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Practical Finance Strategies in Immunization

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<http://dx.doi.org/10.5772/intechopen.78719>

Abstract

My first goal is to present the basic immunization problem (BIP) as it is understood in finance. BIP relies on a construction of such a bond portfolio (BP), meaning a selection of individual bonds, that the single liability to pay L dollars q years from now will be discharged by means of BP (a patient will return to health at time q), no matter what random shift $a(t)$ (a particular disease) will occur in the future. What kind of a function is a shift of interest rates is critically important because both present and future values of BP depend solely on underlying interest rates. Having identified shifts (diseases) against which a BP is immunized, the natural question arises how to find among such immunized (immune) portfolios the best ones. In the context of finance, it means bond portfolios with maximal unanticipated rate of return. My second goal is to trigger interest among medical scientists by suggesting that certain finance notions, such as duration and convexity of a bond portfolio, might give extra insight to medical researchers working in the immunization area both into BIP and into similar problems in medicine. A considerable attention is also paid to certain mathematical notions (base of a linear space, a Hilbert space, triangular functions) because of their successful applications to problem-solving occurring in bond portfolio immunization.

Keywords: immunization, immunity, active immunity, passive immunity, best immunization strategy, duration, convexity, barbell immunization strategy, focused immunization strategy

1. Introduction

In this chapter, I present one of the research areas existing in finance called bond portfolio immunization (BPI). My goal is to make it known to medical researchers dealing with immunity (resistance) of human organisms to diseases. I feature not only basic notions, problems,

and solutions occurring in BPI, but also selected mathematical concepts and tools which proved to be instrumental in developing BPI. I do believe that such information has a good chance to be useful in creation of immunity against particular diseases. Bond investors are called immunizers if, possessing C dollars today, they must achieve an investment goal of L dollars q years from now (a human organism or a particular human organ must achieve a certain level of health q years from now); here L is the future value of C at time q under the current interest rates. This investment goal must be accomplished by means of an appropriately selected bond portfolio, even despite unfavorable sudden change (shift) in interest rates (appearance of a disease), having in mind that the present and future prices of all bonds depend solely on interest rates.

Although, as it will be demonstrated in Sections 3.1, 3.2, and 3.3, immunization against all shifts is never possible, there are many results giving sufficient, or necessary and sufficient, conditions for immunization against a certain classes of shifts (certain diseases). It is worth to know that in the financial immunization, there is no such thing as *acquired* immunity (immunity that develops in a human after exposure to a suitable agent) and *active* immunity (acquired through production of antibodies within the organism in response to the presence of antigens).

Such types of immunization might theoretically take place on a bond market only if a bond holder had the right to change the coupon payments, which is completely out of the question. In other words, the immunization in financial reality has features of *passive* immunity, being in fact a short-acting immunity. On the other hand, however, a BP manager can achieve the state of a BP being all the time immune against a specific class of shifts, provided the manager regularly (every week or so) performs (if necessary) subsequent adjustments of his/her BP according to their expertise in the area of immunization theory.

Theorem 1 (Section 2.4.2) as well as Theorems 3 and 4 allows one to look at immunization from a different perspective. They enable one to identify all shifts $a(t)$ (diseases) of the term structure $s(t)$ of interest rates against which BP is already (fully) immunized, that is, protected against loss of its value at time q . Finally, having identified immunized (immune) bond portfolios, the natural question arises how to find among them the best ones. This topic is dealt with in Section 4.

Below, I shall present (i) what are bonds and bond portfolios; (ii) what is meant by standard (and general) immunization problem; (iii) historical development of immunization theory; (iv) overview of some recent results; (v) the concept of a Hilbert space, and a base in a linear space; (vi) application of orthogonal polynomials to description of the class IMMU of all shifts (diseases) against which a given bond portfolio is immunized; (vii) triangular functions as a base for the linear space IMMU; and (viii) the crucial role of the notions of duration and convexity in choosing the “best” immunized (immune) bond portfolios.

2. Immunization in finance

Below, we will introduce the concept of bonds, formulate the standard and general immunization problem, and outline the development of immunization theory in finance, from the beginning to the latest achievements.

2.1. What are bonds?

Each bond with a face value (par value) of F dollars is a financial instrument which generates to its buyer (holder) specified payments every 6 months (or every quarter, every year) in the form of coupons, plus par value paid only at the termination (maturity) of the bond. The face value represents the amount borrowed by the seller (issuer) of a bond from the bond buyer. The coupons represent a predetermined percentage, say 3%, of face value F ; if $F = \$10,000$, then all coupons paid per year sum up to \$300. Each bond has its own life span (maturity) of n years (3, 5, 10, 20 years, etc.)

