

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



On Conformal Anti-Invariant Submersions Whose Total Manifolds Are Locally Product Riemannian

Mehmet Akif Akyol

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/intechopen.80337>

Abstract

The aim of this chapter is to study conformal anti-invariant submersions from almost product Riemannian manifolds onto Riemannian manifolds as a generalization of anti-invariant Riemannian submersion which was introduced by B. Sahin. We investigate the integrability of the distributions which arise from the definition of the new submersions and the geometry of foliations. Moreover, we find necessary and sufficient conditions for this submersion to be totally geodesic and in order to guarantee the new submersion, we mention some examples of such submersions.

Keywords: conformal submersion, almost product Riemannian manifold, vertical distribution, conformal anti-invariant submersion

2010 Mathematics Subject Classification: primary 53C15; secondary 53C40

1. Introduction

Immersion and submersions, which are special tools in differential geometry, also play a fundamental role in Riemannian geometry, especially when the involved manifolds carry an additional structure (such as contact, Hermitian and product structure). In particular, Riemannian submersions (which we always assume to have connected fibers) are fundamentally important in several areas of Riemannian geometry. For instance, it is a classical and important problem in Riemannian geometry to construct Riemannian manifolds with positive or non-negative sectional curvature. Riemannian submersions between Riemannian manifolds are important geometric structures. Riemannian submersions between Riemannian manifolds were studied by O'Neill [1] and Gray [2]. In [3], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions.

In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds was initiated by Şahin [4]. In this case, the fibers are anti-invariant with respect to the almost complex structure of the total manifold. This notion extended to different total spaces see: [5–14].

On the other hand, as a generalization of Riemannian submersion, horizontally conformal submersions are defined as follows [15]: Suppose that (M, g_M) and (B, g_B) are Riemannian manifolds and $\pi : M \rightarrow B$ is a smooth submersion, then π is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(X, Y) = g_B(\pi_* X, \pi_* Y)$$

for every $X, Y \in \Gamma((\ker \pi_*)^\perp)$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [16] and Ishihara [17]. We also note that a horizontally conformal submersion $\pi : M \rightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(\text{grad} \lambda) = 0 \tag{1}$$

at $p \in M$, where \mathcal{H} is the projection on the horizontal space $(\ker \pi_*)^\perp$. For conformal submersion, see: [15, 18, 19].

One can see that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

Recently, we introduced conformal anti-invariant submersions [20] and conformal semi-invariant submersions [21] from almost Hermitian manifolds, and gave examples and investigated the geometry of such submersions (see also [22, 23]). We showed that the geometry of such submersions is different from their counterpart anti-invariant Riemannian submersions and semi-invariant Riemannian submersions. In the present paper, we define and study conformal anti-invariant submersions from almost product Riemannian manifolds, give examples and investigate the geometry of the total space and the base space for the existence of such submersions.

Our work is structured as follows: Section 2 is focused on basic facts for conformal submersions and almost product Riemannian manifolds. The third section is concerned with definition of conformal anti-invariant submersions, investigating the integrability conditions of the horizontal distribution and the vertical distribution. In Section 4, we study the geometry of leaves of the horizontal distribution and the vertical distribution. In Section 5, we find necessary and

sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. The last section, we give some examples of such submersions.

2. Preliminaries

In this section we recall several notions and results which will be needed throughout the chapter.

Let M be a m -dimensional manifold with a tensor F of a type $(1,1)$ such that

$$F^2 = I, (F \neq I).$$

Then, we say that M is an almost product manifold with almost product structure F . We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F).$$

Then we get

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$

Thus P and Q define two complementary distributions P and Q . We easily see that the eigenvalues of F are $+1$ or -1 . If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{2}$$

for any vector fields X and Y on M , then M is called an almost product Riemannian manifold, denoted by (M, g, F) . Denote the Levi-Civita connection on M with respect to g by ∇ . Then, M is called a locally product Riemannian manifold [24] if F is parallel with respect to ∇ , i.e.,

$$\nabla_X F = 0, \quad X \in \Gamma(TM). \tag{3}$$

Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

Definition 2.1 ([15]) *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called horizontally weakly conformal or semi conformal at x if either*

(i) $d\varphi_x = 0$, or

(ii) $d\varphi_x$ maps horizontal space $\mathcal{H}_x = (\ker(d\varphi_x))^\perp$ conformally onto $T_{\varphi_*}N$, i.e., $d\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that

$$h(d\varphi_x X, d\varphi_x Y) = \Lambda(x)g(X, Y) \quad (X, Y \in \mathcal{H}_x). \tag{4}$$

Note that we can write the last equation more succinctly as

$$(\varphi^*h)_x|_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x)g_x|_{\mathcal{H}_x \times \mathcal{H}_x}.$$

A point x is of type (i) in Definition if and only if it is a critical point of φ ; we shall call a point of type (ii) a *regular point*. At a critical point, $d\varphi_x$ has rank 0; at a regular point, $d\varphi_x$ has rank n and φ is submersion. The number $\Lambda(x)$ is called the *square dilation* (of φ at x); it is necessarily non-negative; its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the *dilation* (of φ at x). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M . It is clear that if φ has no critical points, then we call it a (*horizontally*) conformal submersion.

Next, we recall the following definition from [18]. Let $\pi : M \rightarrow N$ be a submersion. A vector field E on M is said to be projectable if there exists a vector field \check{E} on N , such that $d\pi(E_x) = \check{E}_{\pi(x)}$ for all $x \in M$. In this case E and \check{E} are called π -related. A horizontal vector field Y on (M, g) is called basic, if it is projectable. It is well known fact, that is, \check{Z} is a vector field on N , then there exists a unique basic vector field Z on M , such that Z and \check{Z} are π -related. The vector field Z is called the horizontal lift of \check{Z} .

