Chapter 3

# A Formal Perturbation Theory of Carleman Operators

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Additional information is available at the end of the chapter

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**Abstract**

In this chapter, we introduce a multiplication operation that allows us to give to the Carleman integral operator of second class the form of a multiplication operator. Also we establish the formal theory of perturbation of such operators.

**Keywords:** Carleman kernel, defect indices, integral operator, formal series

## 1. Introduction

In this chapter, we shall assume that the reader is familiar with the fundamental results and the standard notation of the integral operators theory [1–3, 5, 6, 8–12]. Let $X$ be an arbitrary set, $\mu$ be a $\sigma-$ finite measure on $X$ ($\mu$ is defined on a $\sigma-$ algebra of subsets of $X$; we do not indicate this $\sigma-$ algebra), and $L_2(X,\mu)$ the Hilbert space of square integrable functions with respect to $\mu$. Instead of writing "$\mu-$ measurable," "$\mu-$ almost everywhere," and "$(d\mu(x))$," we write "measurable," "a e," and "$dx$."

A linear operator $A$: $D(A) \longrightarrow L_2(X,\mu)$, where the domain $D(A)$ is a dense linear manifold in $L_2(X,\mu)$, is said to be integral if there exist a measurable function $K$ on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x,y)f(y)dy \quad \text{a.e..} \tag{1}$$

A kernel $K$ on $X \times X$ is said to be Carleman, if $K(x,y) \in L_2(X,\mu)$ for almost every fixed $x$, that is to say

$$\int_X |K(x,y)|^2 dy \quad < \infty \quad \text{a.e..} \tag{2}$$

An integral operator $A$ (1) with a kernel $K$ is called Carleman operator, if $K$ is a Carleman kernel (2). Every Carleman kernel $K$ defines a Carleman function $k$ from $X$ to $L_2(X, \mu)$ by $k(x) = \overline{K(x, .)}$ for all $x$ in $X$ for which $K(x, .) \in L_2(X, \mu)$..

Now we consider the Carleman integral operator (1) of second class [3, 8] generated by the following symmetric kernel:

$$K(x,y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)}, \tag{3}$$

where the overbar in (3) denotes the complex conjugation and $\left\{\psi_p(x)\right\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X, \mu)$ such that

$$\sum_{p=0}^{\infty} \left|\psi_p(x)\right|^2 < \infty \ \text{a.e.,} \tag{4}$$

and $\{a_p\}_{p=0}^{\infty}$ is a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2 \left|\psi_p(x)\right|^2 < \infty \text{a.e..} \tag{5}$$

We call $\left\{\psi_p(x)\right\}_{p=0}^{\infty}$ a Carleman sequence.

Moreover, we assume that there exist a numeric sequence $\left\{\gamma_p\right\}_{p=0}^{\infty}$ such that

$$\sum_{p=0}^{\infty} \gamma_p \psi_p(x) = 0 \ \text{a.e.,} \tag{6}$$

and

$$\sum_{p=0}^{\infty} \left|\frac{\gamma_p}{a_p - \lambda}\right|^2 < \infty. \tag{7}$$

With the conditions (6) and (7), the symmetric operator $A = (A^*)^*$ admits the defect indices $(1, 1)$ (see [3]), and its adjoint operator is given by

$$A^* f(x) = \sum_{p=0}^{\infty} a_p \left(f, \psi_p\right) \psi_p(x), \tag{8}$$

$$D(A^*) = \left\{ f \in L^2(X, \mu) : \sum_{p=0}^{\infty} a_p \left(f, \psi_p\right) \psi_p(x) \in L^2(X, \mu) \right\}. \tag{9}$$

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Moreover, we have

$$\begin{cases} \varphi_\lambda(x) = \sum_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p(x) \in \mathfrak{N}_{\overline{\lambda}}, & \lambda \in \mathbb{C}, \quad \lambda \neq a_k, \quad k = 1, 2, \dots \\ \varphi_{a_k}(x) = \psi_k(x), \end{cases} \tag{10}$$

$\mathfrak{N}\overline{\lambda}$ being the defect space associated with $\lambda$ (see [3, 4])..

