Chapter

Three Solutions to the Nonlinear Schrödinger Equation for a Constant Potential

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Abstract

We introduce three sets of solutions to the nonlinear Schrödinger equation for the free particle case. A well-known solution is written in terms of Jacobi elliptic functions, which are the nonlinear versions of the trigonometric functions sin, cos, tan, cot, sec, and csc. The nonlinear versions of the other related functions like the real and complex exponential functions and the linear combinations of them is the subject of this chapter. We also illustrate the use of these functions in Quantum Mechanics as well as in nonlinear optics.

Keywords: new nonlinear exponential-like functions, superpositions of nonlinear functions, nonlinear optics, nonlinear quantum mechanics

1. Introduction

Since the nonlinear Schrödinger equation appears in many fields of physics, including nonlinear optics, thus, there is interest in finding its solutions, in particular, its eigenfunctions. A set of eigenfunctions, for the free particle, is given in terms of Jacobi's elliptic functions [1–4], which are real periodic functions, and they have been used in order to find the eigenstates of the particle in a box [5, 6] and in a double square well [7].

Jacobi's elliptic functions are needed in subjects like the description of pulse narrowing nonlinear transmission lines [8].

Interestingly, there is a way to linearly superpose Jacobi's elliptic functions by means of adding constant terms to their arguments [3]. So, we ask ourselves if there are other ways to achieve nonlinear superposition of nonlinear functions.

Besides, the linear equation has complex solutions with a current density flux different from zero, and we expect that the nonlinear equation should also have this type of solutions at least for small nonlinear interaction.

In this chapter, we introduce three other sets of functions which are also solutions to the Gross-Pitaevskii equation; they all are nonlinear superpositions of functions. The modification of the elliptic functions allows us to consider the nonlinear equivalent of the linear superposition of exponential, real and complex, and trigonometric functions found in nonrelativistic linear quantum mechanics.

The functions we are about to introduce can be used, for instance, in the case of a free Bose-Einstein condensate reflected by a potential barrier. One might be able to further analyze nonlinear tunneling [7] and nonlinear optics phenomena with the help of these functions.

2. Nonlinear complex exponential functions

The definitions of the functions and their properties are similar to those used in Jacobi's elliptic functions [1, 2, 4]. Let us start with the definition of our complex exponential nonlinear functions:

$$\mathrm{cnc}(u,\alpha) = a\ e^{ix} + b\ e^{-ix}, \qquad \mathrm{snc}(u,\alpha) = a\ e^{ix} - b\ e^{-ix}, \tag{1}$$

$$\mathrm{dnc}(u,\alpha) = \sqrt{1-\alpha|\mathrm{cnc}(u)|^2}, \qquad \mathrm{ncc}(u,\alpha) = \frac{1}{\mathrm{cnc}(u,\alpha)}, \tag{2}$$

$$\mathrm{nsc}(u,\alpha) = \frac{1}{\mathrm{snc}(u,\alpha)}, \qquad \mathrm{ndc}(u,\alpha) = \frac{1}{\mathrm{dnc}(u,\alpha)}, \tag{3}$$

$$\mathrm{tac}(u,\alpha) = \frac{\mathrm{snc}(u,\alpha)}{\mathrm{cnc}(u,\alpha)}, \qquad \mathrm{coc}(u,\alpha) = \frac{\mathrm{cnc}(u,\alpha)}{\mathrm{snc}(u,\alpha)}. \tag{4}$$

where $\alpha, a, b \in \mathbb{R}$, and they are such that $\alpha < 1/\max\left[(a \pm b)^2\right]$. With these choices, the function dnc is always positive, and we do not have to worry about branch points in the relation between the variables x and u. The variables u and x are related as

$$u = \int_0^x \frac{dt}{\sqrt{1-\alpha|\mathrm{cnc}(t,\alpha)|^2}}. \tag{5}$$

A plot of these functions is found in **Figure 1** for a particular set of values of the parameters. These functions behave like the usual superposition of complex exponential functions ($\alpha = 0$), changing behavior as the value of α increases until

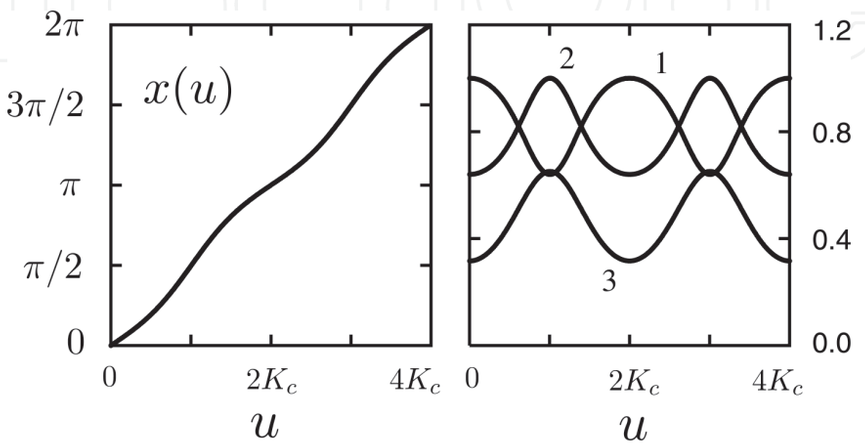

Figure 1.
Nonlinear complex exponential functions with $a = 0.1$, $b = 0.9$, and $\alpha = 0.9$. The curves correspond to 1, $|cnc(u,\alpha)|^2$; 2, $|snc(u,\alpha)|^2$; and 3, $dnc(u,\alpha)$.

it reaches the soliton value, $\alpha = 1/\max\left[(a \pm b)^2\right]$. The functions become concentrated around the origin for the soliton value of α.

