Chapter 8

Invariants of Generalized Fifth Order Non-Linear Partial Differential Equation

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Additional information is available at the end of the chapter

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Abstract

The fifth order non-linear partial differential equation in generalized form is analyzed for Lie symmetries. The classical Lie group method is performed to derive similarity variables of this equation and the ordinary differential equations (ODEs) are deduced. These ordinary differential equations are further studied and some exact solutions are obtained.

Keywords: generalized fifth order non-linear partial differential equation, lie symmetries, exact solutions

1. Introduction

The theories of modern physics mainly include a mathematical structure, defined by a certain set of differential equations and extended by a set of rules for translating the mathematical results into meaningful statements about the physical work. Theories of non-linear science have been widely developed over the past century. In particular, non-linear systems have fascinated much interest among mathematicians and physicists. A lot of study has been conducted in the area of non-linear partial differential equations (NLPDEs) that arise in various areas of applied mathematics, mathematical physics, and many other areas. Apart from their theoretical importance, they have sensational applications to various physical systems such as hydrodynamics, non-linear optics, continuum mechanics, plasma physics and so on. A large variety of physical, chemical, and biological phenomena is governed by nonlinear partial differential equations (NLPDEs). A number of methods has been introduced for finding solutions of these equations such as Homotopy method [1], G'/G expansion method [2, 3], variational iteration method [4], sub-equation method [5], exp. function method [6], and Lie symmetry method [7–10]. Although solutions of such equations can be obtained easily by numerical computation. However, in order to obtain good understanding of the physical phenomena described by NLPDEs it is important to study the exact solutions of the NLPDEs. Exact solutions of mathematical equations play an major role in the proper understanding of qualitative features of many phenomena and processes in different areas of natural and applied sciences. Exact solutions of non-linear differential equations graphically demonstrate and allow unraveling the mechanisms of many complex non-linear phenomena. However, finding exact solutions of NLPDEs representing some physical phenomena is a tough task. However, because of importance of exact solutions for describing physical phenomena, many powerful methods have been introduced for finding solitons and other type of exact solutions of NLPDEs [2, 11–13]. Comparing to other approximate and numerical methods, which provides approximate solutions [14–16], the Lie group method provides the exact and analytic solutions of the differential structure (**Figures 1–3**).

Lie group method is one of the most effective methods for finding exact solutions of NLPDEs [17, 18]. This method was basically initiated by Norwegian mathematician Sophus Lie [19]. He developed the theory of "Continuous Groups" known as Lie groups. This method is orderly used in various fields of non-linear science. Shopus Lie was the first who arranged differential equations in terms of their symmetry groups, thereby analyzing the set of equations, which could be integrated or reduced to lower order equations by group theoretic algorithms. The Lie group analysis is a mathematical theory that synthesizes symmetry of differential equations. In this method, the differential structure is studied for their invariance by acting one or several parameter continuous group of transformations on the space of dependent and independent variables. We observe a plenty books and research article about Lie group method [17, 18, 20–22].

Wazwaz [23] introduced a fifth order non-linear evolution equation as follows:

$$u_{ttt} - u_{txxxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0. \tag{1}$$

In this chapter, he obtained multiple soliton solutions of this equation.

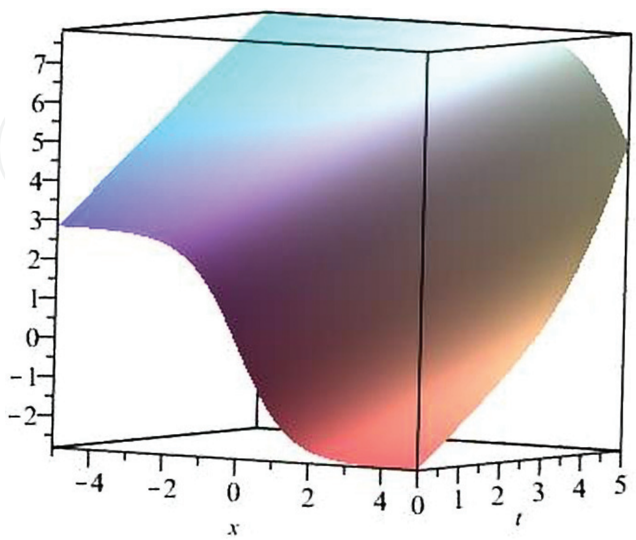

Figure 1. Kink wave solution (17) for $\alpha = \beta = \lambda = \mu = 1, b_1 = b_3 = 0$.