The fundamental tensors of a submersion were introduced in [1]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A defined for vector fields E, G on M by

$$A_E G = \mathcal{V}\nabla_{\mathcal{H}E}^{M_1} \mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}E}^{M_1} \mathcal{V}G \tag{5}$$

$$T_E G = \mathcal{H}\nabla_{\mathcal{V}E}^{M_1} \mathcal{V}G + \mathcal{V}\nabla_{\mathcal{V}E}^{M_1} \mathcal{H}G \tag{6}$$

where \mathcal{V} and \mathcal{H} are the vertical and horizontal projections (see [25]). On the other hand, from (5) and (6), we have

$$\nabla_{\mathcal{V}}^{M_1} W = T_{\mathcal{V}}W + \hat{\nabla}_{\mathcal{V}}W \tag{7}$$

$$\nabla_{\mathcal{V}}^{M_1} X = \mathcal{H}\nabla_{\mathcal{V}}^{M_1} X + T_{\mathcal{V}}X \tag{8}$$

$$\nabla_X^{M_1} V = A_X V + \mathcal{V}\nabla_X^{M_1} V \tag{9}$$

$$\nabla_X^{M_1} Y = \mathcal{H}\nabla_X^{M_1} Y + A_X Y \tag{10}$$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $V, W \in \Gamma(ker\pi_*)$, where $\hat{\nabla}_{\mathcal{V}}W = \mathcal{V}\nabla_{\mathcal{V}}^{M_1} W$. If X is basic, then $\mathcal{H}\nabla_{\mathcal{V}}^{M_1} X = A_X V$. It is easily seen that for $x \in M$, $X \in \mathcal{H}_x$ and \mathcal{V}_x the linear operators $T_{\mathcal{V}}$, $A_X : T_x M \rightarrow T_x M$ are skew-symmetric, that is

$$g(T_V E, G) = -g(E, T_V G) \text{ and } g(A_X E, G) = -g(E, A_X G)$$

for all $E, G \in T_x M$. We also see that the restriction of T to the vertical distribution $T|_{V \times V}$ is exactly the second fundamental form of the fibers of π . Since T_V is skew symmetric, we get π which has totally geodesic fibers if and only if $T \equiv 0$. For the special case when π is horizontally conformal we have the following:

Proposition 2.1 ([18]) *Let $\pi : (M^m, g) \rightarrow (N^n, h)$ be a horizontally conformal submersion with dilation ∇ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2} \left\{ \mathcal{V}[X, Y] - \lambda^2 g(X, Y) \text{grad}_{\mathcal{V}} \left(\frac{1}{\lambda^2} \right) \right\}. \quad (11)$$

We see that the skew-symmetric part of $A|_{(\ker \pi_*)^\perp \times (\ker \pi_*)^\perp}$ measures the obstruction integrability of the horizontal distribution $(\ker \pi_*)^\perp$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\pi : M \rightarrow N$ is a smooth map between them. The differential of π_* of π can be viewed a section of the bundle $\text{Hom}(TM, \pi^{-1}TN) \rightarrow M$, where $\pi^{-1}TN$ is the pullback bundle which has fibers $(\pi^{-1}TN)_p = T_{\pi(p)}N$, $p \in M$. $\text{Hom}(TM, \pi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of π is given by

$$\nabla \pi_* : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TN)$$

defined by

$$(\nabla \pi_*)(X, Y) = \nabla_X^\pi \pi_*(Y) - \pi_*(\nabla_X^M Y) \quad (12)$$

for $X, Y \in \Gamma(TM)$, where ∇^π is the pullback connection. It is known that the second fundamental form is symmetric.

Lemma 2.1. [26] *Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \rightarrow N$ is a smooth map between them. Then we have*

$$\nabla_X^\varphi \varphi_*(Y) - \nabla_Y^\varphi \varphi_*(X) - \varphi_*([X, Y]) = 0 \quad (13)$$

for $X, Y \in \Gamma(TM)$.

Finally, we recall the following lemma from [15].

Lemma 2.2. *Suppose that $\pi : M \rightarrow N$ is a horizontally conformal submersion. Then, for any horizontal vector fields X, Y and vertical fields V, W we have.*

- (i) $(\nabla\pi_*)(X, Y) = X(\ln \lambda)\pi_*Y + Y(\ln \lambda)\pi_*X - g_M(X, Y)\pi_*(grad \ln \lambda)$;
(ii) $(\nabla\pi_*)(V, W) = -\pi_*(T_VW)$;
(iii) $(\nabla\pi_*)(X, V) = -\pi_*(\nabla_X^M V) = -\pi_*(A_XV)$.

3. Conformal anti-invariant submersions from almost product Riemannian manifolds

In this section, we define conformal anti-invariant submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigating the geometry of distributions $(ker\pi_*)$ and $(ker\pi_*)^\perp$ and obtain the integrability conditions for the distribution $(ker\pi_*)^\perp$ for such submersions.

Definition 3.1. Let (M_1, g_1, F) be an almost product Riemannian manifold and (M_2, g_2) be a Riemannian manifold. A horizontally conformal submersion $\pi : M_1 \rightarrow M_2$ with dilation λ is called conformal anti-invariant submersion if the distribution $ker\pi_*$ is anti-invariant with respect to F , i.e., $F(ker\pi_*) \subseteq (ker\pi_*)^\perp$.

Let $\pi : (M_1, g_1, F) \rightarrow (M_2, g_2)$ is a conformal anti-invariant submersion from an almost product Riemannian manifold (M_1, g_1, F) to a Riemannian manifold (M_2, g_2) . First of all, from Definition 3.1, we have $F(ker\pi_*)^\perp \cap ker\pi_* \neq 0$. We denote the complementary orthogonal distribution to $F(ker\pi_*)$ in $(ker\pi_*)^\perp$ by μ . Then we have

$$(ker\pi_*)^\perp = F(ker\pi_*) \oplus \mu. \quad (14)$$

Proposition 3.1. Let (M_1, g_1, F) be an almost product Riemannian manifold and (M_2, g_2) be a Riemannian manifold. Then μ is invariant with respect to F .

