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Nonlinear Response on External Electric Field and Nonlinear Generalization of Fluctuation-Dissipation Theorem for Levy Flights

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Additional information is available at the end of the chapter

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Abstract

As well known, the fluctuation-dissipation theorem (FDT) establishes the relation between two different physical phenomena: the fluctuations and the dissipation. The fluctuations or the stochastic motion are determined by random stochastic forces. The dissipation or the directed motion is determined by regular forces. Nevertheless in the linear case, they are related by the FDT. One of the first and well-known examples of the FDT is Einstein's relation between diffusion coefficient and mobility of particle. It has been shown that a particle's velocity depends on electrical field in a nonlinear way in arbitrary weak fields due to anomalous super-diffusion character of Levy flight. The relation between two different critical indexes, describing Levy flight diffusion and dependence of current on electric field, has been established. This relation is the generalization of fluctuation-dissipation theorem for such a nonlinear Levy flight case. The physical interpretation of these results is given.

Keywords: fluctuation-dissipation theorem, nonlinear response, generalization of Einstein relation, random walks, Levy flights, diffusion on self-similar clusters

1. Introduction

As well known, the fluctuation-dissipation theorem (FDT) establishes the relation between two different physical phenomena: the fluctuations and the dissipation. The fluctuations or the stochastic motion are determined by random stochastic forces. The dissipation or the directed motion is determined by regular forces. Nevertheless in the linear case, they are related by the

fluctuation-dissipation theorem (FDT). One of the first and well-known examples of this FDT is Einstein's relation between diffusion coefficient D and mobility of particle η :

$$qD = \eta kT \quad (1)$$

Here T is the temperature of the system, k is Boltzmann's constant, and q is the charge of the particle.

We first recall the well-known Einstein's arguments [1]. Let the diffusion current be J_d and the field current be J_f in the system. In the equilibrium state, the diffusion current is compensated by the field current:

$$J_d + J_f = 0 \quad (2)$$

and the particles are in the equilibrium state and are described by Boltzmann's distribution function:

$$N_{eq} = N_0 \exp\left(-\frac{U}{kT}\right) \quad (3)$$

where U is the potential energy, T is the temperature, and k is Boltzmann's constant, N_0 is the initial number of particles. Let us consider in more details the assumptions, which are used. There are three assumptions:

- i. Boltzmann's statistics
- ii. Fick's law for the diffusion current:

$$J_d = -D\nabla n \quad (4)$$

It also means that the root-mean-square displacement depends on time in a linear way and it is characterized by diffusion coefficient D :

$$\langle X^2(t) \rangle \sim D t \quad (5)$$

- iii. Ohm's law, which describes a linear dependence on electric field

$$J_f = n\eta E \quad (6)$$

Consequently, if one of these above assumptions does not hold, then we expect that Einstein's relation is broken and the new generalized relation will be appeared.

Subsequently, we consider the case, when diffusion has an anomalous power character:

$$\langle X^2(t) \rangle \sim t^k \quad (7)$$

These anomalous stochastic processes were intensively studied [2]. The value $k = 1$ corresponds to the usual ordinary diffusion, the value of exponent $k < 1$ corresponds to the sub-diffusion case, and the value of exponent $k > 1$ corresponds to the super-diffusion or Levy flights case. Usually, anomalous sub-diffusion random walks were observed in disordered materials as fractals and percolations clusters [3–5]. Another anomalous super-diffusion, that is, Levy flights, was observed in the chaotic dynamics problem [6–10].

In this chapter, the Levy flights diffusion in an external weak electric field is considered. The problem consists of that the diffusion coefficient for Levy flight, which is determined in a usual way, has an infinite value:

$$D = \lim_{t \rightarrow \infty} \frac{\langle X^2(t) \rangle}{t} \rightarrow \infty$$

It occurs due to the possibility of diffusing particle to move for an arbitrary distances at every step. So, if we apply the usual Einstein relation (1), then we obtain the infinite value for a mobility of particle η at arbitrary weak fields:

$$\eta \rightarrow \infty$$

But it is not possible to have infinite value of mobility from the physical point of view. What does it means? We believe that it means Einstein's relation in its usual form does not apply. Furthermore, we show that instead of linear response—Ohm's law—another new nonlinear response is appeared in the studied problem. Namely, the drift velocity depends on a weak electric field in a nonlinear way:

$$V \sim E^\nu \quad (8)$$

Here, ν is the critical exponent of new nonlinearity. The relation between the exponent of nonlinearity ν and the exponent of anomalous super-diffusion μ has been established:

$$\nu = \mu - 1 \quad (9)$$

It is necessary to emphasize that this nonlinearity occurs in arbitrary weak fields and it was a consequence of the anomalous Levy super-diffusion. In other words, Ohm's law (the linear response to a field) holds in the case of usual diffusion and Ohm's law does not apply at all for case of Levy flight super-diffusion.