## 2. Position operator

Let $\psi = \{\psi_n\}_{n=0}^{\infty}$ be a fixed Carleman sequence in $L^2(X, \mu)$. It is clear from the foregoing that $\psi$ is not a complete sequence in $L^2(X, \mu)$. We denote by $\mathfrak{L}_\psi$ the closure of the linear span of the sequence $\left\{\psi_p(x)\right\}_{p=0}^{\infty}$:

$$\mathfrak{L}_\psi = \overline{span\{\psi_n, n \in \mathbb{N}\}}. \tag{11}$$

We start this section by defining some formal spaces.

### 2.1. Formal elements

**Definition 1.** *(see* [7]*) We call formal element any expression of the form*

$$f = \sum_{n \in \mathbb{N}} a_n \psi_n, \tag{12}$$

where the coefficients $a_n (n \in \mathbb{N})$ are scalars.

*The sequence $(a_n)_n$ is said to generate the formal element $f$.*

**Definition 2.** *We say that $f$ is the zero formal element, and we note $f = 0$ if $a_n = 0$ for all $n \in \mathbb{N}$.*

*We say that two formal elements $f = \sum_{n \in \mathbb{N}} a_n \psi_n$ and $g = \sum_{n \in \mathbb{N}} b_n \psi_n$ are equal if $a_n = b_n$ for all $n \in \mathbb{N}$.*

If $\varphi$ is a scalar function defined for each $a_n$, we set

$$\varphi\left(\sum_n a_n \psi_n\right) = \sum_n \varphi(a_n) \psi_n, \tag{13}$$

or in another form,

$$\varphi(a_1, a_2, \dots, a_n, \dots) = (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n), \dots). \tag{14}$$

For example, let

$$\varphi(x) = \frac{1}{x}, \quad (x \neq 0). \tag{15}$$

If $a_n \neq 0$ for all $n \in \mathbb{N}$, then the formal element

$$\varphi\left(\sum_n a_n \psi_n\right) = \sum_n \frac{1}{a_n} \psi_n \tag{16}$$

is called inverse of the formal element $f = \sum_n a_n \psi_n$.

Furthermore, we define the conjugate of a formal element $f$ by

$$\bar{f} = \sum_n \overline{a_n} \psi_n. \tag{17}$$

Denote by $\mathcal{F}_\psi$ the set of all formal elements (12).

On $\mathcal{F}_\psi$, we define the following algebraic operations:

the sum

$$\begin{array}{rcccl} + & : & \mathcal{F}_\psi \times \mathcal{F}_\psi & \to & \mathcal{F}_\psi \\ & & \left(\sum_n a_n \psi_n\right) + \left(\sum_n b_n \psi_n\right) & = & \sum_n (a_n + b_n) \psi_n \end{array} \tag{18}$$

and the product

$$\begin{array}{rcccl} \cdot & : & \mathbb{C} \times \mathcal{F}_\psi & \to & \mathcal{F}_\psi \\ & & \lambda \cdot \left(\sum_n a_n \psi_n\right) & = & \sum_n (\lambda . a_n) \psi_n. \end{array} \tag{19}$$

Hence, we obtain a complex vector space structure for $\mathcal{F}_\psi$.

### 2.2. Bounded formal elements

**Definition 3.** *A formal element* $f = \sum_{n \in \mathbb{N}} a_n \psi_n$ *is bounded if its sequence* $(a_n)_n$ *is bounded.*

We denote by $\mathcal{B}_\psi$ the set of all bounded formal elements.

It is clear that $\mathcal{B}_\psi$ is a subspace of $\mathcal{F}_\psi$.

We claim that:

1. $\mathcal{L}_\psi$ is a subspace of $\mathcal{B}_\psi$.