The quarter period of these functions is defined as

$$Kc = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - \alpha|\mathrm{cnc}(t,\alpha)|^2}}. \tag{6}$$

If we call

$$n_0 = a^2 + b^2, \qquad n_1 = 1 - 2\alpha(a^2 + b^2), \tag{7}$$

$$n_2 = 1 - \frac{3}{2}\alpha(a^2 + b^2), \qquad n_3 = 1 - \alpha(a^2 + b^2), \tag{8}$$

$$n_4 = 1 - \frac{\alpha}{2}(a^2 + b^2), \qquad n_5 = 1 + \frac{\alpha}{2}(a^2 + b^2), \tag{9}$$

$$n_6 = 1 + \alpha(a^2 + b^2), \qquad n_7 = 1 + \frac{3}{2}\alpha(a^2 + b^2), \tag{10}$$

the squares of the nonlinear functions are written as

$$\mathrm{cnc}^2(u,\alpha) - \mathrm{snc}^2(u,\alpha) = 4ab, \tag{11}$$

$$|\mathrm{cnc}(u,\alpha)|^2 + |\mathrm{snc}(u,\alpha)|^2 = 2n_0, \tag{12}$$

$$\mathrm{dnc}^2(u,\alpha) = 1 - \alpha|\mathrm{cnc}(u,\alpha)|^2 \tag{13}$$

$$= n_1 + \alpha|\mathrm{snc}(u,\alpha)|^2, \tag{14}$$

$$\mathrm{tac}^2(u,\alpha) = 1 - 4ab\ \mathrm{ncc}^2(u,\alpha), \tag{15}$$

$$\mathrm{coc}^2(u,\alpha) = 1 + 4ab\,\mathrm{nsc}^2(u,\alpha). \tag{16}$$

Some derivatives of these functions are

$$\mathrm{cnc}'(u,\alpha) = i\ \ \mathrm{snc}(u,\alpha)\ \ \mathrm{dnc}(u,\alpha), \tag{17}$$

$$\mathrm{snc}'(u,\alpha) = i\ \ \mathrm{cnc}(u,\alpha)\ \ \mathrm{dnc}(u,\alpha), \tag{18}$$

$$\mathrm{dnc}'(u,\alpha) = \alpha\,\Im\{\mathrm{cnc}^*(u,\alpha)\ \ \mathrm{snc}(u,\alpha)\}, \tag{19}$$

$$\mathrm{ncc}'(u,\alpha) = -i\,\mathrm{tac}(u,\alpha)\ \ \mathrm{ncc}(u,\alpha)\ \ \mathrm{dnc}(u,\alpha), \tag{20}$$

$$\mathrm{nsc}'(u,\alpha) = -i\,\mathrm{coc}(u,\alpha)\ \ \mathrm{nsc}(u,\alpha)\ \ \mathrm{dnc}(u,\alpha), \tag{21}$$

$$\mathrm{ndc}'(u,\alpha) = -\alpha\ \ \mathrm{ndc}^2(u,\alpha)\ \ \Im\{\mathrm{cnc}^*(u,\alpha)\ \ \mathrm{snc}(u,\alpha)\}, \tag{22}$$

$$\mathrm{tac}'(u,\alpha) = i\left[1 + \mathrm{tac}^2(u,\alpha)\right]\mathrm{dnc}(u,\alpha) \tag{23}$$

$$\mathrm{coc}'(u,\alpha) = -i4ab\,\mathrm{nsc}^2(u,\alpha)\mathrm{dnc}(u,\alpha), \tag{24}$$

where $\Im$ indicates to take the imaginary part of the quantity.

We also have that the derivative of the inverse functions is given by

$$\frac{d}{dy}\mathrm{cnc}^{-1}(y) = \pm\frac{i}{\sqrt{(y^2 - 4ab)\left(1 - \alpha|y|^2\right)}}, \tag{25}$$

$$\frac{d}{dy}\mathrm{snc}^{-1}(y) = \pm\frac{i}{\sqrt{(4ab+y^2)\left(n_1+\alpha|y|^2\right)}}, \tag{26}$$

$$\frac{d}{dy}\mathrm{ncc}^{-1}(y) = \pm\frac{i}{y\sqrt{(1-4aby^2)\left(1-\alpha/|y|^2\right)}}, \tag{27}$$

$$\frac{d}{dy}\mathrm{nsc}^{-1}(y) = \pm\frac{i}{y\sqrt{(1+4aby^2)\left(n_1+\alpha/|y|^2\right)}}. \tag{28}$$

Now, the second derivatives are as follows

$$\mathrm{cnc}''(u,\alpha) = \Big[2\alpha\ \ |\mathrm{cnc}(u,\alpha)|^2 - n_6\,\mathrm{cnc}(u,\alpha) - 2\alpha ab\ \ \mathrm{cnc}^*(u,\alpha), \tag{29}$$

$$\begin{aligned}\mathrm{snc}''(u,\alpha) = &\Big(\big[3\alpha\left(a^2+b^2\right)-1\big] - 2\alpha|\mathrm{snc}(u,\alpha)|^2\Big)\\ &\mathrm{snc}(u,\alpha) - 2\alpha ab\ \ \mathrm{snc}^*(u,\alpha),\end{aligned} \tag{30}$$

$$\mathrm{dnc}''(u,\alpha) = 2\big[\mathrm{dnc}^2(u,\alpha) - \alpha n_0\big]\mathrm{dnc}(u,\alpha), \tag{31}$$

$$\mathrm{ncc}''(u,\alpha) = n_6\mathrm{ncc}(u,\alpha) - 2\alpha\,\mathrm{cnc}^*(u,\alpha) + 2\alpha ab\,\mathrm{ncc}^2(u,\alpha)\,\mathrm{cnc}^*(u,\alpha), \tag{32}$$

$$\begin{aligned}\mathrm{nsc}''(u,\alpha) = &(n_3 + 8abn_1\mathrm{nsc}^2(u,\alpha))\mathrm{nsc}(u,\alpha)\\ &-\alpha[1+10ab\,\mathrm{nsc}^2(u,\alpha)]\mathrm{snc}^*(u,\alpha),\end{aligned} \tag{33}$$

$$\begin{aligned}\mathrm{ndc}''(u,\alpha) = &2\alpha^2\mathrm{ndc}^3(u,\alpha)\left(\Im\mathrm{cnc}(u,\alpha)\mathrm{snc}(u,\alpha)\right)^2\\ &+2\alpha\left(a^2+b^2\right)\mathrm{ndc}(u,\alpha),\end{aligned} \tag{34}$$

$$\begin{aligned}\mathrm{tac}''(u,\alpha) = &[1+\mathrm{tac}^2(u,\alpha)]\\ &\Big\{\alpha[2\,\mathrm{tac}(u,\alpha) + i\,\Im tac(u,\alpha)]|\mathrm{cnc}(u,\alpha)|^2 - 2\,\mathrm{tac}(u,\alpha)\Big\},\end{aligned} \tag{35}$$

$$\begin{aligned}\mathrm{coc}''(u,\alpha) = &2[1-\mathrm{coc}^2(u,\alpha)]\mathrm{coc}(u,\alpha)\mathrm{dnc}^2(u,\alpha)\\ &-2\alpha ab\left[\frac{\mathrm{cnc}^*(u,\alpha)}{\mathrm{cnc}(u,\alpha)} - \frac{\mathrm{snc}^*(u,\alpha)}{\mathrm{snc}(u,\alpha)}\right]\mathrm{coc}(u,\alpha).\end{aligned} \tag{36}$$

The first three of the above equations can be thought of as modifications of the Gross-Pitaevskii equation, which allows for solutions of the form cnc (u,α), snc (u,α), and dnc (u,α). However, when a or b vanishes, we get the Gross-Pitaevskii form.