Proof. For $Z \in \Gamma(\mu)$ and $V \in \Gamma(ker\pi_*)$, by using (2), we have $g_1(FZ, FV) = 0$, which show that FZ is orthogonal to $Fker\pi_*$. On the other hand, since FV and Z are orthogonal we get $g_1(FV, Z) = g_1(V, FZ) = 0$ which shows that FZ is orthogonal to $ker\pi_*$. This completes proof. \square

For $Z \in \Gamma((ker\pi_*)^\perp)$, we have

$$FZ = \mathcal{B}Z + \mathcal{C}Z, \quad (15)$$

where $\mathcal{B}Z \in \Gamma(ker\pi_*)$ and $\mathcal{C}Z \in \Gamma(\mu)$. On the other hand, since $\pi_*((ker\pi_*)^\perp) = TM_2$ and π is a conformal submersion, using (15) we derive $\frac{1}{\lambda^2}g_2(\pi_*FV, \pi_*\mathcal{C}Z) = 0$ for any $Z \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$, which implies that

$$TM_2 = \pi_*(Fker\pi_*) \oplus \pi_*(\mu). \tag{16}$$

Lemma 3.1. *Let π be a conformal anti-invariant submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then we have*

$$g_1(CW, FV) = 0 \tag{17}$$

and

$$g_1(\nabla_Z^{M_1} CW, FV) = -g_1(CW, FA_Z V) \tag{18}$$

for $Z, W \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$.

Proof. For $W \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$, using (2) we have

$$g_1(CW, FV) = g_1(FW - BW, FV) = g_1(FW, FV)$$

due to $BW \in \Gamma(ker\pi_*)$ and $FV \in \Gamma((ker\pi_*)^\perp)$. Hence $g_1(FW, FV) = g_1(W, V) = 0$ which is (17).

Since M_1 is a locally product Riemannian manifold, differentiating (3.4) with respect to Z , we get

$$g_1(\nabla_Z^{M_1} CW, FV) = g_1(CW, F\nabla_Z^{M_1} V)$$

for $Z, W \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$. Then using (9) we have

$$g_1(\nabla_Z^{M_1} CW, FV) = -g_1(CW, FA_Z V) - g_1(CW, FV\nabla_Z^{M_1} V).$$

Since $FV\nabla_Z^{M_1} V \in \Gamma(Fker\pi_*)$, we obtain (18). □

We now study the integrability of the distribution $(ker\pi_*)^\perp$ and then we investigate the geometry of the leaves of $ker\pi_*$ and $(ker\pi_*)^\perp$. We note that it is known that the distribution $ker\pi_*$ is integrable.

Theorem 3.1. *Let $\pi : (M_1, g_1, F) \rightarrow (M_2, g_2)$ is a conformal anti-invariant submersion from an almost product Riemannian manifold (M_1, g_1, F) to a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other;*

(a) $(ker\pi_*)^\perp$ is integrable,

$$(b) \frac{1}{\lambda^2} g_2(\nabla_W^\pi \pi_* CZ - \nabla_Z^\pi \pi_* CW, \pi_* FV) = g_1(A_Z BW - A_W BZ - CW(\ln \lambda)Z + CZ(\ln \lambda)W, FV) \tag{19}$$

for any $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$.

Proof. For $W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$, we see from Definition 3.1, $FV \in \Gamma((\ker \pi_*)^\perp)$ and $FW \in \Gamma(\ker \pi_* \oplus \mu)$. Thus using (2) and (3), for $Z \in \Gamma((\ker \pi_*)^\perp)$ we obtain

$$g_1([Z, W], V) = g_1(\nabla_Z^{M_1} FW, FV) - g_1(\nabla_W^{M_1} FZ, FV).$$

Further, from (15) we get

$$g_1([Z, W], V) = g_1(\nabla_Z^{M_1} \mathcal{B}W, FV) + g_1(\nabla_Z^{M_1} CW, FV) - g_1(\nabla_W^{M_1} \mathcal{B}Z, FV) - g_1(\nabla_W^{M_1} CZ, FV).$$

Using (9), (10) and if we take into account π is a conformal submersion, we arrive at

$$g_1([Z, W], V) = g_1(A_Z \mathcal{B}W - A_W \mathcal{B}Z, FV) + \frac{1}{\lambda^2} g_2(\pi_*(\nabla_Z^{M_1} CW), \pi_* FV) - \frac{1}{\lambda^2} g_2(\pi_*(\nabla_W^{M_1} CZ), \pi_* FV).$$

Thus, from (12) and Lemma 2.2 we derive

$$\begin{aligned} g_1([Z, W], V) &= g_1(A_Z \mathcal{B}W - A_W \mathcal{B}Z, FV) - g_1(\mathcal{H}grad \ln \lambda, Z)g_1(CW, FV) \\ &\quad - g_1(\mathcal{H}grad \ln \lambda, CW)g_1(Z, FV) + g_1(Z, CW)g_1(\mathcal{H}grad \ln \lambda, FV) \\ &\quad + \frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV) + g_1(\mathcal{H}grad \ln \lambda, W)g_1(CZ, FV) \\ &\quad + g_1(\mathcal{H}grad \ln \lambda, CZ)g_1(W, FV) - g_1(W, CZ)g_1(\mathcal{H}grad \ln \lambda, FV) \\ &\quad - \frac{1}{\lambda^2} g_2(\nabla_W^\pi \pi_* CZ, \pi_* FV). \end{aligned}$$

Moreover, using (17), we obtain

$$\begin{aligned} g_1([Z, W], V) &= g_1(A_Z \mathcal{B}W - A_W \mathcal{B}Z - CW(\ln \lambda)Z + CZ(\ln \lambda)W, FV) \\ &\quad - \frac{1}{\lambda^2} g_2(\nabla_W^\pi \pi_* CZ - \nabla_Z^\pi \pi_* CW, \pi_* FV) \end{aligned}$$

which proves (a) \Leftrightarrow (b). □

From Theorem 3.1, we deduce the following characterization.

Theorem 3.2. *Let π be a conformal anti-invariant submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then any two conditions below imply the three;*

- i. $(\ker \pi_*)^\perp$ is integrable.
- ii. λ is a constant on $\Gamma(\mu)$.
- iii. $g_2(\nabla_W^\pi \pi_* CZ - \nabla_Z^\pi \pi_* CW, \pi_* FV) = \lambda^2 g_1(A_Z \mathcal{B}W - A_W \mathcal{B}Z, FV)$

for $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$.

