernel K.

3. Main results

In this section we shall introduce our main results, which assert the existence and uniqueness of solutions to the nonlinear integral equation. That is to say, we derive a probabilistic representation of the solutions to Eq. (2) by employing the star-product functional. As a matter of fact, the solution $u(t,x)$ can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of the below-mentioned branching particle systems and branching models. Let

$$M_\star^{\langle u_0,f\rangle}(\omega) = \prod \star_{[x\tilde{m}]}\Xi_{m_2.m_3}^{m_1}[u_0,f](\omega), \tag{6}$$

be a probabilistic representation in terms of tree-based star-product functional with weight (u_0,f) (see Section 5 for the details of the definition). On the other hand, $M_*^{\langle U,F\rangle}(\omega)$ denotes the associated *-product functional with weight (U,F), which is indexed by the nodes (x_m) of a binary tree. Here, we suppose that $U = U(x)$ (resp. $F = F(t,x)$) is a nonnegative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$), respectively, and also that $F(\cdot,x) \in L^1(\mathbb{R}_+)$ for each x. Indeed, in construction of the *-product functional, the product in question is taken as ordinary multiplication * instead of the star-product $\bigstar$ in the definition of star-product functional.

Theorem 1. *Suppose that* $|u_0(x)| \leqslant U(x)$ *for every* x *and* $|\tilde{f}(t,x)| \leqslant F(t,x)$ *for every* t, x *and also that for some* $T > 0$ ($T > > 1$, *sufficiently large*)

$$E_{T,x}\left[M_*^{\langle U,F\rangle}(\omega)\right] < \infty, \quad \text{a.e.} - x \tag{7}$$

holds. Then, there exists a (u_0, f)*-weighted tree-based star* $\bigstar$*-product functional* $M_{\bigstar}^{\langle u_0, f\rangle}(\omega)$*, indexed by a set of node labels accordingly to the tree structure which a binary critical branching process* $Z^{K_x}(t)$ *determines. Furthermore, the function*

$$u(t,x) = E_{t,x}\left[M_{\bigstar}^{\langle u_0, f\rangle}(\omega)\right] \tag{8}$$

gives a unique solution to the integral equation (Eq. (2))*. Here,* $E_{t,x}$ *denotes the expectation with respect to a probability measure* $P_{t,x}$ *as the time-reversed law of* $Z^{K_x}(t)$*.*

4. Branching model and its associated treelike structure

In this section we consider a continuous time binary critical branching process $Z^{K_x}(t)$ on D_0 [9], whose branching rate is given by a parameter $\lambda|x|^2$, whose branching mechanism is binary with equiprobability, and whose descendant branching particle behavior is determined by the kernel K_x (cf. [10]). Next, taking notice of the tree structure which the process $Z^{K_x}(t)$ determines, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \tag{9}$$

by Ω (see [11]). We also consider the time-reversed law of $Z^{K_x}(t)$ being a probability measure on Ω as $P_{t,x} \in \mathcal{P}(\Omega)$. Here, t denotes the birth time of common ancestor, and the particle x_m dies when $\eta_m = 0$, while it generates two descendants x_{m1}, x_{m2} when $\eta_m = 1$. On the other hand,

$$\mathcal{V} = \bigcup_{\ell \geq 0} \{1, 2\}^{\ell}$$

is a set of all labels, namely, finite sequences of symbols with length ℓ, which describe the whole tree structure given [12]. For $\omega \in \Omega$ we denote by $\mathcal{N}(\omega)$ the totality of nodes being the branching points of tree; let $N_+(\omega)$ be the set of all nodes m being a member of $\mathcal{V} \setminus \mathcal{N}(\omega)$, whose direct predecessor lies in $\mathcal{N}(\omega)$ and which satisfies the condition $t_m(\omega) > 0$, and let $N_-(\omega)$ be the same set as described above but satisfying $t_m(\omega) \leqslant 0$. Finally, we put

$$N(\omega) = N_+(\omega) \cup N_-(\omega). \tag{10}$$

5. Star-product functional

This section treats a tree-based star-product functional. First of all, we denote by the symbol $\mathrm{Proj}^z(\cdot)$ a projection of the objective element onto its orthogonal part of the z component in $\mathbb{C}^3$, and we define a $\bigstar$-product of β, γ for $z \in D_0$ as

$$\beta \bigstar_{[z]} \gamma = -i(\beta \cdot e_z)\mathrm{Proj}^z(\gamma). \tag{11}$$