With these results at hand, we can see that the probability current densities associated with cnc (u,α) and snc (u,α) are given by

$$\begin{aligned}j_c(u) &= \mathrm{Re}\left\{\mathrm{cnc}^*(u,\alpha)\left[-i\frac{d}{du}\mathrm{cnc}(u,\alpha)\right]\right\}\\ &= \left(a^2-b^2\right)\ \ \mathrm{dnc}(u,\alpha),\end{aligned} \tag{37}$$

$$\begin{aligned}j_s(u) &= \mathrm{Re}\left\{\mathrm{snc}^*(u,\alpha)\left[-i\frac{d}{du}\mathrm{snc}(u,\alpha)\right]\right\}\\ &= \left(a^2-b^2\right)\ \ \mathrm{dnc}(u,\alpha),\end{aligned} \tag{38}$$

respectively. The nonlinear term causes that the quantum flux be no longer constant (as is the case for linear interaction) but modulated by dnc (u,α) instead.

The differential equations for cnc (u,α) and snc (u,α) would have the Gross-Pitaevskii equation form if any of α, a, or b becomes zero or when $a=b$ (which is the case of real functions, i.e., Jacobi's functions). The case of α, a, or b zero corresponds to the cases when there is no nonlinear interaction or when there is total reflection or only transmission in a quantum system.

2.1 The potential step

A straight forward application of the functions introduced in this section is the finding of the eigenfunctions of the Gross-Pitaevskii equation for a step potential:

$$V(u)=\begin{cases}0, & \text{when } u<0,\\ V_0, & \text{when } u\geq 0,\end{cases} \tag{39}$$

and a chemical potential μ larger than the potential height V_0. The Gross-Pitaevskii equation is written as

$$\frac{d^2\psi(u)}{du^2}+\frac{2ML^2}{\hbar^2}(\mu-V_0)\ \psi(u)-\frac{2ML^2}{\hbar^2A^2}NU_0|\psi(u)|^2\psi(u)=0, \tag{40}$$

where $\psi(u)$ is the unnormalized eigenfunction for the Bose-Einstein condensate (BEC), M is the mass of a single atom, N is the number of atoms in the condensate, $U_0=4\pi\hbar^2a/M$ characterizes the atom-atom interaction, a is the scattering length, L is a scaling length, A is the integral of the magnitude squared of the wave function, u is a dimensionless length, μ is the chemical potential, and V_0 is an external constant potential.

For $u<0$ (we call it the region I, $V_0=0$), we use the cnc function with $a=1$, i.e.,

$$\psi_I(u,\alpha)=\mathrm{cnc}(k_Iu,\alpha_I), \tag{41}$$

with parameters

$$k_I^2=\frac{2ML^2\mu}{\hbar^2\left[1+\alpha_I\left(a^2+b^2\right)\right]}, \tag{42}$$

$$\alpha_I=\frac{ML^2NU_0}{\hbar^2A^2k_I^2} \tag{43}$$

From these equations, we obtain

$$\alpha_I=\frac{NU_0}{2\mu A^2-NU_0\left(a^2+b^2\right)}, \tag{44}$$

and

$$\mu=\frac{\hbar^2k_I^2}{2ML^2}+\frac{NU_0}{2A^2}\left(a^2+b^2\right). \tag{45}$$

This last result for μ is in agreement with the conjecture formulated by D'Agosta et al. in Ref. [9], with the last term being the self-energy of the condensate, which is independent of k_I.

For $u>0$, we use the nonlinear plane wave ($a=T$, $b=0$)

$$\psi_{II}(u)=\mathrm{cnc}(k_{II}u,\alpha_{II}), \tag{46}$$

with

$$1+\alpha_{II}T^2=\frac{2ML^2}{\hbar^2k_{II}^2}(\mu-V_0),\qquad \frac{2ML^2NU_0}{\hbar^2A^2k_{II}^2}=2\alpha_{II}, \tag{47}$$

i.e.,

$$\mu=V_0+\frac{\hbar^2k_{II}^2}{2ML^2}+\frac{NU_0}{2A^2}T^2,\quad k_{II}^2=\frac{2ML^2(\mu-V_0)}{\hbar^2\left(1+\alpha_{II}T^2\right)}. \tag{48}$$

By combining the expressions for the αs in both regions, we find that

$$\alpha_Ik_I^2=\alpha_{II}k_{II}^2, \tag{49}$$

and since the chemical potential should be the same on both regions, we also get

$$V_0=\frac{\hbar^2\left(k_I^2-k_{II}^2\right)}{2ML^2}+\frac{NU_0}{2A^2}\left(a^2+b^2-T^2\right). \tag{50}$$

The equal flux condition results in

$$k_I\left(a^2-b^2\right)=k_{II}T^2. \tag{51}$$

Now, equating the functions and their derivatives at $u=0$, we find two relations for the parameters:

$$a+b=T, \tag{52}$$

$$(a-b)k_I\sqrt{1-\alpha_I(a+b)^2}=Tk_{II}\sqrt{1-\alpha_{II}T^2}, \tag{53}$$

i.e.,

$$\frac{k_{II}}{k_I}=\frac{(a-b)}{(a+b)}\sqrt{\frac{1-\alpha_I(a+b)^2}{1-\alpha_I(a+b)^2k_I^2/k_{II}^2}}. \tag{54}$$

We show these values in **Figure 2**. We observe a behavior similar to the linear system; when $\mu\gg V_0$ ($k_{II}\to k_I$), which means very high energies, the step is just a small perturbation on the evolution of the wave.