Proof. From Theorem 3.1, we have

$$g_1([Z, W], V) = g_1(A_Z B W - A_W B Z - C W (\ln \lambda) Z + C Z (\ln \lambda) W, FV) - \frac{1}{\lambda^2} g_2(\nabla_W^\pi \pi_* C Z - \nabla_Z^\pi \pi_* C W, \pi_* FV).$$

for $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$. Now, if we have (i) and (iii), then we arrive at

$$-g_1(\mathcal{H}grad \ln \lambda, C W) g_1(Z, FV) + g_1(\mathcal{H}grad \ln \lambda, C Z) g_1(W, FV) = 0. \quad (20)$$

Now, taking $W = FV$ in (20) for $V \in \Gamma(\ker \pi_*)$, using (17), we get

$$-g_1(\mathcal{H}grad \ln \lambda, C(FV)) g_1(Z, FV) + g_1(\mathcal{H}grad \ln \lambda, C Z) g_1(FV, FV) = 0.$$

Hence λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

We say that a conformal anti-invariant submersion is a conformal Lagrangian submersion if $F(\ker \pi_*) = (\ker \pi_*)^\perp$. From Theorem 3.1, we have the following result.

Corollary 3.1. *Let π be a conformal Lagrangian submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other:*

- i. $(\ker \pi_*)^\perp$ is integrable
- ii. $A_Z F W = A_W F Z$
- iii. $(\nabla \pi_*)(Z, F W) = (\nabla \pi_*)(W, F Z)$

for $Z, W \in \Gamma((\ker \pi_*)^\perp)$.

Proof. For $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$, we see from Definition 3.1, $FV \in \Gamma((\ker \pi_*)^\perp)$ and $FW \in \Gamma(\ker \pi_*)$. From Theorem 3.1 we have

$$g_1([Z, W], V) = g_1(A_Z B W - A_W B Z - C W (\ln \lambda) Z + C Z (\ln \lambda) W, FV) - \frac{1}{\lambda^2} g_2(\nabla_W^\pi \pi_* C Z - \nabla_Z^\pi \pi_* C W, \pi_* FV).$$

Since π is a conformal Lagrangian submersion, we derive

$$g_1([Z, W], V) = g_1(A_Z B W - A_W B Z, FV)$$

which shows (i) \Leftrightarrow (ii). On the other hand, using Definition 3.1 and (9) we arrive at

$$\begin{aligned}
g_1(A_Z \mathcal{B}W, FV) - g_1(A_W \mathcal{B}Z, FV) &= \frac{1}{\lambda^2} g_2(\pi_* A_Z \mathcal{B}W, \pi_* FV) - \frac{1}{\lambda^2} g_2(\pi_* A_W \mathcal{B}Z, \pi_* FV) \\
&= \frac{1}{\lambda^2} g_2(\pi_* (\nabla_Z^{M_1} \mathcal{B}W), \pi_* FV) - \frac{1}{\lambda^2} g_2(\pi_* (\nabla_W^{M_1} \mathcal{B}Z), \pi_* FV).
\end{aligned}$$

Now, using (12) we obtain

$$\begin{aligned}
&\frac{1}{\lambda^2} \{g_2(\pi_* (\nabla_Z^{M_1} \mathcal{B}W), \pi_* FV) - g_2(\pi_* (\nabla_W^{M_1} \mathcal{B}Z), \pi_* FV)\} \\
&= \frac{1}{\lambda^2} g_2(-(\nabla_{F_*}) (Z, \mathcal{B}W) + \nabla_Z^\pi \pi_* \mathcal{B}W, \pi_* FV) - \frac{1}{\lambda^2} g_2(-(\nabla_{F_*}) (W, \mathcal{B}Z) + \nabla_W^\pi \pi_* \mathcal{B}Z, \pi_* FV).
\end{aligned}$$

Since $\mathcal{B}Z, \mathcal{B}W \in \Gamma(\ker \pi_*)$, we derive

$$g_1(A_Z \mathcal{B}W, FV) - g_1(A_W \mathcal{B}Z, FV) = \frac{1}{\lambda^2} g_2((\nabla_{F_*}) (W, \mathcal{B}Z) - (\nabla_{F_*}) (Z, \mathcal{B}W), \pi_* FV)$$

which tells that (ii) \Leftrightarrow (iii). □

4. Totally geodesic foliations

In this section, we shall investigate the geometry of leaves of $(\ker \pi_*)$ and $(\ker \pi_*)^\perp$. For the geometry of leaves of the horizontal distribution $(\ker \pi_*)^\perp$, we have the following theorem.

Theorem 4.1. *Let $\pi : (M_1, g_1, F) \rightarrow (M_2, g_2)$ is a conformal anti-invariant submersion from an almost product Riemannian manifold (M_1, g_1, F) to a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other;*

i. $(\ker \pi_*)^\perp$ defines a totally geodesic foliation on M_1 .

ii. $-\frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV) = g_1(A_Z \mathcal{B}W - CW(\ln \lambda)Z + g_1(Z, CW) \ln \lambda, FV)$

for $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$.

Proof. For $Z, W \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(\ker \pi_*)$, by using (3), (9), (10), (14) and (15) we have

$$g_1(\nabla_Z^{M_1} W, V) = g_1(A_Z \mathcal{B}W, FV) + g_1(\nabla_Z^{M_1} CW, FV).$$

Since π is a conformal submersion, using (12) and Lemma 2.2 we arrive at

$$\begin{aligned} g_1(\nabla_Z^{M_1} W, V) &= g_1(A_Z BW, FV) - \frac{1}{\lambda^2} g_1(\mathcal{H}grad \ln \lambda, Z) g_2(\pi_* CW, \pi_* FV) \\ &\quad - \frac{1}{\lambda^2} g_1(\mathcal{H}grad \ln \lambda, CW) g_2(\pi_* Z, \pi_* FV) \\ &\quad + \frac{1}{\lambda^2} g_1(Z, CW) g_2(\pi_*(\mathcal{H}grad \ln \lambda), \pi_* FV) \\ &\quad + \frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV). \end{aligned}$$

Moreover, using Definition 3.1 and (17) we obtain

$$g_1(\nabla_Z^{M_1} W, V) = g_1(A_Z BW - CW(\ln \lambda)Z + g_1(Z, CW) \ln \lambda, FV) + \frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV)$$

which proves (i) \Leftrightarrow (ii). □

From Theorem 4.1, we also deduce the following characterization.