This chapter was organized as follows. In Section 2, the preliminary generalization of Einstein's relation for a Levy flights was obtained. The qualitative estimations for drift velocity in two cases of super diffusion and usual diffusion were obtained too in Section 2. In Section 3, the one-dimensional discrete Levy flight diffusion was studied. The stable non-Gaussian distribution was deduced. The problem of Levy random walks in an external electric field or anisotropic Levy diffusion was studied in Section 4. The numerical simulations of Levy flights in an electric field were presented in Section 5. In Section 6, obtained new results for particle mobility were represented in the scaling form. The fluctuation-dissipation theorem for Levy flight case was rewritten in the scaling form also in Section 7. Section 8 concludes the chapter and the discussion of results was given in this section.

2. Qualitative estimation and generalization of Einstein relation for Levy flight case

Let us briefly remind the Levy flights diffusion. A feature of Levy flight random walks consists of the possibility for a diffusing particle to move on arbitrary large distances at every step, so

that the root-mean-square displacement appears to be infinite. The numerical simulation of Levy hops diffusion has shown that the points, visited during Levy flights diffusion, have formed spatially well-defined clusters. "For more in-depth consideration it makes easy to see that each of clusters consists of a collection of clusters, in turn, so a structure of self-similar clusters was appeared due to Levy flights" [6]. The probability distribution function P in (k, t) -representation is

$$P(k, t) = \exp(-A|k|^\mu t) \quad (10)$$

where A and μ are positive quantities, $1 < \mu < 2$. Such distributions are called as stable Levy distributions. For more information about diffusion, see also [7, 8].

Let us check the above three assumptions for Einstein's relation—formulae (2–4) in the case of anomalous Levy flights super-diffusion. The first assumption about Gibbs-Boltzmann's statistics keeps the same, because the type of statistics—Gibbs-Boltzmann's classical statistics—was determined by the statistical properties of the system in the equilibrium and it does not depend on the kinetic properties of the system. (The kinetic phenomena as relaxation and diffusion describe the processes or ways, which lead to the equilibrium state, only.) So we use Gibbs-Boltzmann's distribution function too. But the second assumption about Fick's law for diffusion current is broken. The diffusion current has another form in the Levy flights case, and we write it in a general operator form:

$$J_d = -\hat{K}n \quad (11)$$

Here, n is the concentration of diffusing particles, the operator \hat{K} in the k -representation is equal to

$$(\hat{K})_k = ik|k|^{\mu-2} \quad (12)$$

And in the r -presentation, it is equal to

$$\hat{K} = \vec{\nabla} |\Delta^2|^{\mu-2/4} \quad (13)$$

where Δ is the Laplace operator and K is the fractional order operator—see, for example [10]. And finally, we use the general form for the field current instead of linear Ohm's law approximation as

$$\vec{J}_f = n \vec{V} \quad (14)$$

where \vec{V} is the drift velocity. In the general case, this velocity depends on electric field in an arbitrary way: a linear or may be a nonlinear way. Repeating the same reasons for equilibrium stated as above, we obtain the general formula for the drift velocity:

$$\vec{V} = \frac{\hat{K}N_{eq}}{N_{eq}} = \exp\left(\frac{U}{kT}\right) \hat{K} \exp\left(-\frac{U}{kT}\right) \quad (15)$$

In the case of the anomalous diffusion, we obtain

$$\hat{K} = \vec{V} |\Delta^2|^{\mu-2/4}$$

By taking a definition for the derivative of the fractional order in the form of the set [10]:

$$\vec{V} = \exp\left(\frac{U}{kT}\right) \lim_{\varepsilon \rightarrow 0} (\Delta^2 + \varepsilon)^{(\mu-2)/4} \vec{p} \exp\left(\frac{U}{kT}\right) \quad (16)$$

we recover that the drift velocity depends on the homogeneous electric field $U = -q \vec{E} \cdot \vec{r}$ in a nonlinear way:

$$\vec{V} \propto \vec{E}^\nu \quad (17)$$

It should be emphasized that this nonlinearity occurs in arbitrarily weak fields, and it was a result of the unusual anomalous character of Levy flights diffusion. The exponent of this nonlinearity relates with the critical exponent of the Levy hop diffusion as above (9): $\nu = \mu - 1$. We consider this relation between two critical exponents, which describe the nonlinear mobility on one hand and the anomalous super-diffusion on the other hand, as generalization of FDT for Levy flight diffusion case.