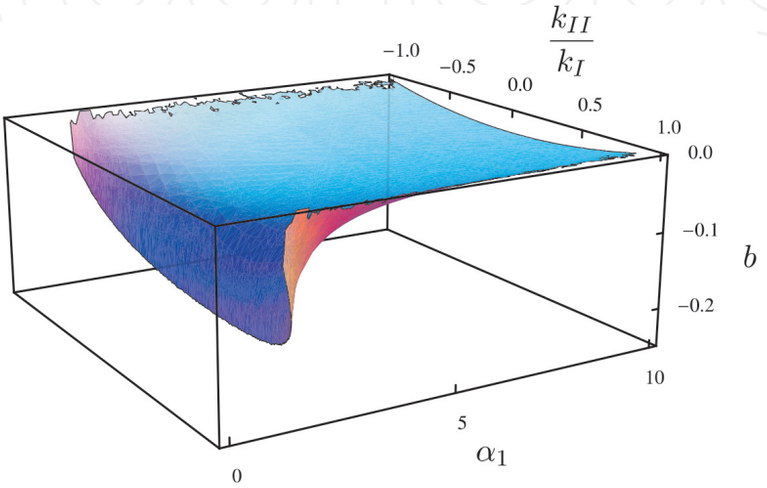

Figure 2.
A three-dimensional plot of the values of k_{II}/k_I for the potential step. Dimensionless units.

3. Nonlinear superposition of trigonometric functions

A second set of nonlinear functions is the nonlinear version of the superposition of trigonometric functions, which is the subject of this section. We only mention some results; more details are found in Ref. [10].

Let us consider the change of variable from θ to u defined by the Jacobian

$$\mathrm{dna}(u) = \frac{d\theta}{du} = \sqrt{1 + \frac{\alpha}{2}\left(a^2 - b^2\right)\cos(2\theta) + \alpha ab \sin(2\theta)}, \tag{55}$$

where $\alpha.a, b \in \mathbb{R}$, and $|\alpha| < 4|ab|/\left(a^2 + b^2\right)^2$, a plot of $4|ab|/\left(a^2 + b^2\right)^2$, is shown in **Figure 3**. Thus, the relationship between θ and u is

$$u = \int_0^{\theta} \frac{d\theta}{\sqrt{1 + \frac{\alpha}{2}\left(a^2 - b^2\right)\cos(2\theta) + \alpha ab \sin(2\theta)}}. \tag{56}$$

We also define the nonlinear functions

$$\mathrm{sna}(u) := a \sin(\theta) - b \cos(\theta), \tag{57}$$

$$\mathrm{cna}(u) := a \cos(\theta) + b \sin(\theta), \tag{58}$$

$$\mathrm{osa}(u) := \frac{1}{\mathrm{sna}(u)}, \quad \mathrm{oca}(u) := \frac{1}{\mathrm{cna}(u)}, \quad \mathrm{oda}(u) := \frac{1}{\mathrm{dna}(u)}, \tag{59}$$

$$\mathrm{csa}(u) := \frac{\mathrm{cna}(u)}{\mathrm{sna}(u)}, \quad \mathrm{sca}(u) := \frac{\mathrm{sna}(u)}{\mathrm{cna}(u)}, \quad \mathrm{dsa}(u) := \frac{\mathrm{dna}(u)}{\mathrm{sna}(u)}, \tag{60}$$

$$\mathrm{dca}(u) := \frac{\mathrm{dna}(u)}{\mathrm{cna}(u)}, \quad \mathrm{sda}(u) := \frac{\mathrm{sna}(u)}{\mathrm{dna}(u)}, \quad \mathrm{cda}(u) := \frac{\mathrm{cna}(u)}{\mathrm{dna}(u)}. \tag{61}$$

A plot of these functions can be found in **Figure 4**, for a set of values of α, a, b. The algebraic relationships between the above functions are

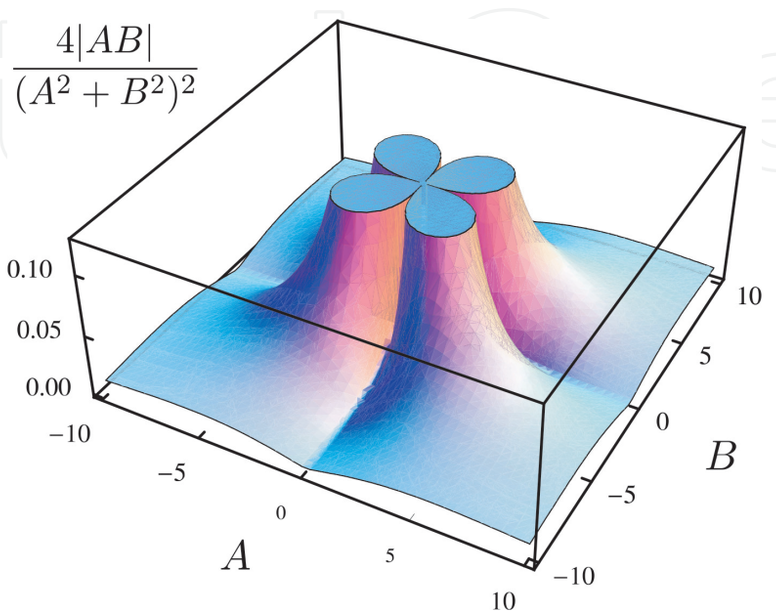

Figure 3.
Three-dimensional plot of $4|ab|/\left(a^2 + b^2\right)^2$.

$$a^2 + b^2 = \mathrm{sna}^2(u) + \mathrm{cna}^2(u), \tag{62}$$

$$\mathrm{dna}^2(u) = 1 - \frac{\alpha}{2}\left(\mathrm{sna}^2(u) - \mathrm{cna}^2(u)\right) \tag{63}$$

$$= n_4 + \alpha\ \mathrm{cna}^2(u) \tag{64}$$

$$= n_5 - \alpha\ \mathrm{sna}^2(u), \tag{65}$$

$$\mathrm{sda}^2(u) = n_0\,\mathrm{oda}^2(u) - \mathrm{cda}^2(u), \tag{66}$$

$$1 - \mathrm{oda}^2(u) = \frac{\alpha}{2}\left[\mathrm{cda}^2(u) - \mathrm{sda}^2(u)\right], \tag{67}$$

$$1 + \alpha\,\mathrm{sda}^2(u) = n_5\,\mathrm{oda}^2(u), \tag{68}$$

$$1 - \alpha\,\mathrm{cda}^2(u) = n_4\,\mathrm{oda}^2(u), \tag{69}$$

$$\mathrm{sca}^2(u) = n_0\,\mathrm{oca}^2(u) - 1, \tag{70}$$

$$\mathrm{dca}^2(u) = n_4\,\mathrm{oca}^2(u) + \alpha, \tag{71}$$

$$\mathrm{csa}^2(u) = n_0\,\mathrm{osa}^2(u) - 1. \tag{72}$$

The derivatives of these functions are

$$\mathrm{sna}'(u) = \mathrm{cna}(u)\ \ \mathrm{dna}(u), \tag{73}$$

$$\mathrm{cna}'(u) = -\mathrm{sna}(u)\ \ \mathrm{dna}(u), \tag{74}$$

$$\mathrm{dna}'(u) = -\alpha\,\mathrm{sna}(u)\mathrm{cna}(u), \tag{75}$$

$$\mathrm{osa}'(u) = -\mathrm{cna}(u)\ \ \mathrm{dna}(u)\mathrm{osa}^2(u), \tag{76}$$

$$\mathrm{oca}'(u) = \mathrm{sna}(u)\ \ \mathrm{dna}(u)\mathrm{oca}^2(u), \tag{77}$$

$$\mathrm{oda}'(u) = \alpha\ \ \mathrm{cna}(u)\ \ \mathrm{sna}(u)\mathrm{oda}^2(u). \tag{78}$$