Notice that this product $\bigstar$ is noncommutative. This property will be the key point in defining the star-product functional below, especially as far as the uniqueness of functional is concerned. We shall define $\Theta^m(\omega)$ for each $\omega \in \Omega$ realized as follows. When $m \in N_+(\omega)$, then $\Theta^m(\omega) = \tilde{f}(t_m(\omega), x_m(\omega))$, while $\Theta^m(\omega) = u_0(x_m(\omega))$ if $m \in N_-(\omega)$. Then, we define

$$\Xi^{m_1}_{m_2.m_3}(\omega) \equiv \Xi^{m_1}_{m_2,\, m_3}[u_0,f](\omega) \coloneqq \Theta^{m_2}(\omega)\bigstar_{[x_{m_1}]}\Theta^{m_3}(\omega), \tag{12}$$

whereas for the product order in the star-product $\bigstar$, when we write $m \prec m'$ lexicographically with respect to the natural order $\prec$, the term Θ^m labeled by m necessarily occupies the left-hand side, and the other $\Theta^{m'}$ labeled by m' occupies the right-hand side by all means. And besides, as abuse of notation, we write

$$\Xi^{\varnothing}_{m,\,\varnothing}(\omega) \equiv \Xi^{\varnothing}_{m,\,\varnothing}[u_0,f](\omega) \coloneqq \Theta^m(\omega), \tag{13}$$

especially when $m \in \mathcal{V}$ is a label of single terminal point in the restricted tree structure in question.

Under these circumstances, we consider a random quantity which is obtained by executing the star-product $\bigstar$ inductively at each node in $\mathcal{N}(\omega)$, we call it a tree-based $\bigstar$-product functional, and we express it symbolically as

$$M_{\bigstar}{}^{\langle u_0 f\rangle}(\omega) = \Pi\bigstar_{[x_{\tilde{m}}]}\Xi^{m_1}_{m_2\cdot m_3}[u_0,f](\omega), \tag{14}$$

where $m_1 \in \mathcal{N}(\omega)$ and $m_2, m_3 \in N(\omega)$, and by the symbol $\prod\bigstar$ (as a product relative to the star-product), we mean that the star-products $\bigstar$'s should be succeedingly executed in a lexicographical manner with respect to $x\tilde{m}$ such that $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$ when $|m_1| = \ell$. At any rate it is of the extreme importance that once a branching pattern $\omega(\in\Omega)$ is realized, its tree structure is uniquely determined, and there can be found the *unique* explicit representation of the corresponding star-product functional $M_{\bigstar}^{\langle u_0 f\rangle}(\omega)$.

Example 2. Let us consider a typical realization $\omega \in \Omega$. Suppose that we have $\mathcal{N}(\omega_2) = \{\phi, 1, 2, 11, 12, 22\}$, $N_+(\omega_2) = \{21, 112, 221\}$, and $N_-(\omega_2) = \{111, 121, 122, 222\}$. This case is nothing but an all-the-members participating type of game. For the case of particle located at x_{111} and x_{112} (with nodes of the level $|m| = \ell = 3$) with its pivoting node x_{11}, we have

$$\begin{aligned}\Xi^{11}_{111,112}(\omega_2) &= \Theta^{111}(\omega_2)\bigstar_{[x_{11}]}\Theta^{112}(\omega_2)\\ &= u_0(x_{111}(\omega_2))\bigstar_{[x_{11}]}\tilde{f}(t_{112}(\omega_2), x_{112}(\omega_2)).\end{aligned}$$

Similarly, for the pair of particles x_{121} and x_{122}, we have

$$\begin{aligned}\Xi^{12}_{121,122}(\omega_2) &= \Theta^{121}(\omega_2)\bigstar_{[x_{12}]}\Theta^{122}(\omega_2)\\ &= u_0(x_{121}(\omega_2))\bigstar_{[x_{12}]}u_0(x_{122}(\omega_2)).\end{aligned}$$

For the pair of particles x_{221} and x_{222}, we also have

$$\begin{aligned}\Xi^{22}_{221,222}(\omega_2) &= \Theta^{221}(\omega_2)\bigstar_{[x_{22}]}\Theta^{222}(\omega_2)\\ &= \tilde{f}(t_{221}(\omega_2), x_{221}(\omega_2))\bigstar_{[x_{22}]}u_0(x_{222}(\omega_2)).\end{aligned}$$