2. Furthermore we have the strict inclusions:

$$\mathcal{L}_\psi \subset \mathcal{B}_\psi \subset \mathcal{F}_\psi. \tag{20}$$

We define a linear form $\langle .,. \rangle$ on $\mathcal{F}_\psi$ by setting:

$$\left\langle \sum_n a_n \psi_n, \sum_n b_n \psi_n \right\rangle = \sum_n a_n \overline{b_n} \tag{21}$$

with the series converging on the right side of (21).

**Proposition 4.** *The form (21) verifies the properties of scalar product.*

**Proof.** Indeed, let

$$f = \sum_n a_n \psi_n, g = \sum_n b_n \psi_n, f_1 = \sum_n a_n^1 \psi_n \text{and } f_2 = \sum_n a_n^2 \psi_n$$

in $\mathcal{F}_\psi$.

We have then:

1. $\langle f, g \rangle = \sum_n a_n \overline{b_n} = \overline{\sum_n a_n \overline{b_n}} = \overline{\langle f, g \rangle}$.

2. $$\langle \lambda f, g \rangle = \left\langle \lambda \left( \sum_n a_n \psi_n \right), \sum_n a_n \psi_n \right\rangle = \left\langle \sum_n (\lambda a_n) \psi_n, \sum_n b_n \psi_n \right\rangle$$
$$= \sum_n (\lambda a_n) \overline{b_n} = \lambda \left\langle \sum_n a_n \psi_n, \sum_n b_n \psi_n \right\rangle = \lambda \langle f, g \rangle.$$

3. $$\langle f_1 + f_2, g \rangle = \left\langle \sum_n (a_n^1 + a_n^2) \psi_n, \sum_n b_n \psi_n \right\rangle$$
$$= \sum_n (a_n^1 + a_n^2) \overline{b_n} = \sum_n a_n^1 \overline{b_n} + \sum_n a_n^2 \overline{b_n} = \langle f_1, g \rangle + \langle f_2, g \rangle.$$

4. $\langle f, f \rangle = \sum_n |a_n|^2 \geq 0 \text{and} \langle f, f \rangle > 0$ if $f \neq 0$.

■

**Remark 5.** *On $\mathcal{L}_\psi$, the scalar product $\langle .,. \rangle$ coincides with the scalar product $(.,.)$ of $L^2(X, \mu)$.*

### 2.3. The multiplication operation

Here, we introduce the crucial tool of our work.

**Definition 6.** *We call multiplication with respect to the Carleman sequence $\{\psi_n\}_n$, the operation denoted "$\circ$" and defined by:*

$$f \circ g = \sum_n \langle f, \psi_n \rangle \langle g, \psi_n \rangle = \sum_n a_n b_n \psi_n, \quad \forall (f, g) \in \mathcal{F}_\psi^2. \tag{22}$$

**Definition 7.** *We call position operator in $\mathcal{L}_\psi$ any unitary self-adjoint (see [1]) operator satisfying*

$$U(f \circ g) = (Uf \circ Ug), \quad \text{for all } f, g \in \mathcal{L}_\psi. \tag{23}$$

The term "position operator" comes from the fact (as it will be shown in the following theorem) that for the elements of the sequence $\psi = \{\psi_n\}_n$, the operator $U$ acts as operator of change of position of these elements.

### 2.4. Main results

**Theorem 8.** *A linear operator defined on $\mathcal{L}_\psi$ is a position operator if and only if there exist an involution $j$ (i.e., $j^2 = Id$) of the set $\mathbb{N}$ such that for all $n \in \mathbb{N}$*

$$U\psi_n = \psi_{j(n)}. \tag{24}$$

**Proof.**

1. It is easy to see that if (24) holds, then $U$ is a position operator.
2. Let $U$ be a position operator. According to 1, we can write

$$U\psi_n = \sum_k \alpha_{n,k} \ \psi_k \ \ \text{with} \sum_k |\alpha_{n,k}|^2 = 1 \tag{25}$$

since $U\psi_n \in \mathcal{L}_\psi$..