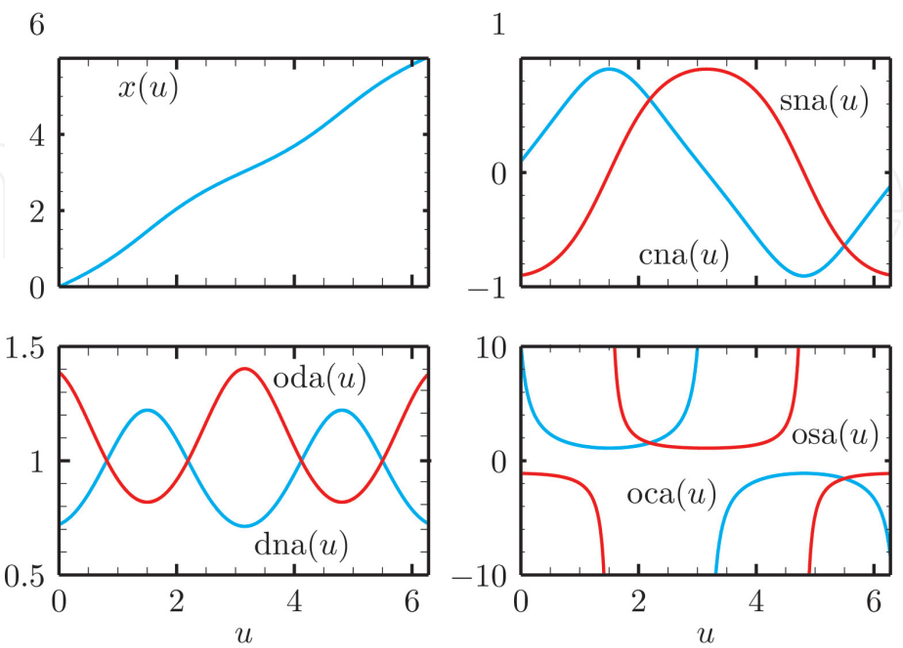

Figure 4.
Plots of the nonlinear functions for $a = 0.1$, $b = 0.9$, and $\alpha = 1.2$. Note that the functions cna and sna have different shapes, and, thus, they are not just the other function shifted by some amount.

Another property is the eliminant equation, also known as energy or Liapunov function,

$$[\mathrm{sna}'(u)]^2 + n_7\,\mathrm{sna}^2(u) - \alpha\ \ \mathrm{sna}^4(u) = n_0 n_5, \tag{79}$$

$$[\mathrm{cna}'(u)]^2 + n_2\,\mathrm{cna}^2(u) + \alpha\ \ \mathrm{cna}^4(u) = n_0 n_4, \tag{80}$$

$$[\mathrm{dna}'(u)]^2 - 2\mathrm{dna}^2(u) + \mathrm{dna}^4(u) = -n_4 n_5, \tag{81}$$

$$[\mathrm{osa}'(u)]^2 + n_7\,\mathrm{osa}^2(u) - n_0 n_5 \mathrm{osa}^4(u) = \alpha, \tag{82}$$

$$[\mathrm{oca}'(u)]^2 + n_2\,\mathrm{oca}^2(u) - n_0 n_4\,\mathrm{oca}^4(u) = -\alpha, \tag{83}$$

$$[\mathrm{oda}'(u)]^2 - 2\,\mathrm{oda}^2(u) + n_4 n_5\,\mathrm{oda}^4(u) = -1. \tag{84}$$

Second derivatives of the functions lead to the differential equations similar to the Gross-Pitaevskii nonlinear differential equation. For sna, cna, and dna, we have that

$$\mathrm{sna}''(u) + n_7\,\mathrm{sna}(u) - 2\alpha\ \ \mathrm{sna}^3(u) = 0, \tag{85}$$

$$\mathrm{cna}''(u) + n_2\,\mathrm{cna}(u) + 2\alpha\ \ \mathrm{cna}^3(u) = 0, \tag{86}$$

$$\mathrm{dna}''(u) + 2\ \ \mathrm{dna}(u)\left[\mathrm{dna}^2(u) - 1\right] = 0, \tag{87}$$

$$\mathrm{osa}''(u) + n_7\,\mathrm{osa}(u) - 2n_0 n_5\,\mathrm{osa}^3(u) = 0, \tag{88}$$

$$\mathrm{oca}''(u) + n_2\,\mathrm{oca}(u) - 2n_0 n_4\,\mathrm{oca}^3(u) = 0, \tag{89}$$

$$\mathrm{oda}''(u) - 2\ \ \mathrm{oda}(u) + 2n_4 n_5\,\mathrm{oda}^3(u) = 0. \tag{90}$$

Quarter period of these functions is defined as

$$\mathrm{K}a(\alpha, a, b) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 + \alpha\left(a^2 - b^2\right)\cos(2t)/2 + \alpha a b\ \ \sin(2t)}}. \tag{91}$$

A plot of Ka (α, a, b) can be found in **Figure 5** for $\alpha = 1.2$.

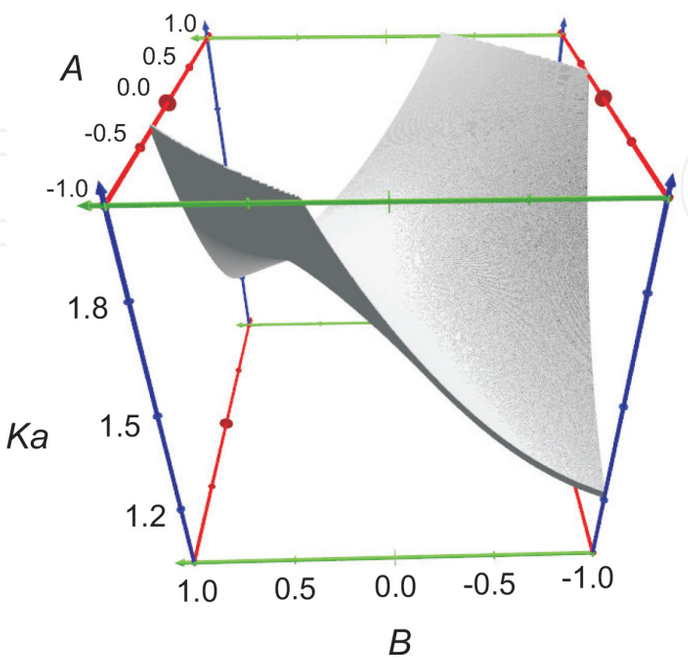

Figure 5.
Some of the values of nonlinear quarter period Ka (α, a, b), for $\alpha = 1.2$.