Next, when we take a look at the groups of particles with nodes of the level $|m| = \ell = 2$. For instance, as to a pair of particles located at x_{11} and x_{12} with its pivoting node x_1, we get an expression

$$\begin{aligned}\Xi^{1}_{11,\,12}(\omega_2) &= \Theta^{11}(\omega_2)\bigstar_{[x_1]}\Theta^{12}(\omega_2)\\ &= \Xi^{11}_{111,112}(\omega_2)\bigstar_{[x_1]}\Xi^{12}_{121,122}(\omega_2)\\ &= \Big(u_0(x_{111})\bigstar_{[x_{11}]}\tilde{f}(t_{112}, x_{112})\Big)\bigstar_{[x_1]}\big(u_0(x_{121})\bigstar_{[x_{12}]}u_0(x_{122})\big).\end{aligned}$$

Therefore, it follows by a similar argument that the explicit representation of star-product functional for ω_2 is given by

$$M_{\bigstar}^{\langle u_0 f\rangle}(\omega_2) = \Big\{ \Big(u_0(x_{111}) \bigstar_{[x_{11}]} \tilde{f}(t_{112}, x_{112}) \Big) \bigstar_{[x_1]} \big(u_0(x_{121}) \bigstar_{[x_{12}]} u_0(x_{122}) \big) \Big\}$$
$$\bigstar_{[x_\phi]} \Big\{ \tilde{f}(t_{21}, x_{21}) \bigstar_{[x_2]} \big(u_0(x_{221}) \bigstar_{[x_{22}]} u_0(x_{222}) \big) \Big\}$$

6. The *-product functional and existence

In this section we first begin with constructing a (U, F)-weighted tree-based *-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes (x_m) of a binary tree. Recall that $U = U(x)$ (resp. $F = F(t,x)$) is a nonnegative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$), respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each x. Moreover, in construction of the functional, the product is taken as ordinary multiplication * instead of the star-product $\bigstar$.

In what follows we shall give an outline of the existence in Theorem 1. We need the following lemma, which is essentially important for the proof.

Lemma 3. *For* $0 \leqslant t \leqslant T$ *and* $x \in D_0$, *the function* $V(t,x) = E_{t,x}\big[M_*^{\langle U,F\rangle}(\omega)\big]$ *satisfies*

$$e^{\lambda t|x|^2} V(t,x) = U(x) + \int_0^t \mathrm{d}s \frac{|x|^2}{2} e^{\lambda s|x|^2} \left\{ F(s,x) + \int V(s,y)V(s,z)K_x(\mathrm{d}y, z) \right\}. \quad (15)$$

Proof of Lemma 3. By making use of the conditional expectation, we may decompose $V(t,x)$ as follows:

$$\begin{aligned} V(t,x) &= E_{t,x}\big[M_*^{U,F}(\omega)\big] \\ &= E_{t,x}\big[M_*^{\langle U,F\rangle}(\omega),\ t_\phi \leqslant 0\big] + E_{t,x}\big[M_*^{\langle U,F\rangle}(\omega), t_\phi > 0\big] \\ &= E_{t,x}\big[M_*^{U,F}(\omega),\ t_\phi \leqslant 0\big] + E_{t,x}\big[M_*^{\langle U,F\rangle}(\omega), t_\phi > 0, \eta_\phi = 0\big] \\ &\quad + E_{t,x}\big[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0, \eta_\phi = 1\big]. \end{aligned} \quad (16)$$

We are next going to take into consideration an equivalence between the events $t_\phi \leqslant 0$ and $T \notin [0,t]$. Indeed, as to the first term in the third line of Eq. (16), since the condition $t_\phi \leqslant 0$ implies that T never lies in an interval $[0,t]$, and since $m = \phi \in N_-(\omega)$ leads to a nonrandom expression

$$M_* = \Theta^\phi = U(x),$$

the tree-based *-product functional is allowed to possess a simple representation:

$$\begin{aligned} E_{t,x}\big[M_*^{\langle U,F\rangle}, t_\phi \leqslant 0\big] &= E_{t,x}\Big[M_*^{\langle U,F\rangle} \cdot 1_{\{t_\phi \leqslant \leqslant 0\}}\Big] = U(x) \cdot P_{t,x}\big(t_\phi \leqslant 0\big) \\ &= U(x) \cdot P(T \notin [0,t]) = U(x) \cdot P(T \in (t, \infty)) \\ &= U(x) \int_t^\infty f_T(s)\mathrm{d}s = U(x) \int_t^\infty \lambda |x|^2 e^{-\lambda s|x|^2} \mathrm{d}s \\ &= U(x) \cdot \exp\left\{-\lambda t|x|^2\right\}. \end{aligned} \quad (17)$$