2.1. Qualitative estimations for particle velocity

Subsequently, we want to confirm the result (17), which was obtained from the phenomenological approach, in another way. For this aim, we consider the problem of diffusion in an electric field in more details. When we introduce the electric field into the diffusion problem, then the new “field” length, governed by external electric field, was appeared:

$$L_E = \frac{kT}{qE} \quad (18)$$

To understand physical sense of this new “field” length and to make necessary estimations for drift velocity, let us imagine that the medium was partitioned into the boxes of size L_E [11]. Further, we proceed the particle motion inside of this box. After leaving this box, the particle goes along the electric field direction with the probability W_+ , which is approximately equal to the unity, ($W_+ \propto 1$) and the particle leaves this box with the approximately zero probability W_- in the opposite direction ($W_- \propto 0$). It means that at these scales of L_E , the directed motion prevails over the random diffusion motion. So we can estimate the particle velocity as follows:

$$V = \frac{L_E}{T_E} \quad (19)$$

where T_E is a diffusion time for a length L_E .

In the case of usual diffusion, this diffusion time equals to: $T_E = \frac{L_E^2}{D}$ and we obtain Ohm’s law with well-known Einstein relation between diffusion and mobility:

$$\vec{V} = \frac{q^2 D \vec{E}}{kT} \quad (20)$$

In the case of Levy flight, the diffusion time is proportional to powers of “field” length: $T_E \propto L_E^\mu$ according to Eq. (10). Repeating the same estimation, we obtain the same nonlinear relation as formula (17):

$$\vec{V} = \frac{q^2 \tilde{D}}{kT} \left| \frac{qE}{kT} \right|^{\mu-2} \vec{E} \quad (21)$$

Here, \tilde{D} is the constant diffusion coefficient for Levy hop diffusion. Correspondingly for a case of two different diffusion regimes, we obtain two different laws for drift velocity: nonlinear behavior (21) and Ohm’s law (20).

We want to stress that these preliminary generalizations of Einstein’s relation in Section 2, see formulae (17, 21), only reveal the possibility of new nonlinear behavior for drift velocity in the anomalous super-diffusion case. To prove this result in an exact way, we need to study the microscopic model.

3. The Levy flight diffusion

To prove the fluctuation-dissipation for Levy flights diffusion case, let us consider the one-dimensional Levy flights diffusion in more details. Briefly, we remind how the Levy stable law (10) for distribution function has been obtained. Let us denote the probability of particle to occupy l -site after n steps as $P_n(l)$ and the probability of hops on length l at every step as $f(l)$. So we obtain the following master equation for a discrete case:

$$P_{n+1}(l) = \sum_{m=-\infty}^{\infty} f(|l-m|) P_n(m) \quad (22)$$

Here, l and m are integer numbers, which describe positions of sites. In the case of usual diffusion, when the particle hops on the nearest (left or right) sites only, this function $f(l)$ is equal to

$$2f(l) = \delta_{l,b} + \delta_{l,-b} \quad (23)$$

where $\delta_{l,b}$ is the Kronecker’s delta symbol. And the known main equation describing diffusion on the nearest sites is received:

$$P_{n+1}(l) = \frac{1}{2} (P_n(l+1) + P_n(l-1)) \quad (24)$$

To simulate a Levy flight, the following Weierstrass function has been used as $f(l)$

$$f(l) = \sum_{n=0}^{\infty} a^{-n} (\delta_{l, -b^n} + \delta_{l, b^n}) \quad (25)$$

Here, parameter b is a length of hop, parameter $\frac{1}{a}$ is a possibility to make hop of length b (e.g., a possibility to hop for a distance b^2 is equal to $\frac{1}{a^2}$ and so on). The value of parameter a is confined by the bound values: $b < a < b^2$, consequently

$$1 < \mu = \frac{\ln a}{\ln b} < 2 \quad (26)$$

Let us shortly discuss the physical picture of Levy flight diffusion. Due to power distribution of hops over the lengths according to (25), the diffusing particle prefers to hop at nearest sites due to the biggest probability for nearest sites, to create the cluster from the nearest visited sites. But there is a small possibility to make a long hop from time to time. After this long hop, the new cluster of another nearest visited sites has formed at new place. So finally, the structure of self-similar clusters appears [6]. So we can say that Levy diffusion is the random walks along self-similar clusters.