The derivatives of the inverse functions are

$$\frac{d}{dy}\mathrm{sna}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_0 - y^2)(n_5 - \alpha y^2)}}, \tag{92}$$

$$\frac{d}{dy}\mathrm{cna}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_0 - y^2)(n_4 + \alpha y^2)}}, \tag{93}$$

$$\frac{d}{dy}\mathrm{dna}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_5 - y^2)(y^2 - n_4)}}, \tag{94}$$

$$\frac{d}{dy}\mathrm{osa}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_0 y^2 - 1)(n_5 y^2 - \alpha)}}, \tag{95}$$

$$\frac{d}{dy}\mathrm{oca}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_0 y^2 - 1)(n_4 y^2 + \alpha)}}, \tag{96}$$

$$\frac{d}{dy}\mathrm{oda}^{-1}(y) = \frac{\pm 1}{\sqrt{(n_5 y^2 - 1)(1 - n_4 y^2)}}. \tag{97}$$

Then, as expected, we can see that these functions also invert the same integrals that Jacobi's functions invert.

We also introduce the integral

$$\mathrm{E}a(u) = \int_0^u dv \ \ \mathrm{dna}^2(v) \tag{98}$$

$$= n_5 u - \alpha \int_0^u dv \ \ \mathrm{sna}^2(v) \tag{99}$$

$$= n_4 u + \alpha \int_0^u dv \ \ \mathrm{cna}^2(v), \tag{100}$$

which resembles Jacobi's elliptic integral of the second kind. This function is shown in **Figure 6**, for a set of values of the parameters.

This is the minimum set of properties for these functions. Fortunately, we can still introduce another set of nonlinear functions.

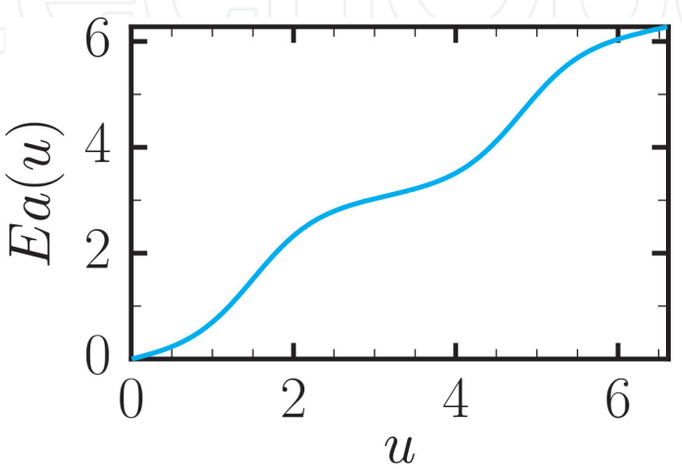

Figure 6.
Plot of Ea (u) for A = 0.1, B = 0.9, and α = 1.2.

4. Nonlinear exponential-like functions

It is possible to define still another set of nonlinear functions inspired on Jacobi's elliptic functions [11]. Let us consider the following set of nonlinear functions of exponential type:

$$\mathrm{pn}(u) = e^{x}, \quad \mathrm{mn}(u) = e^{-x}, \quad \mathrm{fn}(u) = a\ e^{x} + b\ e^{-x}, \tag{101}$$

$$\mathrm{gn}(u) = a\ e^{x} - b\ e^{-x}, \quad \mathrm{rn}(u) = \sqrt{1 + m(a\ e^{x} - b\ e^{-x})^{2}}, \tag{102}$$

$$\mathrm{nf}(u) = \frac{1}{\mathrm{fn}(u)}, \quad \mathrm{ng}(u) = \frac{1}{\mathrm{gn}(u)}, \quad \mathrm{nr}(u) = \frac{1}{\mathrm{rn}(u)}, \tag{103}$$

with u and x related as

$$u = \int_{0}^{x} \frac{dt}{\sqrt{1 + m(a\ e^{t} - b\ e^{-t})^{2}}}, \tag{104}$$

where $a, b \in \mathbb{R}$ and $m > 0$. The required values of a, b, m causes that the radical is positive and then there is no need to consider branching points.

Note that rn $(u) \neq$, rn $(-u)$, and, then, $mn(u)$ is not the mirror image of $pn(u)$, i.e., $mn(u) \neq pn(-u)$ unless $a = b$. A plot of these functions is found in **Figure 7** for a set of values of the parameters a, b, and m. The values of a and b are related to the mirror symmetry between the functions pn (u) and mn (u), being $b = a$ the more symmetric case (which would be the case of Jacobi's elliptic functions with complex arguments). The value of m causes that these functions decay or increase more rapidly with respect to the regular exponential functions. The domain of these functions is finite unless $m = 0$; in fact, increasing the magnitude of x beyond, for instance, ln $\left(10^{4}/2a\sqrt{m}\right)$, does not increase the magnitude of u significantly. One can extend the domain of these functions by setting the value of the function to zero