A bond portfolio (BP), by its definition, is a collection of different bonds with various maturities. Thus, each BP generates a more complicated cash flow pattern than a single bond does. A cash flow generated by a BP consists of various size payments c_i , $1 \leq i \leq m$, (coupons and par values generated by all kinds of bonds forming that portfolio) at certain dates $t_1, t_2, t_3, \dots, t_m$ from an interval $[t_0, T]$, where t_0 is the date when BP was purchased, while T stands for the highest maturity of all bonds tradable on a given debt market D .

The present and future value of each bond, and consequently each bond portfolio, depends solely on current interest rates $s(t)$, which in the simplest case are identical for all maturities t , that is, $s(t) \equiv s$, $t \in [t_0; T]$. By the term structure of interest rates, one understands a schedule of spot interest rates $s(t)$ which are estimated from the yields (returns) of all coupon-bearing bonds. It is well known that interest rates are shaped under various random market forces.

2.2. Standard and general immunization problem

The standard immunization problem relies on a construction of such a bond portfolio with the present value of C dollars that the single liability to pay L dollars q years from now (L is the future value of C) by means of the cash flow generated by BP will be secured regardless of how adverse changes in interest rates will occur in a future. This nontrivial problem is automatically solved by each zero-coupon bond maturing at time q with par value of L dollars. Thus, using medical terminology, one may say that such a zero-coupon bond possesses an innate (natural) immunity. Unfortunately, in practice, such zero-coupon bonds rarely exist.

Besides, an investor may already possess bonds and would like to buy additional ones so that the created, in this way, portfolio BP with the present value of C dollars would secure the payment of L dollars q years from now. Having built such a portfolio, the investor would immunize (hedge) their own investment against a loss of its value at time q . We assume that the new term structure will always be of the form $s^*(t) = s(t) + a(t)$, where $a(t)$ belongs to a certain class of shifts (diseases).

On the other hand, the general immunization problem relies on a construction of such a bond portfolio BP with the present value of C dollars that multiply liabilities to pay L_i dollars at specified instances of time will be secured by means of the cash flow generated by BP regardless of adverse changes/shifts $a(t)$ of interest rates in a future.

2.3. Beginnings of immunization

Immunization as a concept dates back as far as to articles [1, 2]. However, not until work [3] of Fisher and Weil was the impact of interest shifts on the design of immunization strategies

rigorously studied. In a vast majority of publications, immunization was based on a specific stochastic process governing interest rate shifts $a(t)$. In [2], Redington discussed immunization in the context of an actuarial company which had projected liability outflows $L(t)$ at some finite number M of instances (dates) t_k and anticipated inflows $A(t_i)$ at N (typically) different dates t_i . It was assumed that interest rates were flat, that is, $s(t) \equiv s$, and shocks $a(t)$ of interest rates $s(t)$ meant their parallel movements, that is, $a(t) \equiv \lambda$.

In such a situation, the company's task was to choose inflows $A(t)$ in such a manner that the outflows $L(t)$ would be discharged if the interest rates $s(t) \equiv s$ moved to their new constant level $s^*(t) \equiv s + \lambda$. To recall Redington's main result, let us note that $V = \sum_{t=1}^{t=N} \frac{A(t)}{(1+s)^t}$ represents the present value of inflows $A(t)$ occurring at instances t_i ; a similar formula holds for liabilities $L(t)$. Redington introduced the notion of a "mean term" having in mind the weighted average of the dates when the flows are to be received (in case of assets) or have to be discharged (in case of liabilities). This "mean term" was nothing different than the concept of *duration* introduced by Macaulay in [1]. These two authors understood duration as:

$$D = \sum_{t=1}^{t=N} tw_t; w_t = \frac{A(t)}{V(1+s)^t}; \sum_{t=1}^{t=N} w_t = 1 \quad (1)$$

where w_t tells us what portion (weight) of the entire cash flow is represented by $A(t)$ in terms of today's money. It was proved in [2] that any *parallel* movement (shift) of the *flat* term structure $s(t) \equiv s$ of interest rates would affect the value of the assets in the same way as it would affect the value of liabilities if duration D_A of assets $A(t)$ were equal to duration D_L of liabilities $L(t)$, and additionally, the so-called *convexity* of the assets would exceed that of the liabilities.

2.4. Assumptions concerning term structure of interest rates and admissible shifts: historical development

Twenty years later, Fisher and Weil [3] restricted themselves to a single liability at a specified date q , but significantly weakened the adopted so far assumption that the term structure was flat, that is, $s(t) \equiv s$. Denoting current interest rates $s(t)$ as $h(0,t)$, they allowed $h(0,t)$ to be a function of arbitrary shape with $h(0,t)$, $0 \leq t \leq N$, meaning annualized returns on zero-coupon default-free bonds tradable on a debt market D . However, they upheld the strong assumption concerning the admissible shifts $a(t)$ by supposing that $h(0,t)$ was subject only to a random additive shift of the form $h^*(0,t) = h(0,t) + \lambda$ for $0 \leq t \leq N$ appearing instantly after the acquisition of a bond portfolio.