As to the third term, we need to note the following matters. A particle generates two offsprings or descendants x_1, x_2 with probability $\frac{1}{2}$ under the condition $\eta_\phi = 1$; since $t_\phi > 0$, when the branching occurs at $t_\phi = s$, then, under the conditioning operation at t_ϕ, the Markov property [13] guarantees that the lower tree structure below the first-generation branching node point x_1 is independent to that below the location x_2 with realized $\omega \in \Omega$; hence, a tree-based *-product functional branched after time s is also probabilistically independent of the other tree-based *-product functional branched after time s, and besides the distributions of x_1 and x_2 are totally controlled by the Markov kernel K_x. Therefore, an easy computation provides with an impressive expression:

$$E_{t,x}\left[M_*^{\langle U,F\rangle}, t_\phi > 0, \eta_\phi = 1\right] = \frac{1}{2}\int_0^t \mathrm{d}s\, \lambda|x|^2 e^{-\lambda|x|^2(t-s)}. \\ \times \iint E_{s,x_1}[M_*]\cdot E_{s,x_2}[M_*]K_x(\mathrm{d}x_1, \mathrm{d}x_2).$$

Note that as for the second term, it goes almost similarly as the computation of the above-mentioned third one. Finally, summing up we obtain

$$\begin{aligned} V(t,x) &= E_{t,x}\left[M_*^{\langle U,F\rangle}(\omega)\right] \\ &= U(x)r^{-\lambda t|x|^2} + \int_0^t \frac{\lambda|x|^2}{2} e^{-\lambda|x|^2(t-s)} F(s,x)\,\mathrm{d}s \\ &\quad + \int_0^t \frac{\lambda|x|^2}{2} e^{-\lambda|x|^2(t-s)} \iint V(s,y)V(s,z)K_x(dy,dz)ds. \end{aligned} \tag{18}$$

On this account, if we multiply both sides of Eq. (18) by $\exp\left\{\lambda t|x|^2\right\}$, then the required expression Eq. (15) in Lemma 3 can be derived, which completes the proof. □

By a glance at the expression Eq. (15) obtained in Lemma 3, it is quite obvious that, for each $x \in D_0$, the mapping $[0,T]\ni t \mapsto e^{\lambda|x|^2 t} V(t,x) \in \overline{\mathbb{R}}_+$ is a nondecreasing function. Taking the above fact into consideration, we can deduce with ease that

$$E_{t,x}\left[M_*^{\langle U,F\rangle}(\omega)\right] < \infty \tag{19}$$

holds for $\forall t \in [0,T]$ and $x \in E_c$, where the measurable set E_c denotes the totality of all the elements x in D_0 such that $E_{T,x}\left[M_*^{\langle U,F\rangle}\right] < \infty$ holds for a.e.-x, namely, it is the same condition Eq. (7) appearing in the assertion of Theorem 1. Another important aspect for the proof consists in establishment of the M_*-control inequality, which is a basic property of the star-product ★. That is to say, we have.

Lemma 4. (M_*-control inequality) *The following inequality*

$$|M_\star^{\langle u_0,f\rangle}(\omega)| \leqslant M_*^{\langle U,F\rangle}(\omega). \tag{20}$$

holds $P_{t,x}$-a.s.

This inequality enables us to govern the behavior of the star-product functional with a very complicated structure by that of the *-product functional with a rather simplified structure. In fact, the M_*-control inequality yields immediately from a simple fact:

$$|w \star_{[x]} v| \leqslant |w|\cdot|v| \quad \text{for every} \quad w, v \in \mathbb{C}^3 \quad \text{and every} \quad x \in D_0.$$

Next, we are going to derive the space of solutions to Eq. (2). If we define

$$u(t,x) := \begin{cases} E_{t,x}\left[M_\star^{\langle u_0,f\rangle}(\omega)\right], & \text{on} \quad E_c, \\ 0, & \text{otherwise}, \end{cases}$$

then $u(t,x)$ is well defined on the whole space D_0 under the assumptions of the main theorem (Theorem 1). Moreover, it follows from the M_*-control inequality (Eq. (20)) that

$$|u(t,x)| \leqslant V(t,x) \quad \text{on} \quad [0,T]\times D_0. \tag{21}$$

On this account, from Eq. (15) in Lemma 3, by finiteness of the expectation of tree-based *-product functional $M_*^{\langle U,F\rangle}(\omega)$, by the M_*-control inequality, and from Eq. (21), it is easy to see that

$$\int_0^T \mathrm{d}s \int |u(s,y)|\cdot|u(s,z)|K_x(\mathrm{d}y,\mathrm{d}z) < \infty \qquad \text{for} \quad x \in E_c. \tag{22}$$