Then the structural function for such random walks is equal to

$$\lambda(k) = \int f(l) \exp(ikl) dl = \sum_{n=0}^{\infty} a^{-n} \cos(kb^n) \quad (27)$$

Note too that the structural function of Levy flight satisfies the functional equation:

$$\lambda(k) = a\lambda(kb) + \cos(k) \quad (28)$$

Therefore, for $k \rightarrow 0$, it has a power behavior:

$$\lambda(k) \approx k^\mu, \text{ where } \mu = \ln a / \ln b \quad (29)$$

Exactly, the nonanalytic power behavior for k has been established by means of Mellin's transformation or by formulas of Poisson's set summation. In more detail, see [7].

4. Introduction of field in the Levy flight problem and nonlinear response on electric field

Let us introduce an anisotropy into the random walk on self-similar clusters, formed during Levy flights diffusion. By virtue of specific nature of Levy hops, a particle can move for an arbitrary distance b^n at every step. For this reason, a small anisotropy $(1 + \alpha)$ for small displacements s (with $\alpha = \frac{qEs}{kT}$) has an exponential large value at large distances b^n as $(1 + \alpha)^{b^n}$. Since at each step, a diffusing particle certainly leaves a site, so the sum of probability of motion along the electric field direction W_+ and probability of motion in opposite direction W_- must be equal to 1:

$$W_- + W_+ = 1 \quad (30)$$

Hence, we obtain the following expressions for these probabilities:

$$W_{\pm} = \frac{(1 \pm \alpha)^{b^n}}{(1 + \alpha)^{b^n} + (1 - \alpha)^{b^n}} \quad (31)$$

Therefore, the structural function $\lambda(k; E)$ for Levy flights diffusion in an electrical field is equal to

$$2\lambda(k, E) = \sum a^{-n} [\cos(kb^n) + i \sin(kb^n)(W_+ - W_-)] \quad (32)$$

As well as for the usual ordinary diffusion, the second member with anisotropy for small $k \rightarrow 0$ contains the expression for the drift velocity:

$$V = i \frac{\partial \lambda(k, E)}{\partial k} \Big|_{k \rightarrow 0} = \sum \left(\frac{b}{a}\right)^n \frac{(1 + \alpha)^{b^n} - (1 - \alpha)^{b^n}}{(1 + \alpha)^{b^n} + (1 - \alpha)^{b^n}} \approx \sum \left(\frac{b}{a}\right)^n th(\alpha b^n) \quad (33)$$

here $th(x)$ is the hyperbolic tangent.

It is easy to see that the drift velocity satisfies the following functional equation:

$$V(\alpha) = \frac{b}{a} V(\alpha b) + c th(\alpha) \quad (34)$$

It means that at weak fields $\alpha \rightarrow 0$, the velocity depends on the electric field in a power-like way:

$$V(\alpha) \propto \alpha^\nu \quad (35)$$

with exponent $\nu = (\mu - 1)$.

To calculate the velocity by exact way, we used Poisson's formula:

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int f(t) dt + 2 \sum_{m=1}^{\infty} f(t) \cos(2\pi m t) dt \quad (36)$$

After calculations, we obtain the formula for the velocity:

$$V(\alpha) = \frac{\alpha}{2} + \alpha^{(\mu-1)} \left[\sum_{m=-\infty}^{\infty} \int th(z) z^{-\gamma_m} dz \right] + \int_0^{\alpha} th(z) z^{-\gamma_m} dz \quad (37)$$

where a power exponent is equal to $\gamma_m = \mu + 2\pi m i / \ln b$. It is easy to see that for arbitrary weak fields α , the first term has been neglected in comparison with the second term in the brackets. Thus, in arbitrary weak electric fields, the nonlinear dependence on electrical field of velocity (35) has appeared.