They applied (popular already at that time) continuous compounding of cash flows $A(t)$ and $L(t)$, which in their approach represented instantaneous rate of payments per one unit of time rather than payments themselves, so that the present value of the assets V_A could be expressed by means of an integral

$$V_A = \int_0^N A(t) \exp[-h(0,t)] dt \quad (2)$$

while the duration D_A was given by

$$D_A = \int_0^N w_t t dt = \int_0^N \frac{A(t) \exp[-h(0, t)]}{V_A} t dt. \quad (3)$$

It was stated in [3] that immunization was secured if the duration of the assets D_A equaled the duration of the single liability to be discharged at time q : $D_A = \int_0^N w_t t dt = q$. Clearly, duration of any single inflow or outflow at time q equals q since all weights, except for the one at date q , are equal to 0 (zero).

2.4.1. Further developments of immunization theory

In subsequent 20–30 years of development of immunization theory, the strong assumption made so far that interest rates were subject to random shifts of the form $h^*(0, t) = h(0, t) + \lambda$ was being dropped. Many authors began to study shifts governed by some specific stochastic processes. For example, it was proved in [4] that some alternative stochastic processes permitted immunization, and others did not. When immunization can occur, formulas for calculating the resulting duration differ depending on the underlying stochastic process.

Few years later, it was demonstrated in [5] that these differences may be significant. For example, when a multiplicative stochastic process λ is used, that is, $h^*(0, t) = \lambda h(0, t)$ instead of additive stochastic process $h^*(0, t) = h(0, t) + \lambda$, one would obtain the implicit formula for the immunizing duration shown as Eq. (13) in [6] on p. 29. On the other hand, both the multiplicative shift

$$h^*(0, t) = \left[1 + \frac{\lambda \ln(1 + \alpha t)}{\alpha t} \right] h(0, t); \quad (4)$$

and the additive one

$$h^*(0, t) = h(0, t) + \frac{\lambda \ln(1 + \alpha t)}{\alpha t}, \quad (5)$$

were studied in [7], where suitable implicit formulas for the respective immunizing durations were derived.

Another approach, called contingent immunization, was developed in [8]. It consists of building a bond portfolio with a duration shorter or longer than the investor's planning horizon, taking into account "personal" expectations of a bond manager with regard interest rates. The idea standing behind such approach is to take advantage of the manager's ability to forecast interest rate movements (diseases) as long as his/her predictions are accurate.

2.4.2. Latest developments of immunization theory

The contingent immunization was implemented in many situations for various term structures of interest rates; see [9]. Other than mentioned in Eqs. (4) and (5), stochastic process governing admissible shifts was analyzed in [10].

Striving to offer a more general approach, the authors of article [11] did not confine ourselves to a specific process governing shifts, but allowed them to belong to a certain class of functions, such as, for example, polynomials of degree less than some specified number n (pp. 858–861). In this way, they did not expose themselves to any model misspecification risk, similarly as Zheng in [12].

The larger class of shifts (diseases) against which the immunization will work, the better. Having this in mind, the interval $[t_0; T]$ was divided in [6] into n equal nonoverlapping sub-intervals I_k , $1 \leq k \leq n$, and set $a_k(t) = 1$ when $t \in I_k$ and $a_k(t) = 0$ otherwise (p. 34). The admissible shifts were assumed to be piecewise constant functions of the form $\sum_{k=1}^{k=n} \lambda_k a_k(t)$. The authors made a general assumption stating that a BP generates inflows

$$A(t) = A_0(t) + \sum_{k=1}^{k=m} c_k \delta(t - t_k), \quad (6)$$

with $A_0(t)$ representing an instantaneous rate of cash payout, while s_k standing for single payment at instances $t_1, t_2, t_3, \dots, t_m$. The expression $\delta(t - t_k)$, with $\delta(t)$ standing for a Dirac delta function, was employed in order to make integration possible. The following result (Theorem 3, pp. 34–35 in [6]) was then proved.

Theorem 1. If q denotes the date when the single liability of L dollars has to be discharged by means of the cumulative value of assets $A(t)$, then the immunization is secured against all adverse piecewise constant shifts $\sum_{k=1}^{k=n} \lambda_k a_k(t)$ of interest rates $h(0, t)$ if and only if

$$a_k(q)q = \int_0^T \frac{A(t) \exp[-h(0, t)]t}{V_A} a_k(t) dt, \quad 1 \leq k \leq n, \quad (7)$$

where V_A stands for the present value of the portfolio represented by assets $A(t)$.