Hence, taking Eq. (22) into consideration, we define the space $\mathcal{D}$ of solutions to Eq. (2) as follows:

$$\begin{aligned}\mathcal{D} := \{&\varphi : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3;\ \varphi \quad \text{is continuous in} \quad t \\ &\text{and measurable such that} \\ &\textstyle\int_0^\infty \mathrm{d}s \int e^{\lambda|x|^2 s}|\varphi(s,y)|\cdot|(s,z)|K_x(\mathrm{d}y,\mathrm{d}z) < \infty \\ &\qquad\qquad\qquad \text{holds a.e.} - x\}\end{aligned} \tag{23}$$

By employing the Markov property [13] with respect to time t_ϕ and by a similar technique as in the proof of Lemma 3, we may proceed in rewriting and calculating the expectation for $\forall t > 0$ and $x \in E_c$:

$$\begin{aligned} u(t,x) &= E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right] \\ &= E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega), t_\phi \leqslant 0\right] + E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) t_\phi > 0\right] \\ &= E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega), t_\phi \leqslant 0\right] + E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) t_\phi > 0, \eta_\phi = 0\right] \\ &\quad + E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) t_\phi > 0\ \eta_\phi = 1\right] \\ &= e^{-t|x|^2}u_0(x) + \textstyle\int_0^t s\ |x|^2 e^{-(t-s)|x|^2} \\ &\quad \times \frac{1}{2}\left\{\tilde{f}(s,x) + \iint E_{s,x_1}[M_\star]\star_{[x]}E_{s,x_2}[M_\star]K_x(\mathrm{d}x_1,\mathrm{d}x_2)\right\}. \end{aligned} \tag{24}$$

Furthermore, we may apply the integral equality Eq. (4) in the assumption on the Markov kernel for Eq. (24) to obtain

$$\begin{aligned} E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right] &= e^{-\lambda t|x|^2}u_0(x) + \int_0^t \mathrm{d}s\,\lambda\,|x|^2 e^{-\lambda(t-s)|x|^2} \\ &\quad \times \frac{1}{2}\left\{\tilde{f}(s,x) + \iint E_{s,x_1}[M_\star]\star_{[x]}E_{s,x_2}[M_\star]K_x(\mathrm{d}x_1,\mathrm{d}x_2)\right\} \\ &= e^{-\lambda t|x|^2}u_0(x) + \int_0^t \mathrm{d}s\,\lambda\,|x|^2 e^{-\lambda(t-s)|x|^2} \\ &\quad \times \frac{1}{2}\left\{\tilde{f}(s,x) + \iint u(s,y)\star_{[x]}u(s,z)K_x(\mathrm{d}y,\mathrm{d}z)\right\} \\ &= e^{-\lambda t|x|^2}\left\{u_0(x) + \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2}f(s,x)\mathrm{d}s\ + \frac{\lambda}{2}\int_0^t \mathrm{d}s\int e^{\lambda s|x|^2}p(s,x,y;u)n(x,y)\mathrm{d}y\right\}, \end{aligned} \tag{25}$$

because in the above last equality we need to rewrite its double integral relative to the space parameters into a single integral. Finally, we attain that $u(t,x) = E_{t,x}\left[M_\star^{\langle u_0,f\rangle}(\omega)\right]$ satisfies the integral equation Eq. (2), and this $u(t,x)$ is a solution lying in the space $\mathcal{D}$. This completes the proof of the existence.

7. Uniqueness

First of all, note that we can choose a proper measurable subset $F_0 \subset D_0$ with $m(F_0^c) = m(D_0 \setminus F_0) = 0$ (meaning that its complement F_0^c is a null set with respect to Lebesgue measure $m(x)$), such that

$$E_{t,x}\left[M_*^{\langle U,F\rangle}(\omega)\right] < \infty \quad \text{on} \quad F_0 \tag{26}$$

and

$$\int_0^T \mathrm{d}s \iint e^{\lambda|x|^2 s}|u(s,y)| \cdot |u(s,z)| K_x(\mathrm{d}y, \mathrm{d}z), \quad \text{for} \quad \forall T > 0,$$