On the other hand, we have

$$\sum_k \alpha_{n,k} \ \psi_k = \sum_k \alpha_{n,k}^2 \ \psi_k \tag{26}$$

as

$$U\psi_n = U(\psi_n \circ \psi_n) = U\psi_n \circ U\psi_n.$$

The equalities (26) lead to the resolution of the system:

$$\begin{cases} \sum_n \alpha_{n,k}^2 = 1, \\ \alpha_{n,k}^2 = \alpha_{n,k}, \quad k \in \mathbb{N}. \end{cases} \tag{27}$$

We get then

$$(\forall n \in \mathbb{N}) \ (\exists! k_n \in \mathbb{N}) : \begin{cases} \alpha_{n,k_n} = 1, \\ \alpha_{n,k} = 0 \qquad \forall k \neq k_n. \end{cases}$$

Let us now consider the following application:

$$\begin{array}{cccc} j & : & \mathbb{N} & \rightarrow & \mathbb{N}, \\ & & n & \mapsto & j(n) = k_n. \end{array}$$

It's clear that $j$ is injective.

Now let $m \in \mathbb{N}$. Since $U^2 = I$, then

$$U(U\psi_m) = U\psi_{j(m)} = \psi_{j(j(m))} = \psi_m.$$

Hence,

$$j(j(m)) = m.$$

Finally $j$ is well defined as involution.

■

**Notation** In the sequel, $j(n)$ will be noted by $n^v$. We write

$$U\psi_n = \psi_{j(n)} = \psi_{\overset{v}{n}} \tag{28}$$

and

$$Uf = U\left(\sum_n a_n \ \ \psi_n\right) = \sum_n a_n \ \ \psi_{\overset{v}{n}} = \overset{v.}{f} \tag{29}$$

**Remark 9.** *The position operator $U$ can be extended over $\mathcal{F}_\psi$ as follows:*

*If $f = \sum_n a_n \ \ \psi_n \in \mathcal{F}_\psi$, then*

$$Uf = \overset{v}{f} = \sum_n a_n \ \ \psi_{\overset{v}{n}}. \tag{30}$$

## 3. Carleman operator in $\mathcal{F}_\psi$

### 3.1. Case of defect indices $(1, 1)$

Let $\alpha = \sum_p \alpha_p \psi_p \in \mathcal{F}_\psi$; we introduce the operator $A^{\circ_\alpha}$ defined in $\mathcal{L}_\psi$ by

$$\mathring{A}_\alpha f = \alpha \circ f = \sum_n \langle \alpha, \psi_n \rangle \langle f, \psi_n \rangle \psi_n. \tag{31}$$

It is clear that $\mathring{A}_\alpha$ is a Carleman operator induced by the kernel

$$k(x, y) = \sum \alpha_n \psi_n(x) \overline{\psi_n(y)}, \tag{32}$$

with domain

$$D\left(\mathring{A}_\alpha\right) = \left\{ f \in \mathcal{L}_\psi : \sum_n \left| \alpha_n (f, \psi_n) \right|^2 < \infty \right\}. \tag{33}$$

Moreover, if $\alpha = \overline{\alpha}$, $\mathring{A}_\alpha$ is self-adjoint.

Now *let* $\Theta = \sum_p \gamma_p \psi_p \in \mathcal{F}_\psi$ and $\Theta \notin \mathcal{L}_\psi \left( i.e., \ \sum_p \left| \gamma_p \right|^2 = \infty \right)$. We introduce the following set

$$\mathcal{H}_\Theta = \{ f + \mu\Theta : f \in \mathcal{L}_\psi, \mu \in \mathbb{C} \} \tag{34}$$

which verifies the following properties.