Remark 1. When $n = 1$, then Theorem 1 gives a sufficient and necessary condition

$$a(q)q = \int_0^T \frac{A(t) \exp[-h(0, t)]t}{V_A} a(t) dt \quad (8)$$

for immunization when the term structure $h(0, t)$ is subject to shifts $h(t, 0) + \lambda a(t)$. In case of parallel shifts, (8) reduces to Fisher-Weil condition $q = \int_0^T \frac{A(t) \exp[-h(0, t)]t}{V_A} dt$, which is well remembered as the following statement: Immunization is secured if the duration of the assets D_A equals the duration of the single liability.

Remark 2. Theorem 1 can also be looked at from a different perspective. Namely, one may be interested in identification of such a set of shifts $a(t)$ of the term structure $s(t)$ of interest rates,

say IMMU, against which a bond portfolio BP is already immunized, that is, protected against loss of its value at time q . In this context, Theorem 1 offers a sufficient and necessary condition for a shift $a(t)$ to belong to set IMMU.

2.5. One cannot immunize against all possible shifts of interest rates development

As of today, no one was successful in building up a bond portfolio BP immunized (immune) against all shifts of interest rates (diseases). What is more, in Section 3, we demonstrate that the set IMMU is always a proper subset of all admissible shifts, being in fact a linear subspace of all shifts.

3. Overview of some recent results

In a recent paper [13], the authors found a strong evidence that momentum across various asset classes is caused by macroeconomic variables. By properly modifying their portfolio, in response to changes in macroeconomic environment, their strategy performed particularly well in times of economic distress. The obtained results allowed them to establish a link between momentum and sophisticated predictive regressions.

Aiming at securing higher effectiveness of their investment in fixed income bonds, the authors of [14] successfully used simulations of the portfolio surplus, measuring the inherent risk by means of the value-at-risk methodology. In another very recent publication [15], the authors studied immunization assuming that shifts were parallel or symmetric. A quite different approach to immunization was proposed in [16]. The authors concentrated on hedging risk inherent in bond portfolio. They divided the entire problem into two parts, by formulating a two-step optimization problem. They focused first on immunization risk, and next maximized the portfolio wealth.

In this section, it is proved that the set of all continuous shocks $a(t)$ against which a bond portfolio BP is immunized is an m -dimensional linear subspace in the $(m + 1)$ -dimensional linear space of all continuous shifts $a(t)$, with m standing for the number of instances when BP promises to pay cash (coupons or par values generated by bonds forming BP). The main mathematical concept used below is the notion of a Hilbert space and the concept of a base in a Hilbert space.

3.1. When polynomials are admissible shifts

From now on, we assume that $A_0(t) \equiv 0$ in Formula 6, so that inflows given by

$$A(t) = \sum_{k=1}^{k=m} c_k \delta(t - t_k), \quad (9)$$

generate only payments c_k at specified instances $t_1, t_2, t_3, \dots, t_m$. In such a situation, the present value of assets $A(t)$ is no longer given by Eq. (2), but by

$$V_A = \sum_{k=1}^{k=m} c_i \exp [-c(t_k)t_k]. \quad (10)$$

One of two classes of admissible shifts studied in [14] was the class of polynomials.

$$a(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^{n-1} = \sum_{j=1}^{j=n} a_{j-1} t^{j-1}, t \in [t_0; T]. \quad (11)$$

The new term structure was assumed to be of the form:

$$s^*(0, t) = s(t) + \lambda a(t), t \in [t_0, T], \quad (12)$$

with $a(t)$ satisfying Formula 11.

Definition 1. (see also [11], p. 859). A set S is said to be a linear space if the sum of its arbitrary 2 elements $a \in S$ and $b \in S$ belongs to S ($a + b \in S$), and for any real number r the product of r and any element $a \in S$ belongs to S as well.

The most well-known linear spaces are probably the set of all real numbers R , a two-dimensional Cartesian plane R^2 , a three-dimensional linear space R^3 , and their generalizations R^n , known as an n -dimensional linear spaces.

Definition 2. A set of k vectors v^1, v^2, \dots, v^k is called linearly independent if each linear combination of these vectors $\lambda^1 v^1 + \lambda^2 v^2 + \dots + \lambda^k v^k$ is a vector different from vector 0; see Definition 2.2 (p. 860).

Definition 3. A set of linearly independent vectors from a linear space S is called a base for S if each vector $a \in S$ is a linear combination of theirs, and this property does not hold any longer after removal of any of these base vectors.

All bases have the same size and there are many of them in each linear space S . R^n is a linear space with a natural addition $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ of two vectors, and natural multiplication $r \cdot x = (rx_1, rx_2, rx_3, \dots, rx_n) \in R^n$ of a vector $x = (x_1, x_2, \dots, x_n)$ by a real number r . The most popular base in R^n is the set of n vectors: $v^1 = (1, 0, 0, \dots, 0)$, $v^2 = (0, 1, 0, \dots, 0)$, $v^3 = (0, 0, 1, 0, \dots, 0)$, and so on until $v^n = (0, 0, 0, \dots, 0, 1)$. Each vector, for example, $(2, 3, -7, 10)$, is the following linear combination of the above base vectors: $2(1, 0, 0, 0) + 3(0, 1, 0, 0) + (-7)(0, 0, 1, 0) + 10(0, 0, 0, 1)$.