is convergent for a.e.-x ($\in F_0$), and $u(t,x)$ satisfies the nonlinear integral equation (Eq. (2)) for a.e.-$x \in F_0$. Suggested by the argument in [7], we adopt here a martingale method in order to prove the uniqueness of the solutions to Eq. (2). The leading philosophy for the proof of uniqueness consists in extraction of the martingale part from the realized tree structure and in representation of the solution u in terms of martingale language. In so doing, we need to construct a martingale term from the given functional and to settle down the required σ-algebra with respect to which its constructed term may become a martingale. Let Ω_+ be the set of all the elements ω's corresponding to time $t_m(\omega) > 0$ for the label m. Next, we consider a kind of the notion like n-section of the set of labels for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define several families of Ω in what follows, in order to facilitate the extraction of its martingale part from our star-product functional $M_\star^{\langle u_0,f\rangle}(\omega)$. For each realized tree ω, $\tilde{\mathcal{N}}_n(\omega)$ is the totality of the labels

$$m \in \bigcup_{0 \leqslant \ell \leqslant n} \{1,2\}^\ell$$

satisfying $t_m(\omega) > 0$ and $\eta_m(\omega) = 1$. Namely, this family $\tilde{\mathcal{N}}_n(\omega)$ is a subset of labels restricted up to the nth generation and limited to the nodes related to branching at positive time. Moreover, let $\tilde{N}_n(\omega)$ be the set of labels lying in $\mathcal{N} \setminus \tilde{\mathcal{N}}_n(\omega)$ whose direct predecessor belongs to $\tilde{\mathcal{N}}_n(\omega)$. By convention, we define $\tilde{N}_n(\omega) = \{\varnothing\}$ if $\tilde{\mathcal{N}}_n(\omega) = \varnothing$. We shall introduce a new family $\tilde{N}_n^{cut}(\omega)$ of cutoff labels, which is determined by the set of labels $m \in \mathcal{V}$ whose direct predecessor belongs to $\tilde{\mathcal{N}}_n(\omega)$ and has length $|m| = n$, and we call this family $\tilde{N}_n^{cut}(\omega)$ the cutoff part of $\tilde{N}_n(\omega)$, while $\breve{N}_n nct(\omega)$ is the non-cutoff part of $\tilde{N}_n(\omega)$, which is defined by

$$\breve{N}_n nct(\omega) := \tilde{N}_n(\omega) \setminus \tilde{N}_n^{cut}(\omega). \tag{27}$$

We are now in a position to introduce a new class $M_\star^{n,\langle u_0,f,u\rangle}(\omega)$ of $\star$-product functional, which should be called the n-section of the star-product functional. In fact, by taking the above argument in Example 2 into account, we can define its

n-section as follows. In fact, if the label m is a member of the cutoff family $\tilde{N}_n^{cut}(\omega)$, the input data of the functional attached to m is given by $u\left(t_{p(m)}(\omega), x_m(\omega)\right)$ instead of the usual initial data $u_0(x_m(\omega))$ or $\tilde{f}(t_m(\omega), x_m(\omega))$, where $p(m)$ indicates the direct ancestor m' of m having length n. On the other hand, if m lies in the non-cutoff family $\breve{N}_n nct(\omega)$, then the input data of the functional attached to m is completely the same as before with no change, that is, we use $u_0(x_m)$ if $t_m \leqslant 0$ and use $\tilde{f}(t_m, x_m)$ if $t_m > 0$. In such a way, we can construct a new $\bigstar$-product functional $M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)$ by the almost sure procedure, and we call it the n-section $\bigstar$-product functional. Similarly, we can also define the corresponding n-section $\bigstar$-product functional $M_*^{n, \langle U,F,V\rangle}(\omega)$. Simply enough, to get the *-product counterpart, we have only to replace those functions $u_0, \tilde{f}$ and u by U, F and V in the definition of $\bigstar$-product functional. As easily imagined, we can also derive an n-section version of M_*^n-control inequality:

Lemma 5. (M_*^n-control inequality) *The following inequality*

$$|M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)| \leqslant M_*^{n, \langle U,F,V\rangle}(\omega) \tag{28}$$

holds $P_{t,x}$-a.s.

because of the domination property: $|u(t,x)| \leqslant V(t,x)$ for $[0,T] \times D_0$, $|u_0(x)| \leqslant U(x)$ for $\forall x$, $|\tilde{f}(t,x)| \leqslant F(t,x)$ for $\forall t, x$, and a simple inequality $|w \bigstar_{[x]} v| \leqslant |w| \cdot |v|$ for $\forall w, v \in \mathbb{C}^3$ and $\forall x \in D_0$.