Theorem 4.2. *Let π be a conformal anti-invariant submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then any two conditions below imply the three;*

- i. $(ker \pi_*)^\perp$ defines a totally geodesic foliation on M_1 .
- ii. π is horizontally homothetic submersion.
- iii. $g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV) = \lambda^2 g_1(A_Z FV, BW)$

for $Z, W \in \Gamma((ker \pi_*)^\perp)$ and $V \in \Gamma(ker \pi_*)$.

Proof. For $Z, W \in \Gamma((ker \pi_*)^\perp)$ and $V \in \Gamma(ker \pi_*)$, from Theorem 4.1, we have

$$g_1(\nabla_Z^{M_1} W, V) = g_1(A_Z BW - CW(\ln \lambda)Z + g_1(Z, CW) \ln \lambda, FV) + \frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV).$$

Now, if we have (i) and (iii), then we obtain

$$-g_1(\mathcal{H}grad \ln \lambda, CW) g_1(Z, FV) + g_1(\mathcal{H}grad \ln \lambda, FV) g_1(Z, CW) = 0. \tag{21}$$

Now, taking $Z = CW$ in (4.1) and using (17), we get

$$g_1(\mathcal{H}grad \ln \lambda, FV) g_M(CW, CW) = 0.$$

Thus, λ is a constant on $\Gamma(Fker \pi_*)$. On the other hand, taking $Z = FV$ in (25) and using (17) we derive

$$g_1(\mathcal{H}grad \ln \lambda, CW)g_1(FV, FV) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. \square

For conformal Lagrangian submersion, we have the following result.

Corollary 4.1. *Let π be a conformal Lagrangian submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other;*

i. $(ker\pi_*)^\perp$ defines a totally geodesic foliation on M_1 .

ii. $A_ZBW = 0$

iii. $(\nabla\pi_*)(Z, FV) = 0$

for $Z, W \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$.

Proof. For $Z, W \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$, from Theorem 4.1, we have

$$g_1(\nabla_Z^{M_1} W, V) = g_1(A_ZBW - CW(\ln \lambda)Z + g_1(Z, CW) \ln \lambda, FV) + \frac{1}{\lambda^2} g_2(\nabla_Z^\pi \pi_* CW, \pi_* FV).$$

Since π is a conformal Lagrangian submersion, we derive

$$g_1(\nabla_Z^{M_1} W, V) = g_1(A_ZBW, FV)$$

which shows (i) \Leftrightarrow (ii). On the other hand, using Definition 3.1 and (9) we arrive at

$$g_1(A_ZBW, FV) = \frac{1}{\lambda^2} g_2(\pi_*(A_ZBW), \pi_* FV) = \frac{1}{\lambda^2} g_2(\pi_*(\nabla_Z^{M_1} BW), \pi_* FV).$$

Now, using (12) we obtain

$$\begin{aligned} \frac{1}{\lambda^2} g_2(\pi_*(\nabla_Z^{M_1} BW), \pi_* FV) &= \frac{1}{\lambda^2} g_2(-(\nabla\pi_*)(Z, BW) + \nabla_Z^\pi \pi_* BW, \pi_* FV) \\ &= -\frac{1}{\lambda^2} g_2((\nabla\pi_*)(Z, BW), \pi_* FV) \end{aligned}$$

which tells that (ii) \Leftrightarrow (iii). \square

For the totally geodesicness of the foliations of the distribution $ker\pi_*$.

Theorem 4.3. *Let $\pi : (M_1, g_1, F) \rightarrow (M_2, g_2)$ is a conformal anti-invariant submersion from an almost product Riemannian manifold (M_1, g_1, F) to a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other;*

i. $\ker\pi_*$ defines a totally geodesic foliation on M_1 .

ii.
$$-\frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi\pi_*FV, \pi_*FCZ) = g_1(T_VFU, \mathcal{B}Z) + g_1(U, V)g_1(\mathcal{H}grad \ln \lambda, FCZ)$$

for $V, U \in \Gamma(\ker\pi_*)$ and $Z \in \Gamma((\ker\pi_*)^\perp)$.

Proof. For $Z \in \Gamma((\ker\pi_*)^\perp)$ and $V, U \in \Gamma(\ker\pi_*)$, by using (2), (3), (8) and (15) we get

$$g_1(\nabla_V^{M_1}U, Z) = g_1(T_VFU, \mathcal{B}Z) + g_1(\mathcal{H}\nabla_V^{M_1}FU, CZ).$$

Since ∇^{M_1} is torsion free and $[V, FU] \in \Gamma(\ker\pi_*)$ we obtain

$$g_1(\nabla_V^{M_1}U, Z) = g_1(T_VFU, \mathcal{B}Z) + g_1(\nabla_{FU}^{M_1}V, CZ).$$

Using (3) and (10) we have

$$g_1(\nabla_V^{M_1}U, Z) = g_1(T_VFU, \mathcal{B}Z) + g_1(\nabla_{FU}^{M_1}FV, FCZ)$$

here we have used that μ is invariant. Since π is a conformal submersion, using (12) and Lemma 2.2 we obtain

$$\begin{aligned} g_1(\nabla_V^{M_1}U, Z) &= g_1(T_VFU, \mathcal{B}Z) + \frac{1}{\lambda^2}g_1(\mathcal{H}grad \ln \lambda, FU)g_2(\pi_*FV, \pi_*FCZ) \\ &\quad - \frac{1}{\lambda^2}g_1(\mathcal{H}grad \ln \lambda, FV)g_2(\pi_*FU, \pi_*FCZ) \\ &\quad + g_1(FU, FV)\frac{1}{\lambda^2}g_2(\pi_*(\mathcal{H}grad \ln \lambda), \pi_*FCZ) \\ &\quad + \frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi\pi_*FV, \pi_*FCZ). \end{aligned}$$

Moreover, using Definition 3.1 and (17), we obtain

$$g_1(\nabla_V^{M_1}U, Z) = g_1(T_VFU, \mathcal{B}Z) + g_1(U, V)g_1(\mathcal{H}grad \ln \lambda, FCZ) + \frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi\pi_*FV, \pi_*FCZ)$$

which proves (i) \Leftrightarrow (ii). □

From Theorem 4.3, we deduce the following result.