5. Numerical simulations

Subsequently, the results of numerical simulations of Levy random walks were reported. Let us briefly explain the algorithm of simulations. Probabilities of left and right walks are determined as probabilities to have a random value from $[0, 0.5]$ and $[0.5, 1]$ correspondingly. The anisotropy of random walks is simulated by the decreasing length of $[0, 0.5]$ for quantity W_- anti-parallel field and increasing $[0.5, 1]$ for quantity W_+ in parallel field case. The simulations are made at different values of parameters a and b . As the probability a^{-n} decreases rather rapidly, so we can confine finite members in the sum (10). For example, at $a = 50$, $b = 10$, $n = 6$ and $a = 6$, $b = 3$, $n = 12$. But we proceed that at every hop, the sum of all probabilities with finite numbers of hops is equal to 1, that is, particle does not stay in the site.

The results of random walks, **Figure 1**, are in accordance with the known results [2].

The step-like dependence of rms as a function of time is easy to understand as follows. The particle diffuses at nearest sites mainly, making the cluster from visited sites, and with a small probability hops at big distance (at next step) and again diffuses at nearest sites and so on.

The electric field leads to the particle drift. The dependence of the average displacement $\langle X \rangle$ as function of the time is represented in **Figure 2** at different values of anisotropy. From linear dependence, it is easy to find the particle velocity by standard way: $V = \langle X \rangle / N$. The value of the nonlinear dependence index is determined from numerical simulation data as.

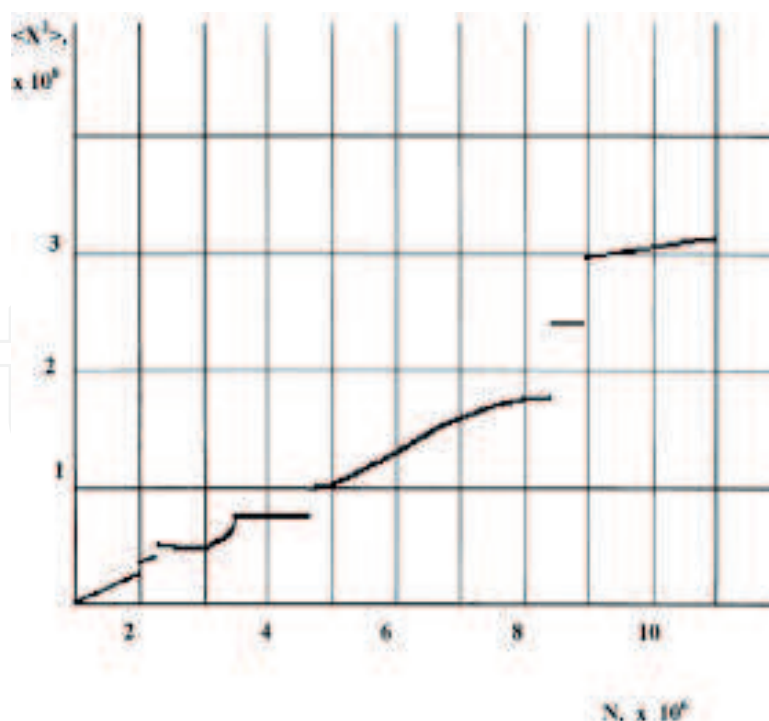


Figure 1. Typical dependence of RMS for Levy flight.

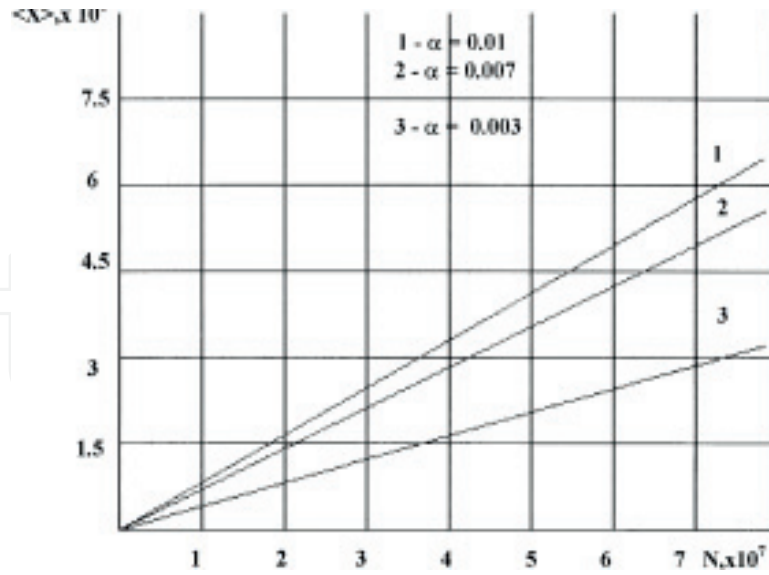


Figure 2. Dependence of the average displacement $\langle X(t) \rangle$ on number of hops N .