**Proposition 10.** *1.* $\mathcal{H}_\Theta$ *is a subset of* $\mathcal{F}_\psi$.

2. $\mathcal{H}_\theta = \mathcal{L}_\psi \oplus \mathbb{C}\theta$, i.e., direct sum of $\mathcal{L}_\psi$ with $\mathbb{C}\theta = \{ \mu\theta : \mu \in \mathbb{C} \}$.

**Proof.** The first property is easy to establish. We show the uniqueness for the second.

Let $g_1 = f_1 + \mu_1\theta$ and $g_2 = f_2 + \mu_2\theta$, two formal elements in $\mathcal{H}_\theta$. Then

$$g_1 = g_2 \Leftrightarrow f_1 - f_2 = (\mu_2 - \mu_1)\theta.$$

This last equality is verified only if $\mu_2 = \mu_1$. Therefore, $f_1 = f_2$. ■

Denote by $Q$ the projector of $\mathcal{H}_\Theta$ on $\mathcal{L}_\psi$, that is to say: if $g \in \mathcal{H}_\Theta$,

$$g = f + \mu\Theta \ \text{ with } \ f \in \mathcal{L}_\psi \ \text{ and } \ \mu \in \mathbb{C}$$

then

$$Qg = f.$$

We define the operator $B_\alpha$ by:

$$B_\alpha f = Q(\alpha \circ f), f \in \mathcal{L}_\psi. \tag{35}$$

It is clear that

$$D(B_\alpha) = \{ f \in \mathcal{L}_\psi : (\alpha \circ f) \in \mathcal{H}_\Theta \}. \tag{36}$$

**Theorem 11** $B_\alpha$ *is a densely defined and closed operator.*

**Proof.**

1. Since

$$span\{ \psi_n, n \in \mathbb{N} \} \subset D(B_\alpha)$$

and that $\{ \psi_n \}_n$ is complete in $\mathcal{L}_\psi$, then

$$\overline{D(B_\alpha)} = \mathcal{L}_\psi.$$

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2. Let $(f_n)_n$ be a sequence of elements in $D(B_\alpha)$. Checking:

$$\begin{cases} f_n & \rightarrow & f \\ B_\alpha f_n & \rightarrow & g \end{cases} \text{(convergence in the } L^2 \text{ sense)}.$$

We have then

$$B_\alpha f_n = Q(\alpha \circ f_n),$$

with

$$\alpha \circ f_n = g_n + \mu\Theta, g_n \in \mathcal{L}_\psi.$$

Then

$$g_n = \alpha \circ f_n - \mu_n \Theta \in \mathcal{L}_\psi,$$

This implies that

$$\langle g_n, \psi_m \rangle = \alpha_m \langle f_n, \psi_m \rangle - \mu_n \gamma_m \psi_m \quad \forall m \in \mathbb{N}.$$

Or, when $n$ tends to $\infty$, we have

$$g_n \rightarrow g \text{ and } f_n \rightarrow f.$$

Therefore, there exist $\mu \in \mathbb{C}$ such that

$$\lim_{n \to \infty} \mu_n = \mu.$$

And as $Q$ is a closed operator, then we can write

$$(\alpha \circ f) \in \mathcal{H}_\Theta \text{ and } g = Q(\alpha \circ f).$$

Finally $f \in D(B_\alpha)$ and $g = B_\alpha f$.

■

It follows from this theorem that the adjoint operator $B_\alpha^*$ exists and $B_\alpha^{**} = B_\alpha$.

Let us denote by $A_\alpha$ the operator adjoint of $B_\alpha$,

$$A_\alpha = B_\alpha^*. \tag{37}$$

In the case $\alpha = \overline{\alpha}$, the operator $A_\alpha$ is symmetric and we have the following results:

**Theorem 12.** *$A_\alpha$ admits defect indices $(1, 1)$ if and only if*

$$\varphi_\lambda = (\alpha - \lambda)^{-1} \circ \Theta \in \mathcal{L}_\psi. \tag{38}$$

*In this case* $\varphi_\lambda \in \mathcal{N}\overline{\lambda}$ *(defect space associated with* $\lambda$, [3]).