Theorem 4.4. *Let π be a conformal anti-invariant submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then any two conditions below imply the three;*

i. $\ker\pi_*$ defines a totally geodesic foliation on M_1

ii. λ is a constant on $\Gamma(\mu)$

iii.
$$-\frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi\pi_*FV, \pi_*FCZ) = g_1(T_VFU, \mathcal{B}Z)$$

for $V, U \in \Gamma(\ker \pi_*)$ and $Z \in \Gamma((\ker \pi_*)^\perp)$.

Proof. For $V, U \in \Gamma(\ker \pi_*)$ and $Z \in \Gamma((\ker \pi_*)^\perp)$, from Theorem 4.3 we have

$$g_1(\nabla_V^{M_1} U, Z) = g_1(T_V F U, \mathcal{B}Z) + g_1(U, V)g_1(\mathcal{H}grad \ln \lambda, FCZ) + \frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi \pi_* FV, \pi_* FCZ).$$

Now, if we have (i) and (iii), then we obtain

$$g_1(U, V)g_1(\mathcal{H}grad \ln \lambda, FCZ) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. \square

If π is a conformal Lagrangian submersion, then (16) implies that $TM_2 = \pi_*(Fker \pi_*)$. Hence we have the following corollary:

Corollary 4.2. *Let π be a conformal Lagrangian submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . Then the following assertions are equivalent to each other;*

- i. $ker \pi_*$ defines a totally geodesic foliation on M_1 .
- ii. $T_V F U = 0$

for $V, U \in \Gamma(\ker \pi_*)$ and $Z \in \Gamma((\ker \pi_*)^\perp)$.

Proof. From Theorem 4.3 we have

$$g_1(\nabla_V^{M_1} U, Z) = g_1(T_V F U, \mathcal{B}Z) + g_1(U, V)g_1(\mathcal{H}grad \ln \lambda, FCZ) + \frac{1}{\lambda^2}g_2(\nabla_{FU}^\pi \pi_* FV, \pi_* FCZ).$$

for $V, U \in \Gamma(\ker \pi_*)$ and $Z \in \Gamma((\ker \pi_*)^\perp)$. Since π is a conformal Lagrangian submersion, we get

$$g_1(\nabla_V^{M_1} U, Z) = g_1(T_V F U, \mathcal{B}Z)$$

which shows (i) \Leftrightarrow (ii). \square

5. Totally geodesicness of the conformal anti-invariant submersion

In this section, we shall examine the totally geodesicness of a conformal anti-invariant submersion. We give a necessary and sufficient condition for a conformal anti-invariant submersion to be totally geodesic map. Recall that a smooth map π between two Riemannian manifolds is called totally geodesic if $\nabla \pi_* = 0$ [15].

Theorem 5.1. Let $\pi : (M_1, g_1, F) \rightarrow (M_2, g_2)$ is a conformal anti-invariant submersion from an almost product Riemannian manifold (M_1, g_1, F) to a Riemannian manifold (M_2, g_2) . π is totally geodesic map if and only if.

- (a) π is a horizontally homothetic map,
 - (b) $T_U FV = 0$ and $\mathcal{H}\nabla_U^{M_1} FV \in \Gamma(\text{Fker}\pi_*)$,
 - (c) $A_Z FV = 0$ and $\mathcal{H}\nabla_Z^{M_1} FV \in \Gamma(\text{Fker}\pi)$
- for $Z, W, Z \in \Gamma((\text{ker}\pi_*)^\perp)$ and $U, V \in \Gamma(\text{ker}\pi_*)$.

Proof. (a) For any $Z, W \in \Gamma(\mu)$, from Lemma 2.2 we derive

$$(\nabla\pi_*)(Z, W) = Z(\ln \lambda)\pi_*W + W(\ln \lambda)\pi_*Z - g_1(Z, W)\pi_*(\text{grad } \ln \lambda).$$

It is obvious that if π is a horizontally homothetic map, it follows that $(\nabla\pi_*)(Z, W) = 0$. Conversely, if $(\nabla\pi_*)(Z, W) = 0$, taking $W = FZ$ in above equation, we get

$$Z(\ln \lambda)\pi_*FZ + FZ(\ln \lambda)\pi_*Z - g_1(Z, FZ)\pi_*(\text{grad } \ln \lambda) = 0. \tag{22}$$

Taking inner product in (31) with π_*FZ , we obtain

$$g_1(\text{grad } \ln \lambda, Z)\lambda^2g_1(FZ, FZ) + g_1(\text{grad } \ln \lambda, FZ)\lambda^2g_1(Z, FZ) - g_1(Z, FZ)\lambda^2g_1(\text{grad } \ln \lambda, FZ) = 0. \tag{23}$$

From (32), λ is a constant on $\Gamma(\mu)$. On the other hand, for $U, V \in \Gamma(\text{ker}\pi_*)$, from Lemma 2.2 we have

$$(\nabla\pi_*)(FU, FV) = FU(\ln \lambda)\pi_*FV + FV(\ln \lambda)\pi_*FU - g_1(FU, FV)\pi_*(\text{grad } \ln \lambda).$$

Again if π is a horizontally homothetic map, then $(\nabla\pi_*)(FU, FV) = 0$. Conversely, if $(\nabla\pi_*)(FU, FV) = 0$, putting U instead of V in above equation, we derive

$$2FU(\ln \lambda)\pi_*FU - g_1(FU, FU)\pi_*(\text{grad } \ln \lambda) = 0. \tag{24}$$

Taking inner product in (33) with π_*FU and since π is a conformal submersion, we have

$$g_1(FU, FU)\lambda^2g_1(\text{grad } \ln \lambda, FU) = 0.$$

From above equation, λ is a constant on $\Gamma(\text{Fker}\pi_*)$. Thus λ is a constant on $\Gamma((\text{ker}\pi_*)^\perp)$.

(b) For any $U, V \in \Gamma(\text{ker}\pi_*)$, using (3) and (12) we have

$$\begin{aligned}
(\nabla\pi_*)(U, V) &= \nabla_U^\pi \pi_* V - \pi_*(\nabla_U^{M_1} V) \\
&= -\pi_*(F\nabla_U^{M_1} FV).
\end{aligned}$$

Then from (7) and (8) we arrive at

$$(\nabla\pi_*)(U, V) = -\pi_*(FT_U FV + \mathcal{C}\mathcal{H}\nabla_U^{M_1} FV).$$

From above equation, $(\nabla\pi_*)(U, V) = 0$ if and only if

$$\pi_*(FT_U FV + \mathcal{C}\mathcal{H}\nabla_U^{M_1} FV) = 0 \quad (25)$$

Since π is non-singular, this implies $T_U FV = 0$ and $\mathcal{H}\nabla_U^{M_1} FV \in \Gamma(\text{Fker}\pi_*)$.