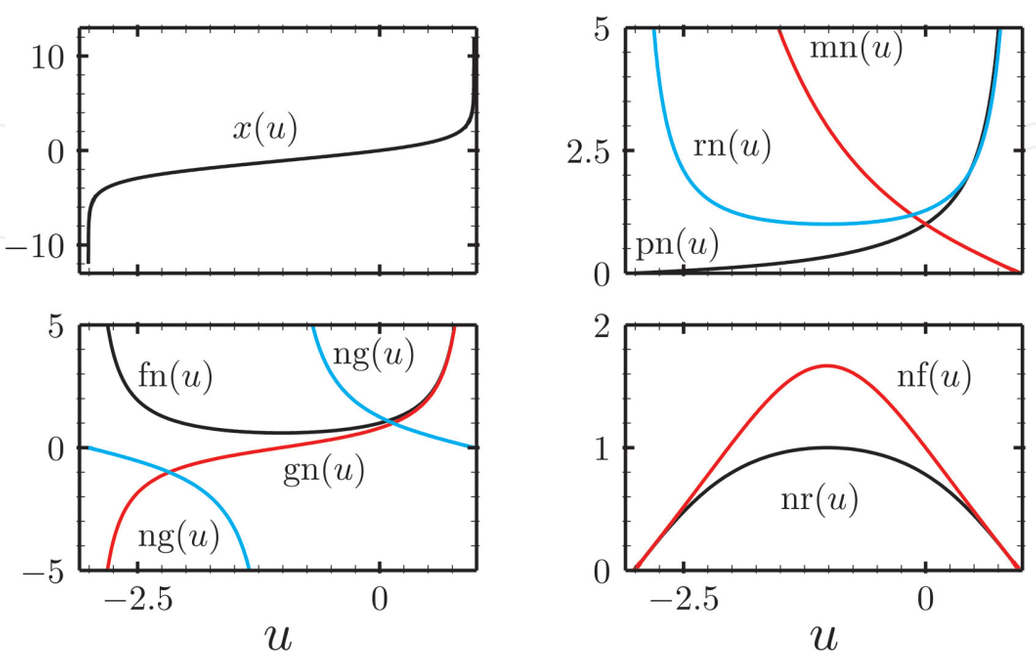

Figure 7.
Nonlinear exponential-like functions for $m = 1$, $a = 0.1$, and $b = 0.9$.

or infinity for larger $|u|$, making them nonperiodic functions on the real axes. We also note that some of these functions are actually bounded.

We can verify easily the following properties which are similar to those for the elliptic functions. The square of these functions are related as

$$4ab = \mathrm{fn}^2(u) - \mathrm{gn}^2(u), \tag{105}$$

$$\mathrm{rn}^2(u) - 1 = m\ \mathrm{gn}^2(u) = m\left[\mathrm{fn}^2(u) - 4ab\right] \tag{106}$$

$$\mathrm{fn}(u)\mathrm{gn}(u) = a^2\mathrm{pn}^2(u) - b^2\mathrm{mn}^2(u), \tag{107}$$

$$\mathrm{fn}^2(u) + \mathrm{gn}^2(u) = 2\left[b^2\ \mathrm{mn}^2(u) + a^2\ \mathrm{pn}^2(u)\right], \tag{108}$$

whereas the derivatives of them are

$$\mathrm{pn}'(u) = \mathrm{pn}(u)\ \mathrm{rn}(u), \qquad \mathrm{mn}'(u) = -\mathrm{mn}(u)\ \mathrm{rn}(u), \tag{109}$$

$$\mathrm{fn}'(u) = \mathrm{gn}(u)\ \mathrm{rn}(u), \qquad \mathrm{gn}'(u) = \mathrm{fn}(u)\ \mathrm{rn}(u), \tag{110}$$

$$\mathrm{rn}'(u) = m\ \mathrm{fn}(u)\ \mathrm{gn}(u), \qquad \mathrm{nf}'(u) = -\mathrm{gn}(u)\ \mathrm{nf}^2(u)\ \mathrm{rn}(u), \tag{111}$$

$$\mathrm{ng}'(u) = -\mathrm{fn}(u)\ \mathrm{ng}^2(u)\ \mathrm{rn}(u), \qquad \mathrm{nr}'(u) = -m\ \mathrm{fn}(u)\ \mathrm{gn}(u)\ \mathrm{nr}^2(u). \tag{112}$$

As we can see from these derivatives, the rate of increase or decrease of the functions is modulated by the rn function; it would be the same as that for the usual exponential functions for the case $m = 0$.

We also have that

$$\frac{d\ \mathrm{pn}^{-1}(y)}{dy} = \frac{1}{\sqrt{y^2 + m(a\ y^2 - b)^2}}, \tag{113}$$

$$\frac{d\ \mathrm{mn}^{-1}(y)}{dy} = -\frac{1}{\sqrt{y^2 + m(a - b\ y^2)^2}}, \tag{114}$$

$$\frac{d\ \mathrm{fn}^{-1}(y)}{dy} = \pm\frac{1}{\sqrt{(y^2 - 4ab)(c_2 + m\ y^2)}}, \tag{115}$$

$$\frac{d\ \mathrm{gn}^{-1}(y)}{dy} = \frac{1}{\sqrt{(y^2 + 4ab)(1 + m\ y^2)}}, \tag{116}$$

$$\frac{d\ \mathrm{rn}^{-1}(y)}{dy} = \pm\frac{1}{\sqrt{(1 - y^2)(c_2 - y^2)}}, \tag{117}$$

$$\frac{d\ \mathrm{nf}^{-1}(y)}{dy} = -\frac{1}{\sqrt{(1 - 4ab\ y^2)[c_2y^2 + m]}}, \tag{118}$$

$$\frac{d\ \mathrm{ng}^{-1}(y)}{dy} = -\frac{1}{\sqrt{(1 + 4ab\ y^2)(y^2 + m)}}, \tag{119}$$

$$\frac{d\ \mathrm{nr}^{-1}(y)}{dy} = -\frac{1}{\sqrt{(1 - y^2)(1 - c_2y^2)}}. \tag{120}$$

As expected, from these derivatives, we can see that these functions also invert the same integral functions that Jacobi was interested on [1, 4].

The second derivatives are

$$\mathrm{pn}''(u) - \mathrm{pn}(u)\left[c_3 + 2ma^2\ \ \mathrm{pn}^2(u)\right] = 0, \tag{121}$$

$$\mathrm{mn}''(u) - \mathrm{mn}(u)\left[c_3 + 2mb^2\ \ \mathrm{mn}^2(u)\right] = 0, \tag{122}$$

$$\mathrm{fn}''(u) - \mathrm{fn}(u)\left[c_1 + 2m\ \ \mathrm{fn}^2(u)\right] = 0, \tag{123}$$

$$\mathrm{gn}''(u) - \mathrm{gn}(u)\left[c_4 + 2m\ \ \mathrm{gn}^2(u)\right] = 0, \tag{124}$$

$$\mathrm{rn}''(u) + 2\mathrm{rn}(u)\left[c_3 - \mathrm{rn}^2(u)\right] = 0, \tag{125}$$

$$\mathrm{nf}''(u) - \mathrm{nf}(u)\left[c_1 - 8abc_2\mathrm{nf}^2(u)\right] = 0, \tag{126}$$

$$\mathrm{ng}''(u) - \mathrm{ng}(u)\left[c_4 + 8ab\ \ \mathrm{ng}^2(u)\right] = 0, \tag{127}$$

$$\mathrm{n}r''(u) - 2\mathrm{n}r(u)\left[c_2\ \ \mathrm{n}r^2(u) - c_3\right] = 0. \tag{128}$$

where

$$c_1 = 1 - 8mab, \quad c_2 = 1 - 4mab, \quad c_3 = 1 - 2mab, \tag{129}$$

$$c_4 = 1 + 4mab. \tag{130}$$

Then, the functions that we have just introduced are solutions of nonlinear second-order differential equations with the one-dimensional Gross-Pitaevskii equation form, for a constant potential and real functions.