Remark 3. The below Formula (13), being a counterpart of Formula 8, gives a necessary and sufficient condition for immunization against shifts $a(t)$ in case when a bond portfolio BP generates payments c_k at instances $t_1, t_2, t_3, \dots, t_m$:

$$a(q)q = \sum_{i=1}^{i=m} t_i w_i a(t_i), \quad (13)$$

with weights

$$w_k = \frac{c_k \exp[-s(t_k)t_k]}{\sum_{i=1}^{i=m} c_i \exp[-s(t_i)t_i]}. \quad (14)$$

Remark 4. The class of polynomials of the form (11), with fixed n , is a linear space. What is more, the subset of these polynomials satisfying Eq. (13) is a linear space, too.

Proof. Assume that $a(t)$ satisfies Eq. (13). For any real number r , Eq. (13) implies $[ra(q)]q = \sum_{i=1}^{i=m} t_i w_i [ra(t_i)]$ because parameters t_i and w_i remain the same. Adding Eq. (13) holding for $b(t)$ to Eq. (13) holding for $c(t)$, one immediately obtains the required relationship.

$$[b(q) + c(q)]q = \sum_{i=1}^{i=m} t_i w_i [b(t_i) + c(t_i)]. \quad (15)$$

Theorem 2. (see [11], Theorem 2.1). Let q denote the date when the single liability of L dollars has to be discharged by means of the cumulative value of assets (9) despite additive adverse shifts (11) (n is fixed) of interest rates $s(t)$ so that the new interest rates will be of the form (12). Then, the subclass of shifts (11) for which immunization is secured is a $(n-1)$ -dimensional linear space, denote it by IMMU , of the space of all polynomials (11), which itself has dimension n .

How to determine IMMU is demonstrated in [11] in pp. 858–860.

3.2. Continuous functions are admissible shifts: a Hilbert space approach

The other class of admissible shifts studied in [11] was the class of all continuous functions (CF) defined as always on interval $[t_0; T]$. As previously, the new interest rates (after a shift) satisfy Eq. (12) with $a(t)$ standing this time for any CF. As previously, assets $A(t)$ are given by Formula 9. It is easy to notice that the class of CF is a linear space with ordinary addition of two functions and ordinary multiplication of a function by a real number. However, it has an infinite number of independent vectors!

We shall demonstrate that the notion of a Hilbert space is very useful in the study of immunization theory. It was named after a German mathematician David Hilbert (1862–1943) who is recognized as one of the most influential and universal mathematicians of the nineteenth century and the first half of twentieth century. By definition, a Hilbert space is a linear space, say H , which is additionally equipped with so-called scalar product (a generalization of the scalar product of two vectors from R^n) defined for any two of its elements (vectors) $h_1 \in H$ and $h_2 \in H$.

A specific Hilbert space H^* of all CF (shifts) defined on interval $[t_1, T]$ was introduced in [11] and it was demonstrated that H^* had dimension m . However, the shifts of interest rates should be considered on interval $[t_0; T]$ because a random and unexpected shift $a(t)$ might appear instantly after the acquisition of BP. In such a case, the dimension of H^* would be $(m + 1)$, which is really the case. So, in this chapter, we correct and simplify the definition of a scalar product of two arbitrary continuous functions (shifts) $f(t)$ and $g(t)$, by letting

$$\langle f, g \rangle = \sum_{k=0}^{k=m} f(t_k)g(t_k). \quad (16)$$

It is good to know that in each Hilbert space H , one can measure a distance between any two elements $h_1 \in H$ and $h_2 \in H$ by the formula $\|h_1 - h_2\|$, where $\|h\| = \sqrt{\langle h, h \rangle}$ is said to be a norm of vector h . In space H^* , the norm is therefore defined as follows:

$$\|h\| = \sqrt{\langle h, h \rangle} = \sqrt{\sum_{k=0}^{k=m} h(t_k)h(t_k)}. \quad (17)$$

Clearly, $\|h\| = 0$ if and only if $h(t_k) = 0$, $0 \leq k \leq m$. A function $h(t)$ belonging to H^* is treated as an element (vector) 0 (zero) if and only if $\|h\| = 0$. Therefore, $\|h_1 - h_2\| = 0$ means, then the two functions $h_1(t)$ and $h_2(t)$ are viewed as same on interval $[t_0, T]$. It holds if and only if they coincide at all instances t_k , $0 \leq k \leq m$, when bond portfolio BP is paying cash.