Let us now introduce a filtration $\{\mathcal{F}_n\}$ for $n \in \mathbb{N}_0$ on Ω_+, according to the discussion in Example 2. As a matter of fact, we define

$$\mathcal{F}_n := \sigma\left(\tilde{\mathcal{N}}_n(\omega); (t_m, x_m), m \in \tilde{\mathcal{N}}_n(\omega) \cup \breve{N}_n^{nct}(\omega); (\eta_m), m \in \tilde{N}_n^{cut}(\omega)\right) \tag{29}$$

for each $n \in \mathbb{N}_0$. Notice that $\tilde{\mathcal{N}}_n(\omega)$ itself determines the other two families $\tilde{N}_n^{cut}(\omega)$ and $\breve{N}_n nct(\omega)$. Then, it is readily observed that both functionals $M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)$ and $M_*^{n, \langle U,F,V\rangle}(\omega)$ are $\mathcal{F}_n$-adapted.

Lemma 6. *For each* $n \in \mathbb{N}_0$, *the equality*

$$M_*^{n, \langle U,F,V\rangle}(\omega) = E_{t,x}\left[M_*^{\langle U,F\rangle}(\omega) | \mathcal{F}_n\right] \tag{30}$$

holds $P_{t,x}$-a.s. *for every* $t \in [0,T]$ *and every* $x \in F_0$.

Proof. By its construction, we can conclude the equality of Eq. (30) from the strong Markov property [13] applied at times (t_m)s for $m \in \mathcal{V}$ of length n on the set $\left\{m \in \tilde{\mathcal{N}}_n(\omega)\right\} \in \mathcal{F}_n$. □

Moreover, an application of Lemma 6 with the n-section M_*^n-control inequality (Eq. (28)) shows the $P_{t,x}$-integrability of $M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)$ for every $t \in [0,T]$ and every $x \in F_0$. Actually, it proves to be true that a martingale part, in question, extracted by the star-product functional relative to those n-section families, is given by the n-section $\bigstar$-product functional $M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)$.

Lemma 7. The n-section $M_{\bigstar}^{n, \langle u_0 f, u\rangle}(\omega)$ of $\bigstar$-product functional with weight functions u_0 and f is an $\{\mathcal{F}_n\}$-martingale [8].

Proof. When we set $= E_{t,x}\left[M_{\bigstar}^n(\omega) | \mathcal{F}_n\right]$, then ξ_n turns out to be a $\{\mathcal{F}_n\}$-martingale, since

$$E_{t,x}[\xi_n | \mathcal{F}_{n-1}] = E_{t,x}\left[E_{t,x}\left[M_{\bigstar}^n | \mathcal{F}_n\right] | \mathcal{F}_{n-1}\right] = E_{t,x}\left[M_{\bigstar}^n\big|_{n-1}\right] = \xi_{n-1}$$

by virtue of the inclusion property of the σ-algebras. Consequently, it suffices to show that

$$E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)|\mathcal{F}_n\right] = M_\star^{n,\langle u_0 f,u\rangle}(\omega) \tag{31}$$

holds a.s. By employing the representation formula (Eq. (8)), an conditioning argument leads to Eq. (31), because the establishment is verified by the Markov property applied at t_m and on the event $\left\{m \in \tilde{\mathcal{N}}_n\right\}$ being $\mathcal{F}_n$-measurable. □

Finally, the uniqueness yields from the following assertion.

Proposition 8. *When* $u(t,x)$ *is a solution to the nonlinear integral equation* (Eq. (2)), *then we have*

$$u(t,x) = E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right] \tag{32}$$

holds for every $t \in [0,T]$ *and for* a.e.-x.

Proof. Our proof is technically due to a martingale method. We need the following lemma.

Lemma 9. *Let* $M_\star^{n,\langle u_0 f,u\rangle}(\omega)$ *be the n-section of* $\star$*-product functional, and let* $u(t,x)$ *be a solution of the nonlinear integral equation* (Eq. (2)). *Then, we have the following identity: for each* $n \in \mathbb{N}_0$

$$u(t,x) = E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega)\right] \tag{33}$$

holds for every t $(0 \leqslant t \leqslant T)$ *and every* $x \in F_0$.