(c) For $Z \in \Gamma(\mu)$ and $V \in \Gamma(\text{ker}\pi_*)$, from (3) and (12) we get

$$\begin{aligned}
(\nabla\pi_*)(Z, V) &= \nabla_Z^\pi \pi_* V - \pi_*(\nabla_Z^{M_1} V) \\
&= -\pi_*(F\nabla_Z^{M_1} FV).
\end{aligned}$$

Using (9) and (10) we have

$$(\nabla\pi_*)(Z, V) = \pi_*(FA_Z FV + \mathcal{C}\mathcal{H}\nabla_Z^{M_1} FV).$$

Thus $(\nabla\pi_*)(Z, V) = 0$ if and only if

$$\pi_*(FA_Z FV + \mathcal{C}\mathcal{H}\nabla_Z^{M_1} FV) = 0.$$

Then, since π is a linear isomorphism between $(\text{ker}\pi_*)^\perp$ and TM_2 , $(\nabla\pi_*)(Z, V) = 0$ if and only if $A_Z FV = 0$ and $\mathcal{H}\nabla_Z^{M_1} FV \in \Gamma(\text{Fker}\pi_*)$. Thus proof is complete. \square

Here we present another result on conformal anti-invariant submersion to be totally geodesic.

Theorem 5.2 *Let π be a conformal anti-invariant submersion from a locally product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2) . If π is a totally geodesic map then*

$$\nabla_Z^\pi \pi_* W_2 = \pi_*(F(A_Z F W_1 + \mathcal{V}\nabla_Z^{M_1} B W_2 + A_Z C W_2) + C(\mathcal{H}\nabla_Z^{M_1} F W_1 + A_Z B W_2 + \mathcal{H}\nabla_Z^{M_1} C W_2))$$

for any $Z \in \Gamma((\text{ker}\pi_*)^\perp)$ and $W = W_1 + W_2 \in \Gamma(TM)$, where $W_1 \in \Gamma(\text{ker}\pi_*)$ and $W_2 \in \Gamma((\text{ker}\pi_*)^\perp)$.

Proof. Using (3) and (12) we have

$$(\nabla\pi_*)(Z, W) = \nabla_Z^\pi \pi_* W - \pi_*(F\nabla_Z^{M_1} FW)$$

for any $Z \in \Gamma((\ker\pi_*)^\perp)$ and $W \in \Gamma(TM_1)$. Then from (9), (10) and (15) we get

$$\begin{aligned} (\nabla\pi_*)(Z, W) = & \nabla_Z^\pi \pi_* W_2 - \pi_*(FA_Z FW_1 + B\mathcal{H}\nabla_Z^{M_1} FW_1 + C\mathcal{H}\nabla_Z^{M_1} FW_1 + BA_Z BW_2 \\ & + CA_Z BW_2 + F\mathcal{V}\nabla_Z^{M_1} BW_2 + FA_Z CW_2 + B\mathcal{H}\nabla_Z^{M_1} CW_2 + C\mathcal{H}\nabla_Z^{M_1} CW_2) \end{aligned}$$

for any $W = W_1 + W_2 \in \Gamma(TM)$, where $W_1 \in \Gamma(\ker\pi_*)$ and $W_2 \in \Gamma((\ker\pi_*)^\perp)$. Thus taking into account the vertical parts, we find

$$\begin{aligned} (\nabla\pi_*)(Z, W) = & \nabla_Z^\pi \pi_* W_2 - \pi_*(F(A_Z FW_1 + \mathcal{V}\nabla_Z^{M_1} BW_2 + A_Z CW_2) \\ & + C(\mathcal{H}\nabla_Z^{M_1} FW_1 + A_Z BW_2 + \mathcal{H}\nabla_Z^{M_1} CW_2)) \end{aligned}$$

which gives our assertion. □

6. Examples

In this section, we now give some examples for conformal anti-invariant submersions from almost product Riemannian manifolds.

Example 6.1. Every anti-invariant Riemannian submersion is a conformal anti-invariant submersion with $\lambda = I$, where I is the identity function [7].

We say that a conformal anti-invariant submersion is proper if $\lambda \neq I$. We now present an example of a proper conformal anti-invariant submersion. Note that given an Euclidean space \mathbb{R}^4 with coordinates (x_1, \dots, x_4) , we can canonically choose an almost product structure F on \mathbb{R}^4 as follows:

$$\begin{aligned} F\left(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} + a_4 \frac{\partial}{\partial x_4}\right) &= a_3 \frac{\partial}{\partial x_1} + a_4 \frac{\partial}{\partial x_2} + a_1 \frac{\partial}{\partial x_3} + a_2 \frac{\partial}{\partial x_4}, \\ a_1, \dots, a_4 &\in \mathbb{R}. \end{aligned} \tag{26}$$

Example 6.2. Let π be a submersion defined by

$$\pi : \begin{matrix} \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) \end{matrix} \rightarrow \begin{matrix} \mathbb{R}^2 \\ (\cos x_1 \sinh x_2, \sin x_1 \cosh x_2) \end{matrix}.$$

Then it follows that

$$\ker\pi_* = \text{span}\{V_1 = \partial x_3, V_2 = \partial x_4\}$$

and

$$(\ker\pi_*)^\perp = \text{span}\{X_1 = \partial x_1, X_2 = \partial x_2\}.$$

Hence, we have $FV_1 = X_1$ and $FV_2 = X_2$ imply that $F(\ker\pi_*) = (\ker\pi_*)^\perp$. Also by direct computations, we get

$$\pi_*X_1 = -\sin x_1 \sinh x_2 \partial y_1 + \cos x_1 \cosh x_2 \partial y_2,$$

$$\pi_*X_2 = \cos x_1 \cosh x_2 \partial y_1 + \sin x_1 \sinh x_2 \partial y_2.$$

Hence, we have

$$g_2(\pi_*X_1, \pi_*X_1) = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2)g_1(X_1, X_1),$$

$$g_2(\pi_*X_2, \pi_*X_2) = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2)g_1(X_2, X_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of \mathbb{R}^4 and \mathbb{R}^2 . Thus π is a conformal anti-invariant submersion with $\lambda^2 = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2)$.