In Theorem 3 to follow, we identify a base for H^* among polynomials. This approach is rather complicated since it involves the use of Gram-Schmidt orthogonalization procedure to determine base polynomials. In Section 3.3, a far more straightforward and easier to implement approach is presented where there is no need to identify base functions (shifts) because they are already given by Formulas (20)–(23).

Theorem 3. (compare Theorem 3.1 in [11]). Suppose a bond portfolio BP has been bought, and admissible shifts $a(t)$ of a term structure $s(t)$ are allowed to be continuous functions on interval $[t_0, T]$. Then, the set of these shifts equipped with the scalar product (16) is an $(m + 1)$ -dimensional Hilbert space H^* , where m is the number of instances when portfolio BP generates cash. The subset of these shifts, say IMM, against which a holder of BP is immune (will be able to discharge the liability of L dollars to be paid at time $q \in [t_0, T]$ by means of the cumulative value of assets (9)) is an m -dimensional subspace (depending to a large extent on BP) of the form

$$a(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + \dots + a_mP_m(t) \quad (18)$$

where the $m + 1$ polynomials $P_k(t)$, $0 \leq k \leq m$, constitute a base of space H^* . This base may be determined by the Gram-Schmidt orthogonalization procedure, while the coefficients $a_0, a_1, a_2, \dots, a_m$ can be identified as solutions to the linear equation

$$[a_0P_0(q) + a_1P_1(q) + a_2P_2(q) + \dots + a_mP_m(q)] \cdot q \cdot \sum_{k=1}^{k=m} c_k \exp[-s(t_k)t_k] = a_0P_0(q) \cdot \sum_{k=1}^{k=m} c_k t_k \exp[-s(t_k)t_k] \quad (19)$$

It is worth to notice that after determination of polynomials $P_k(t)$, $0 \leq k \leq m$, all numbers $P_0(q), P_1(q), P_2(q), \dots, P_m(q)$ are known, as well as parameters $q, c_k, t_k, s(t_k)$, so that $a_0, a_1, a_2, \dots, a_m$ remain the only unknown variables. The readers interesting in identifying subspace IMMUE are referred to Example 4.2 (pp. 863–864).

3.3. Identification of continuous shifts against which a bond portfolio is immunized: the triangular functions approach

A strict definition of triangular functions is given by Eqs. (20)–(23) below. Roughly speaking, a triangular function (sometimes called a tent function, or a hat function) is a function whose graph takes the shape of a triangle. Among our $(m + 1)$ tent functions employed in this chapter, $(m - 1)$ are isosceles triangles with height 1 and base 2, while the other two are perpendicular triangles with height 1 and base 1. Triangular functions have been successfully employed in signal processing as representations of idealized signals from which more realistic signals can be derived, for example, in kernel density estimation.

They also have applications in pulse code modulation as a pulse shape for transmitting digital signals, and as a matched filter for receiving the signals. Triangular functions are used to define the so-called triangular window, also known as the Bartlett window. Since they occur in the formula for Lagrange polynomials used in numerical analysis for polynomial interpolation, they are also called Lagrange functions. Their other applications include the Newton-Cotes method of numerical integration, and Shamir's secret sharing scheme in cryptography.

In the financial context, tent functions were employed in [17] for modeling shifts of the term structure of interest rates. The framework and assumptions made in this section are the same as in Section 3.2. Our purpose is to characterize the subspace IMMUE of the Hilbert space H^* by means of triangular functions based on results presented in [18].

In this section, $t_1, t_2, t_3, \dots, t_m = T$ comprise not only all instances when a given bond portfolio BP generates payments, but also additionally the date q when the liability to pay L dollars has to be discharged. Below, we define $m + 1$ triangular functions $S_0(t), S_1(t), S_2(t), \dots, S_m(t)$ whose graphs are triangles with bases $[t_0, t_1], [t_0, t_2], [t_1, t_3], \dots, [t_{m-2}, t_m], [t_{m-1}, t_m]$. The first one $S_0(t)$ and the last one $S_m(t)$ represent perpendicular triangles, while the remaining ones are isosceles triangles.

$$S_0(t) = \frac{t - t_1}{t_0 - t_1}, t \in [t_0; t_1] \text{ and } S_0(t) = 0 \text{ for } t \in [t_1; t_m], \quad (20)$$

$$S_m(t) = \frac{t - t_{m-1}}{t_m - t_{m-1}}, t \in [t_{m-1}; t_m] \text{ and } S_m(t) = 0 \text{ for } t \in [t_0; t_{m-1}], \quad (21)$$

$$S_k(t) = \frac{t - t_{k-1}}{t_k - t_{k-1}}, t \in [t_{k-1}; t_k], 1 \leq k \leq m - 1, \quad (22)$$

$$S_k(t) = \frac{t - t_{k+1}}{t_k - t_{k+1}}, t \in [t_k; t_{k+1}]; S_k(t) = 0 \text{ elsewhere in } [t_0; T]. \quad (23)$$

The following result is well known.