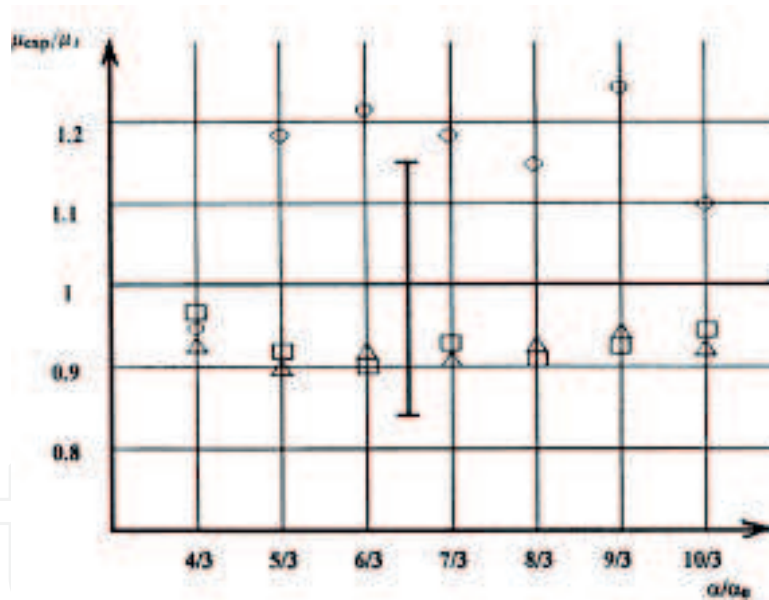


Figure 3. Dependence of relation $\mu_{\text{exp}} / \mu_{\text{theor}}$ at different values of anisotropy.

$$\mu_{\text{exp}} = 1 + \ln(V/V_0) / \ln(\alpha/\alpha_0)$$

These results are represented in **Figure 3**. The main distortion in the simulations is due to the random character of walks, and it was checked in the calculations from values of average displacement at zero fields.

6. Transition from nonlinear response to Ohm's law

6.1. Transition from Levy super-diffusion to ordinary diffusion

In this section, we additionally introduce the usual diffusion on the nearest neighboring sites in the process of random Levy walks. It gives us the possibility to proceed the transition from Levy super-diffusion to the usual diffusion. For this aim, the finite hop length ξ at every step has been introduced. So we construct the complex random walks, in which the Levy super-diffusion alternates with the usual diffusion. Accordingly, the distribution function for lengths of hops has the following form:

$$f(l, \xi) = \sum_{n=0}^{\infty} a^{-n} (\delta_{l, -(b^n + \xi)} + \delta_{l, b^n + \xi}) \quad (38)$$

Hence, the structural function for complex random walks with Levy diffusion and ordinary diffusion is equal to

$$\lambda(k, \xi) = \sum a^{-n} \cos(kb^n + k\xi) \quad (39)$$

In the case of complex alternative diffusion, the main contribution to the root-mean-square displacement was provided by Levy flights on long times, corresponding to big scales. Correspondingly, on small times and at small scales, the main contribution was provided by the usual diffusion. In the limit of the small lengths of hops $b \ll \xi$, we obtain the formula for structural function, which corresponds to the usual diffusion:

$$\lim_{b \rightarrow 0} \lambda(k, \xi) = \frac{a}{a-1} \cos(k\xi) \quad (40)$$

We consider this transition $b \ll \xi$ as transition from the discrete medium to the continuous medium with heterogeneity length as ξ . It is easy to check that the usual diffusion equation has followed from this structural function as a result.

6.2. The drift in the case of both ordinary and Levy diffusion

Let us introduce the anisotropy into these complex random walks as described earlier, but now we replace the hop length b^n to the new quantity: $b^n + \xi$. After this replacement, we obtain the formula for the new structural function in an electric field with finite hop length:

$$2\lambda(k, \xi, E) = \sum_{n=0}^{\infty} a^{-n} [\cos(kb^n + k\xi) + i \sin(kb^n + k\xi)(W_+ - W_-)] \quad (41)$$

Accordingly, the velocity has been described by the following formula:

$$V = i \frac{\partial \lambda(k, \xi, E)}{\partial k} \Big|_{k \rightarrow 0} = \sum \left(\frac{b^n + \xi}{a^n} \right) \tanh(ab^n + a\xi). \quad (42)$$

To calculate this sum in formula (42), Poisson's method of summation has been used again.