Example 6.3. Let π be a submersion defined by

$$\pi : \begin{matrix} \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) \end{matrix} \rightarrow \begin{matrix} \mathbb{R}^2 \\ \left(\frac{e^{x_3} \sin x_4}{\sqrt{2}}, \frac{e^{x_3} \cos x_4}{\sqrt{2}} \right) \end{matrix}.$$

Then it follows that

$$\ker\pi_* = \text{span}\{V_1 = \partial x_1, V_2 = \partial x_2\}$$

and

$$(\ker\pi_*)^\perp = \text{span}\{W_1 = \partial x_3, W_2 = \partial x_4\}.$$

Hence we have $FV_1 = W_1$ and $FV_2 = W_2$ imply that $F(\ker\pi_*) = (\ker\pi_*)^\perp$. Also by direct computations, we get

$$\pi_*W_1 = \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_1 + \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_2,$$

$$\pi_*W_2 = \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_1 - \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_2.$$

Hence, we have

$$g_2(\pi_*W_1, \pi_*W_1) = \left(\frac{e^{\lambda_3}}{\sqrt{2}}\right)^2 g_1(W_1, W_1),$$

$$g_2(\pi_*W_2, \pi_*W_2) = \left(\frac{e^{\lambda_3}}{\sqrt{2}}\right)^2 g_1(W_2, W_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of \mathbb{R}^4 and \mathbb{R}^2 . Thus π is a conformal anti-invariant submersion with $\lambda = \left(\frac{e^{\lambda_3}}{\sqrt{2}}\right)$.

Author details

Mehmet Akif Akyol

Address all correspondence to: mehmetakifakyol@bingol.edu.tr

Department of Mathematics, Faculty of Arts and Sciences, Bingöl University, Bingöl, Turkey

References

- [1] O'Neill B. The fundamental equations of a submersion. Michigan Mathematical Journal. 1966;**13**:458-469
- [2] Gray A. Pseudo-Riemannian almost product manifolds and submersions. Journal of Mathematics and Mechanics. 1967;**16**:715-737
- [3] Watson B. Almost Hermitian submersions. Journal of Differential Geometry. 1976;**11**(1): 147-165
- [4] Şahin B. Anti-invariant Riemannian submersions from almost Hermitian manifolds. Central European Journal of Mathematics. 2010;**3**:437-447
- [5] Beri A, Erken İK, Murathan C. Anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. 2016;**40**:540-552
- [6] Cengizhan M, Erken İK. Anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian submersions. Univerzitet u Nišu. 2015;**29**(7):1429-1444
- [7] Gündüzalp Y. Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds. Journal of Function Spaces and Applications. 2013;**7**. Article ID 720623
- [8] Gündüzalp Y. Anti-invariant Riemannian submersions from almost product Riemannian manifolds. Mathematical Science and Applications E-notes. 2013;**1**(1):58-66

- [9] Lee JW. Anti-invariant ξ^\perp -Riemannian submersions from almost contact manifolds. Hacettepe Journal of Mathematics and Statistics. 2013;**42**(3):231-241
- [10] Lee JC, Park JH, Şahin B, Song DY. Einstein conditions for the base space of anti-invariant Riemannian submersions and Clairaut submersions. Taiwanese Journal of Mathematics. In press. DOI: 10.11650/tjm.19.2015.5283
- [11] Park KS. H-anti-invariant submersions from almost quaternionic Hermitian manifolds. arXiv:1507.04473 [math.DG]
- [12] Shahid A, Tanveer F. Anti-invariant Riemannian submersions from nearly Kählerian manifolds. Univerzitet u Nišu. 2013;**27**(7):1219-1235
- [13] Şahin B. Riemannian submersions from almost Hermitian manifolds. Taiwanese Journal of Mathematics. 2012;**17**(2):629-659
- [14] Taştan HM. On Lagrangian submersions. Hacettepe Journal of Mathematics and Statistics. 2014;**43**(6):993-1000
- [15] Baird P, Wood JC. Harmonic Morphisms between Riemannian Manifolds, London Mathematical Society Monographs, 29. Oxford: Oxford University Press, The Clarendon Press; 2003
- [16] Fuglede B. Harmonic morphisms between Riemannian manifolds. Annales De L'Institut Fourier. 1978;**28**:107-144
- [17] Ishhara T. A mapping of Riemannian manifolds which preserves harmonic functions. Journal of Mathematics of Kyoto University. 1979;**19**:215-229
- [18] Gundmundsson S. The geometry of harmonic morphisms [Ph.D. thesis]. University of Leeds; 1992
- [19] Ornea L, Romani G. The fundamental equations of a conformal submersions. Beitrage Z. Algebra and Geometrie/Contributions Algebra and Geometry. 1993; **34**(2):233-243
- [20] Akyol MA, Şahin B. Conformal anti-invariant submersions from almost Hermitian manifolds. Turkish Journal of Mathematics. 2016;**40**:43-70
- [21] Akyol MA, Şahin B. Conformal semi-invariant submersions. Communications in Contemporary Mathematics. 2017;**19**(2):1650011. DOI: 10.1142/S0219199716500115
- [22] Akyol MA. Conformal anti-invariant submersions from cosymplectic manifolds. Hacettepe Journal of Mathematics and Statistics. 2017;**46**(2):177-192
- [23] Akyol MA. Conformal semi-invariant submersions from almost product Riemannian manifolds. Acta Mathematica Vietnamica. 2017;**42**(3):491-507. DOI: 10.1007/s40306-016-0193-9

- [24] Yano K, Kon M. Structures on Manifolds. Singapore: World Scientific; 1984
- [25] Falcitelli M, Ianus S, Pastore AM. Riemannian Submersions and Related Topics. River Edge, NJ: World Scientific; 2004
- [26] Urakawa H. Calculus of variations and harmonic maps. American Mathematical Society. 1993;132

IntechOpen

IntechOpen