Remark 5. Each continuous function $a(t)$ defined on $[t_0; T]$ attains the same values as the function $b(t) = a(0) \cdot S_0(t) + a(t_1) \cdot S_1(t) + a(t_2) \cdot S_2(t) + \dots + a(t_m) \cdot S_m(t)$ (built up with $(m + 1)$ triangular functions) at all points $t_k, 0 \leq k \leq m$. Therefore, $a(t)$ may be identified in H^* with the piecewise linear function $b(t)$ because the distance between $a(t)$ and $b(t)$ in H^* is zero: $\|b(t) - a(t)\| = 0$.

It is a nice exercise to prove the following result.

Remark 6. The Lagrange functions $S_0(t), S_1(t), S_2(t), \dots, S_m(t)$ given by (20)–(23) constitute a base for Hilbert space H^* of all admissible (continuous) shifts defined on $[t_0; T]$.

Theorem 4. The set IMMU of all shifts (continuous functions) against which a bond portfolio BP with payouts represented by (9) and the new term structure given by (12) is immunized constitutes an m -dimensional linear subspace in the $(m + 1)$ -dimensional Hilbert space H^* .

Two examples illustrating how to identify IMMU are worked out in detail in [18], pp. 531–537. A special attention is given to continuity properties of subspace IMMU; see [18], pp. 534–537.

4. Maximizing the unanticipated rate of return among immunized bond portfolios

The natural question arises of how to select the “best” portfolios among those which are (have been) protected (immunized) against admissible shifts (movements) of interest rates? In finance, by best portfolios are meant those which yield the highest rate of return (the highest increase in the present value of a BP), resulting from a sudden shift of interest rates. Below, we present the results obtained in [19]. Rewriting a sufficient and necessary condition (13) and (14) for immunization of portfolio BP generating payouts (9), one obtains:

$$q = \sum_{i=1}^{i=m} t_i w_i v_i \quad (24)$$

and

$$v_i = \frac{a(t_i)}{a(q)} \text{ provided } a(q) \neq 0. \quad (25)$$

For each vector $v = (v_1, v_2, \dots, v_m) \in R^m$, the class K_v of such continuous shifts $a(t)$ for which (25) holds was defined in [19]. When vector $v = (1, 1, \dots, 1) \in R^m$ is used, then the corresponding class K_v comprises all parallel shifts for which $a(t) = \text{constant}$. We shall call $D_v = \sum_{i=1}^m t_i w_i v_i$ the dedicated (for class K_v) duration. For zero-coupon bearing bond B_k , maturing at time t_k , the dedicated duration $D_v(B_k) = t_k \cdot v_k$ since all weights w_i , except for w_k , are equal to 0.

Theorem 5. The immunization of a bond portfolio BP against shifts $a(t)$ from class K_v is secured if and only if $q = D_v(BP) = \sum_{i=1}^m t_i w_i v_i$.

With $s(t)$ standing for the current interest rates, $PV[s(\cdot)] = \sum_{k=1}^{k=m} c_k \exp[-s(t_k)t_k]$ is the present value of BP. Suppose that immediately after purchasing BP, interest rates $s(t)$ will shift to new levels $s^*(t) = s(t) + a(t)$. Then

$$PV[s(\cdot) + a(\cdot)] = \sum_{k=1}^{k=m} c_k \exp[-s(t_k) - a(t_k)]t_k.$$

4.1. Convexity of a bond portfolio

Set $C_v(BP) = \frac{1}{2} \sum_{k=1}^m t_k^2 w_k v_k^2$ and call it dedicated (for class K_v) convexity of portfolio BP; for more details, see [19], p. 105. It is easy to notice that convexity of a zero-coupon bearing bond maturing at t_k is given by the formula $C_v = \frac{1}{2} t_k^2 v_k^2$. It was proved in [19], p. 106, that so-called unanticipating rate of return resulting from a shift $a(t)$ of interest rates $s(\cdot)$ is given by the formula:

$$\frac{PV[s(\cdot) + a(\cdot)] - PV[s(\cdot)]}{PV[s(\cdot)]} = -D_v(BP)a(q) + C_v(BP)a^2(q) + \sum_{k=1}^{k=m} O[a(t_k)]a(t_k)^2 \quad (26)$$

where $\lim O(a) = 0$ when $a \rightarrow 0$. Taking into account that $a(t_k)$ are small numbers of order $0.1\% = 0.001$, one concludes that the third term in (26) is really very small. Since each immunized bond portfolio BP satisfies $D_v(BP) = q$, the maximal unanticipating rate of return among immunized portfolios will be achieved when dedicated convexity $C_v(BP)$ will be as high as possible.