The following results have obtained. For weak electric fields ($\frac{qE\xi}{kT} \ll 1$), the velocity is a nonlinear function of the electric field:

$$V \sim E^{\mu-1} \quad (43)$$

and in the strong fields ($\frac{qE\xi}{kT} \gg 1$), the velocity became a linear function in the field:

$$V \sim E\xi^{2-\mu} \quad (44)$$

Note that the particle velocity has two asymptotic regimes in accordance with the diffusion limits: Levy hops and usual ordinary diffusion. The Levy flight diffusion leads to the nonlinear response, and the usual diffusion leads to the linear Ohm's law. So the two different power dependencies of particle mobility (43, 44) were obtained for a specific distribution of hops as (38). But before this result was obtained without any assumptions about the nature of hops, only specific form of Levy diffusion current was used as (11). And now, we consider the specific distribution of hops (38) only as microscopic model. We believe that the same nonlinear result will be correct for another hops distribution over lengths.

7. Scaling for particle mobility

We want to remark that above results look similar to the phase transition theory results [12, 13]. First of all, we have the analog of correlation radius for phase transition L_c —in our case, this is the finite length of hop ξ . At scales, which bigger than ξ , we have anomalous super-diffusion and at scales, which are smaller than ξ , we have the usual diffusion. So this length ξ has a role of heterogeneous scale as correlation radius. Second as it is well known that if the correlation radius L_c trends to the infinity at the phase transition point (at threshold), then any characteristic scales in the phase transition theory at threshold are absent, so any response for external fields has the power behavior, which is described by the critical exponents of phase transition theory. Near threshold point, results of the phase transitions theory were easy to understand if they have the scaling form. So we want to present the above-obtained results in the general scaling automodel form too, using the finite hop length ξ instead of correlation length L_c .

So to clarify the obtained results, the expression for the particle mobility $\eta = \frac{V}{E}$ has been rewritten in the scaling form too:

$$\eta \propto \xi^{-\lambda} f\left(\frac{qE\xi}{kT}\right) \quad (45)$$

where λ is the critical exponent of scaling, and the scaling function $f(x)$ has the asymptotic power behavior:

$$f(x) = \begin{cases} 1, & x \ll 1 \\ x^\lambda, & x \gg 1 \end{cases} \quad (46)$$

For our model of Levy flights diffusion, this scaling exponent λ is connected with the exponent of the super-diffusion as

$$\lambda = \mu - 2 \quad (47)$$

At the small scales $\xi \ll \frac{kT}{qE}$, where the usual diffusion dominates, the particle mobility depends on the homogeneity length ξ only (correlation radius in the phase transitions theory). At the large scales, where the Levy super-diffusion dominates, the mobility depends on the electric field E only or, in other words, the mobility became a function of the new "field" length $L_E = \frac{kT}{qE}$ with the same exponent λ (see formula (42) too).

7.1. The scaling form for fluctuation-dissipation theorem for nonlinear case

Usually, the Einstein relation between diffusion and conductivity was considered as a simple example of fluctuation-dissipation theorem (FDT), which was connected by the different characteristics of the considered system: the dissipation, described by the relaxation time τ (the particle mobility $\eta = \frac{q\tau}{m}$), and the fluctuation characteristic, described by the diffusion coefficient D :

$$qD = \frac{q\tau}{m}kT = \eta kT \quad (48)$$

We want to stress that this obtained nonlinearity (43) essentially differs from the usual nonlinearity, and our result means that the relation between the nonlinear mobility and the coefficient of diffusion existed in the new nonlinear form, when the mobility became as nonlinear function of the electric field

$$\eta(E) \sim E^\lambda \quad (49)$$

Here, λ is exponent of the nonlinear dependence of mobility. And new nonlinear generalized fluctuation-dissipation theorem relates the exponent of the nonlinear response λ with the exponent of the anomalous diffusion μ :

$$\lambda = \frac{d \ln \eta(E)}{d \ln E} = \mu - 2 \quad (50)$$

It seems that this investigated case was a first case when the fluctuation-dissipation theorem in the usual form of linear relation between two coefficients was broken. And instead of simple relation between linear coefficients, the new and more general relation between exponents of mobility and exponent of the super-diffusion appeared.

From this point of view, we believe that the case of usual diffusion or Einstein's relation between two coefficients of diffusion and mobility is the limiting case of new generalized

FDT between exponents of mobility of particle in an electric field and exponent of diffusion:
 $\lambda = 0$ ($\mu = 2$).