Additionally, the energy or Liapunov functions are given by

$$\mathrm{pn}'(u)^2 - \mathrm{pn}(u)^2\left[c_3 + ma^2\ \ \mathrm{pn}^2(u)\right] = mb^2, \tag{131}$$

$$\mathrm{mn}'(u)^2 - \mathrm{mn}(u)^2\left[c_3 + mb^2\ \ \mathrm{mn}^2(u)\right] = ma^2, \tag{132}$$

$$\mathrm{fn}'(u)^2 - \mathrm{fn}(u)^2\left[c_1 + m\ \ \mathrm{fn}^2(u)\right] = -4abc_2, \tag{133}$$

$$\mathrm{gn}'(u)^2 - \mathrm{gn}(u)^2\left[c_4 + m\ \ \mathrm{gn}^2(u)\right] = 4ab, \tag{134}$$

$$\mathrm{rn}'(u)^2 + 2\mathrm{rn}(u)^2\left[c_3 - \mathrm{rn}^2(u)\right] = c_2, \tag{135}$$

$$\mathrm{nf}'(u)^2 - \mathrm{nf}(u)^2\left[c_1 - 4abc_2\mathrm{nf}^2(u)\right] = m, \tag{136}$$

$$\mathrm{ng}'(u)^2 - \mathrm{ng}(u)^2\left[c_4 + 4ab\ \ \mathrm{ng}^2(u)\right] = m, \tag{137}$$

$$\mathrm{nr}'(u)^2 - \mathrm{nr}(u)^2\left[-2c_3 + c_2\ \ \mathrm{nr}^2(u)\right] = 1, \tag{138}$$

where we have made use of the relationships between the squares of the functions. Note that, the functions nf and ng have the same energy, whereas that the functions pn (u) and mn (u) would have the same energy if $b = a$.

Some particular cases are the following. When $4mab = 1$ or $2mab = 1$, we can write down explicit expressions of u in terms of trigonometric, hypergeometric, and exponential functions of x. When $4mab = 1$, we get

$$u = \int_0^x \frac{\sqrt{4ab}\,dx}{\left(ae^{2t} + be^{-2t}\right)} = 2\left[\tan^{-1}\left(\sqrt{\frac{a}{b}}e^x\right) - \tan^{-1}\left(\sqrt{\frac{a}{b}}\right)\right], \tag{139}$$

and when $2mab = 1$, we obtain

$$u = \sqrt{2ab}\int_0^x \frac{dt}{\sqrt{a^2e^{2t}+b^2e^{-2t}}}$$
$$= \frac{1}{\sqrt{2ab}\,ab}$$
$$\times\left\{e^{-2x}\sqrt{b^2e^{-2x}+a^2e^{2x}}\left[b^2 - e^{2x}\left(b^2e^{-2x}+a^2e^{2x}\right){}_2F_1\left(\frac{3}{4},1;\frac{1}{4};-\frac{a^2e^{4x}}{b^2}\right)\right]\right.$$
$$\left.-\sqrt{a^2+b^2}\left[b^2-\left(a^2+b^2\right){}_2F_1\left(\frac{3}{4},1;\frac{1}{4};-\frac{a^2}{b^2}\right)\right]\right\}, \tag{140}$$

where ${}_2F_1$ is the hypergeometric function.

When $a = b = 1$, the nonlinear functions reduce to Jacobi's elliptic functions with complex argument:

$$u = \int_0^x \frac{1}{\sqrt{1+4m\ \sin\mathrm{h}^2(t)}}\,dt = -iF(ix|4m), \tag{141}$$

where F is elliptic integral of the first kind.

This is the minimum set of properties of the exponential-type nonlinear functions.

5. Remarks

Thus, we were able to obtain three sets of nonlinear functions which are solutions to the Gross-Pitaevskii equation. With these functions, we have the nonlinear versions of the trigonometric, real, and complex exponential functions and their linear combinations, and a complete set of functions as in the linear counterpart.

Due to the method of solution, which makes use of elliptic functions, these functions will expand the set of solutions that can be given to polynomial nonlinear equations, in general [8, 12–25].

For instance, a well-known optical phenomenon is the nonlinear dispersion in parabolic law medium with Kerr law nonlinearity [24]. This system is described by a nonlinear Schrödinger equation:

$$i\Psi_t + a\Psi_{xx} + b|\Psi|^2\Psi + c|\Psi|^4\Psi + d\left(|\Psi|^2\right)_{xx}\Psi = 0, \tag{142}$$

where a subindex indicates a derivative with respect to that index. The second term of the above equation represents the group velocity dispersion, the third and fourth terms are the parabolic law nonlinearity, and the last term is the nonlinear dispersion. Some solutions of Eq. (142) were found in Ref. [24]. A solution is the traveling wave, with Jacobi's sn function profile, given by

$$\Psi(x,t) = A\ \mathrm{sn}[B(x-vt),m]e^{i\phi}, \tag{143}$$

$$B = \left(\frac{-bA^2}{am(1+m)-2d(m^2+m+2)A^2}\right)^{1/2}, \tag{144}$$

$$\omega = B^2\left(2dA^2 - a(1+m)\right). \tag{145}$$

where $v = -2ak$ is the velocity, k is the soliton frequency, ω is the soliton wave number, θ is the phase constant, and $0 < m < 1$ is the modulus of Jacobi's elliptic function.

A second solution was given as

$$\Psi(x,t) = A\,\mathrm{cn}[B(x-vt),l]e^{i\phi}, \tag{146}$$

$$B = \left(\frac{b}{4d}\right)^{1/2}, \tag{147}$$

$$\omega = B^2\left(2dA^2 - a\right) - ak^2. \tag{148}$$

Since the functions that we have introduced in these chapters comply with differential and algebraic equations similar to the ones for Jacobi's elliptic functions, we can give additional solutions in terms of these new functions, giving rise to new sets of soliton waves.

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