Assumption 1. All zero-coupon bearing bonds B_k , which form a bond portfolio BP and mature at t_k , have mutually different dedicated durations, that is, $D_v(B_j) \neq D_v(B_n)$ if and only if $j \neq n$, that is, $t_j \cdot v_j \neq t_n \cdot v_n \leftrightarrow j \neq n$.

Definition 4. Following [20], p. 552, a bond portfolio BP is said to be a barbell strategy (barbell portfolio) if it is built up of two bonds, say B^1, B^2 with significantly different dedicated durations D_v^1 and D_v^2 . On the other hand, BP is said to be a focused strategy (focused portfolio) if it consists of several bonds whose dedicated durations D_v^j are centered around duration of the liability (q in our context).

Theorem 6. (see [19], Theorem 1). If Assumption 1 holds then the bond portfolio BP^* with the highest unanticipated rate of return is a barbell strategy built up of zero-coupon bearing bonds B^s, B^l with minimal and maximal dedicated durations. The weights \bar{x}_s and \bar{x}_l , expressing the amounts of payments resulting from B^s and B^l , are given by formulas:

$$\bar{x}_s = \frac{t_l v_l - q}{t_l v_l - t_s v_s}, \bar{x}_l = \frac{q - t_s v_s}{t_l v_l - t_s v_s}, \bar{x}_k = 0 \text{ for } k \neq s, k \neq l. \quad (27)$$

Comment 1. Suppose that instead of dedicated duration and dedicated convexity, we employ the classic notions of duration and convexity derived for additive shifts only. Then, $v_l \equiv 1$ and consequently Eq. (27) reduces to simpler, say classic, formulas $\bar{x}_s = \frac{t_l - q}{t_l - t_s}$, $\bar{x}_l = \frac{q - t_s}{t_l - t_s}$. The natural question arises of how much the weights given by Eq. (27) differ from the classic ones.

Finally, another interesting question arises, to what extent does the dedicated duration of the best immunized portfolio BP^* differ from its Macaulay's counterpart? That is, what is the difference between $D_v(BP^*) = t_s w_s v_s + t_l w_l v_l$ and $D(BP^*) = t_s w_s + t_l w_l$ with $v_s = \frac{a(t_s)}{a(q)}$, $v_l = \frac{a(t_l)}{a(q)}$? It is easy to observe that when a shift $a(t)$ affects the current interest rates in a similar manner at all or many points $t_1, t_2, t_3, \dots, t_m$, then there is a good chance that $v_s \approx 1 \approx v_l$, and consequently, the difference between the dedicated duration D_v and the classic one will be very small.

For a specific situation, when shifts $a(t)$ of interest rates $s(t)$ satisfied the "proportionality" condition $\frac{a(t_k)}{1+s(t_k)} = \text{constant}$ (for details, see [21]), the maximal convexity and formula for the best immunizing bond portfolio was determined by means of Kuhn-Tucker conditions (pp. 139–140 in [21]). A formula for the resulting unanticipating rate of return was derived (pp. 141–142) and illustrating with an example (p. 143).

5. Concluding remarks

Let us summarize what we have said so far. Each bond portfolio BP (a human body? or a human body organ?) generates cash at various dates $t_1, t_2, t_3, \dots, t_m$. What should (could) be substituted for cash (payments generated by a BP) in the medical setting remains an important open problem. Maybe, it is something related to a human body's performance; call this mysterious agent by Z .

In bond portfolio theory, the greater payouts generated by BP , the higher is the present value (PV) and future value (FV) of BP . An analogous statement is therefore expected in the medical context. Having settled what is Z , it would be probably easy to find out what is the counterpart

in medicine of the duration concept defined for the first time by Macaulay (in 1938) and independently by Redington (in 1952); see Formula (1).

Let us formulate the following hypothesis: the higher values (levels) of Z , the more healthy is a human body (a human body organ).

In the financial immunization context, there is a fixed date q when BP must attain at least a certain value L , called liability. In the medical context, one might say that there is a fixed date q when the quality of human health must attain at least a certain level L .

In the financial theory context, when interest rates $s(t)$ change due to a shift $a(t)$, that is, $s(t) \rightarrow s(t) + a(t)$, then the FV of BP at date q may fall below L dollars. In the medical context, the appearance of disease may cause a deterioration of health at date q .

We still do not know what should (could) be substituted for interest rates $s(t)$, knowing that changes (movements, shifts) in interest rates mean a disease.

Using the concept of *duration* (and dedicated duration), we identified the set IMMU of all shifts (diseases) $a(t)$ against which BP is immunized. By means of notion of duration and *convexity* (dedicated convexity), we determined the best immunizing portfolios for a large class of shifts (continuous functions). In the financial context, the best portfolios meant portfolios generating the highest (unanticipated) rate of return. In the medical context, the best would probably mean the fastest rate of health improvement.

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