Proof of Lemma 9. Recall that $M_\star^{n,\langle u_0 f,u\rangle}(\omega)$ is a martingale relative to $\{\mathcal{F}_n\}$. For $n = 0$, it follows from the identity (Eq. (31)) and by the martingale property that

$$\begin{aligned} E_{t,x}\left[M_\star^{0,\langle u_0 f,u\rangle}(\omega)\right] &= E_{t,x}\left[E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)|\mathcal{F}_0\right]\right] \\ &= E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right] = u(t,x). \end{aligned} \tag{34}$$

Next, for the case $n = 1$, by the same reason, we can get

$$\begin{aligned} E_{t,x}\left[M_\star^{1,\langle u_0 f,u\rangle}(\omega)\right] &= E_{t,x}\left[E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)|\mathcal{F}_1\right]\right] \\ &= E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right] = u(t,x). \end{aligned} \tag{35}$$

We resort to the mathematical induction with respect to $n \in \mathbb{N}_0$. If we assume the identity (Eq. (33)) for the case of n, then the case of $n+1$ reads at once

$$\begin{aligned} E_{t,x}\left[M_\star^{n+1,\langle u_0 f,u\rangle}(\omega)\right] &= E_{t,x}\left[E_{t,x}\left[M_\star^{n+1,\langle u_0 f,u\rangle}(\omega)|\mathcal{F}_n\right]\right] \\ &= E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega)\right] = u(t,x), \end{aligned} \tag{36}$$

where we made use of the martingale property in the first equality and employed the hypothesis of induction in the last identity. This concludes the assertion. □

To go back to the proof of Proposition 8. We define an $\mathcal{F}_n$-measurable event A_n as the set of $\omega \in \Omega_+$ such that $\tilde{\mathcal{N}}_n(\omega)$ contains some label m of length n. From the definition, it holds immediately that

$$M_\star^{\langle u_0 f\rangle}(\omega) = M_\star^{n,\langle u_0 f,u\rangle}(\omega) \quad \text{on} \quad \Omega_+ \; A_n. \tag{37}$$

Hence, for every $x \in F_0$ and $0 \leqslant t \leqslant T$ and $\forall n \in \mathbb{N}_0$, we may apply Lemma 9 for the expression below with the identity (Eq. (31)) to obtain

$$\begin{aligned}
&|u(t,x) - E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right]| \\
&= |E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega)\right] - E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right]| \\
&\leqslant |E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega) - M_\star^{\langle u_0 f\rangle}(\omega); A_n\right]| \\
&+ |E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega) - M_\star^{\langle u_0 f\rangle}(\omega); A_n^c\right]| \\
&= |E_{t,x}\left[\left(M_\star^{n,\langle u_0 f,u\rangle}(\omega) - M_\star^{\langle u_0 f\rangle}(\omega)\right) \cdot 1_{A_n}\right]|
\end{aligned} \tag{38}$$

where the symbol $E_{t,x}[X(\omega); A]$ denotes the integral of $X(\omega)$ over a measurable event A with respect to the probability measure $P_{t,x}(d\omega)$, namely,

$$E_{t,x}[X(\omega); A] = E_{t,x}[X(\omega) \cdot 1_A] = \int_A X(\omega) P_{t,x}(d\omega).$$

Furthermore, we continue computing

$$\begin{aligned}
(38) &\leqslant |E_{t,x}\left[M_\star^{n,\langle u_0 f,u\rangle}(\omega) 1_{A_n}\right]| + |E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}\right]| \\
&= |E_{t,x}\left[E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) | \mathcal{F}_n\right] 1_{A_n}\right]| + |E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}\right]| \\
&= 2|E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}\right]|.
\end{aligned} \tag{39}$$

Since $\cap_n A_n = \varnothing$ by the binary critical tree structure [12], and since we have an natural estimate

$$\begin{aligned}
&|M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}(\omega)| < M_\star^{\langle U,F\rangle}(\omega), \quad \text{a.s.} \\
&\text{and} \quad \lim_{n\to\infty} M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}(\omega) = 0, \quad \text{a.s.}
\end{aligned} \tag{40}$$

it follows by the bounded convergence theorem of Lebesgue that

$$\lim_{n\to\infty} |E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega) 1_{A_n}\right]| = 0. \tag{41}$$

Consequently, from Eq. (39) and Eq. (41), we readily obtain

$$|u(t,x) - E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right]| \to 0 \quad (\text{as} \quad n \to \infty) \tag{42}$$

holds for every $(t,x) \in [0,T] \times F_0$. Thus, we attain that $u(t,x) = E_{t,x}\left[M_\star^{\langle u_0 f\rangle}(\omega)\right]$, a.e.-$x \in F_0$. This finishes the proof of Proposition 8. $\square$

Concurrently, this completes the proof of the uniqueness.

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