8. Discussion

Let us discuss the results. All the above obtained both the nonanalytic behavior of structural function for small $k \rightarrow 0$ and the nonlinear electric field dependence of the velocity in arbitrarily weak fields which were the asymptotical results. We show that the current (velocity) depends on electric field in a nonlinear way due to the anomalous character of Levy flights and possibility to fly at arbitrary distances:

$$J(E) \sim E^\nu \quad (51)$$

Nonlinear properties of media intensively have been studied. Usually, the nonlinearity has been connected with the expansion of electric current for set in powers of the electric field and with consideration of the cubic nonlinearity [14]:

$$\vec{J} = \sigma \vec{E} + \chi |E|^2 \vec{E} + \dots \quad (52)$$

But our result essentially differs from the results, obtained by this method. We show that in the investigated case of Levy super-diffusion, the nonlinear behavior appeared due to anomalous super-diffusion character and the electric current depends on electric field in a power nonlinear way. It means that Ohm's law or a linear term was absent in the field series expansion of the current (58) in the investigated case.

The generalization of fluctuation-dissipation theorem for a case of Levy flights diffusion was obtained. Instead of well-known Einstein's relation between diffusion coefficient D and mobility η , which is correct in linear Ohm's law case, the new relation between exponents, which describes the nonlinear response of system ν on the hand and anomalous Levy flight diffusion μ on the other hand, was obtained:

$$\nu = \mu - 1$$

It is interesting to note that from the above-obtained results, we understand what two results were contained in Einstein's relation (1). Firstly, we can say that Einstein recovers or proves the existence of Ohm's law (linear response) for any systems with usual diffusion, and secondly, he established the relation between diffusion coefficient and mobility of particle in a linear case.

As for "real" systems, the different theories with different predictions have been existed and numerical simulations have not given a clear answer yet: the non-monotonically dependence with time were founded [15, 16]. We hope that these results may be applied for real disordered systems and in particular also for the problem of hopping in the disordered systems, but we need to make further investigations for it [17].

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References

- [1] Einstein A. Investigations on the Theory of Brownian Movement. New York: Dover; 1956
- [2] Metzler R, Klafter J. The random walks guide to anomalous diffusion: A fractional dynamics approach. *Physics Reports*. 2000;**339**:1-77
- [3] Metzler R, Klafter J. Anomalous stochastic processes in the fractional dynamics framework: Fokker-Planck equation, dispersive transport, and non-exponential relaxation. *Advances in Chemical Physics*. 2001;**116**:223-264
- [4] Isichenko MB. Percolation, statistical topography and transport in random media. *Reviews of Modern Physics*. 1992;**64**:961-1043
- [5] Arkhincheev VE, Baskin EM. Anomalous diffusion and drift in a comb model of percolation clusters. *Soviet Physics - JETP*. 1991;**73**:161-165
- [6] Mandelbrot B. *Fractals: Form, Chance and Dimension*. San-Francisco: Freeman; 1977
- [7] Hughes BD, Shlesinger MF, Montroll EW. Random walks with self-similar clusters. *Proceedings of the National Academy of Sciences of the United States of America*. 1981;**78**:3287-3291
- [8] Barkai E. Fractional Fokker-Planck equation, solution, and application. *Physical Review E*. 2001;**63**:046118
- [9] Uchaikin VV. Anomalous diffusion and fractional stable distributions. *JETP*. 2003;**97**:810-825
- [10] Arkhincheev VE, Baskin EM, Batiev EG. Diffusion and conductivity in percolation systems. *Journal of Non-Crystalline Solids*. 1987;**90**:21-24
- [11] Batyev EG. Private communications
- [12] Pokrovskii VL, Patashinskii AZ. *Fluctuation Theory of Phase Transitions*. Moscow: Nauka; 1982

- [13] Stanley HE. Introduction to Phase Transitions and Critical Phenomena. USA: Oxford University Press; 1987
- [14] Klimontovich Y. Nonlinear Brownian motion. Uspekhi Fiziologicheskikh Nauk. 1994;**164**: 811-844
- [15] Overhoff H, Beyer W. Philosophical Magazine B. 1981;**43**:433-438
- [16] Benjamin J, Adkins C, van Clever J. Journal of Physics C. 1981;**17**:559-564
- [17] Arkhincheev VE. Hopping by Levy flights and nonlinear relation between diffusion and conductivity. SPIE Proceedings. 2004;**5471**:560-567