**Proof.** We know (see [3]) that $A_\alpha$ has the defect indices $(1,1)$ if and only if its defect subspaces $\mathcal{N}\overline{\lambda}$ and $\mathcal{N}_\lambda$ are unidimensional.

We have

$$\mathcal{N}_{\overline{\lambda}} = \ker\left(A_\alpha^* - \lambda I\right) = \ker(B_\alpha - \lambda I).$$

So it suffices to solve the system:

$$\begin{cases} B_\alpha \varphi_\lambda = \lambda \varphi_\lambda \\ \varphi_\lambda \in \mathcal{L}_\psi \end{cases}$$

that is,

$$\begin{cases} Q\left(\alpha \circ \varphi_\lambda\right) = \lambda \varphi_\lambda \\ \varphi_\lambda \in \mathcal{L}_\psi \end{cases} \Leftrightarrow \begin{cases} \left(\alpha \circ \varphi_\lambda\right) = \lambda \varphi_\lambda + \mu\Theta, \mu \in \mathbb{C} \\ \varphi_\lambda \in \mathcal{L}_\psi \end{cases}$$

$$\Leftrightarrow \begin{cases} (\alpha - \lambda) \circ \varphi_\lambda = \Theta \\ \varphi_\lambda \in \mathcal{L}_\psi \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi_\lambda = (\alpha - \lambda)^{-1} \circ \Theta \\ \varphi_\lambda \in \mathcal{L}_\psi \end{cases}.$$

■

### 3.2. Case of defect indices $(\mathbf{m}, \mathbf{m})$

In this section, we give the generalization for the case of defect indices $(m, m)$, $m > 1$.

*Let* $\Theta_1, \Theta_2, \ldots, \Theta_m$, $m$ be formal elements not belonging to $\mathcal{L}_\psi$, and let

$$\mathcal{H}_\Theta = \left\{ f + \sum_{k=1}^{m} \mu_k \Theta_k, \quad f \in \mathcal{L}_\psi, \mu_k \in \mathbb{C}, \quad k = 1, \ldots, m \right\}. \tag{39}$$

We consider the operator $B_\alpha$ defined by

$$\begin{aligned} B_\alpha f = Q(\alpha \circ f) \quad f \in D(B_\alpha), \\ D(B_\alpha) = \left\{ f \in \mathcal{L}_\psi : \alpha \circ f \in \mathcal{H}_\Theta \right\} \end{aligned} \tag{40}$$

We assume that $\alpha = \overline{\alpha}$ and we set

$$A_\alpha = B_\alpha^*. \tag{41}$$

By analogy to the case of defect indices $(1, 1)$, we also have the following:

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**Theorem 13.** *The operator* $B_\alpha$ *is densely defined and closed.*

**Theorem 14.** *The operator* $A_\alpha$ *admits defect indices* $(m, m)$ *if and only if*

$$\varphi_\lambda^{(k)} = (\alpha - \lambda) \circ \Theta_k \in \mathcal{L}_\psi, k = 1, \ldots, m. \tag{42}$$

In this case, the functions $\varphi_\lambda^{(k)}(k = 1, \ldots, m)$ are linearly independent and generate the defect space $\mathcal{N}\overline{\lambda}$.

## 4. Conclusion

We have seen the interest of multiplication operators in reducing Carleman integral operators and how they simplify the spectral study of these operators with some perturbation. In the same way, we can easily generalize this perturbation theory to the case of the non-densely defined Carleman operators:

$$\begin{gathered} H(x,y) = K(x,y) + \sum_{j=1}^{m} b_j \psi_j(x) \varphi_j(y), \\ \left(\varphi_j \in L^2(X, \mu), \psi_j \notin L^2(X, \mu), j = \overline{1, m}\right), \end{gathered} \tag{43}$$

with $K(x, y)$ a Carleman kernel.

It should be noted that this study allows the estimation of random variables.

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