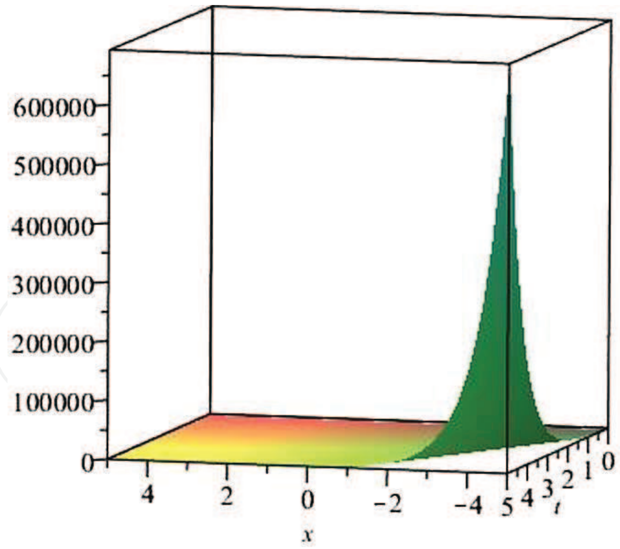

Figure 2. Singularity solution (18) for $\alpha = \lambda = \mu = b_5 = 1, b_2 = b_4 = 0$.

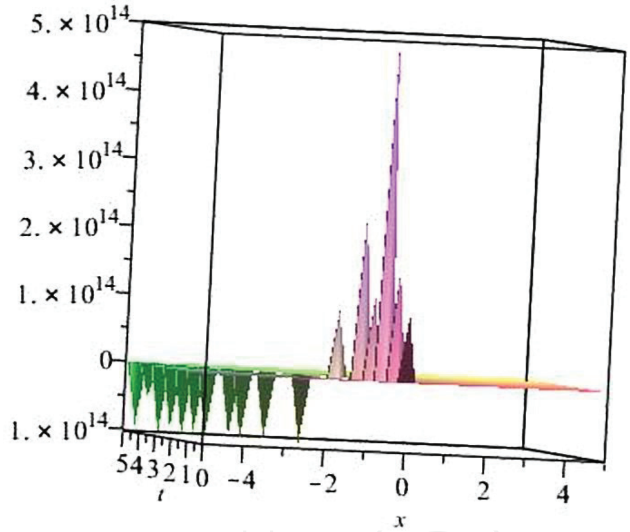

Figure 3. Singularity solution (19) for $\alpha = b_2 = b_4 = 0, b_4 = \lambda = 1, \mu = -1$.

We will consider the generalized fifth order non-linear evolution equation of the form:

$$u_{ttt} - u_{txxxx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0, \tag{2}$$

where α, β are parameters.

In this chapter, we will study the Eq. (2) by the Lie classical method. Firstly, Lie classical method will be used to obtain symmetries of generalized fifth order non-linear evolution Eq. (2). Symmetries will be used to reduce the Eq. (2) to ordinary differential equations (ODEs) and corresponding exact solutions of the generalized fifth order non-linear evolution Eq. (2) will be obtained.

2. Symmetry analysis

Lie classical method of infinitesimal transformation groups reduces the number of independent variables in partial differential equations (PDEs) and reduces the order of ODEs. Lie's method has been widely used in equations of mathematical physics and many other fields [11, 24]. In this chapter, we will perform Lie symmetry analysis [17–19, 24] for the generalized fifth order nonlinear evolution Eq. (2).

Let the group of infinitesimal transformations be defined as:

$$\begin{aligned} t^* &= t + \varepsilon\tau(x,t,u) + O(\epsilon^2) \\ x^* &= x + \varepsilon\xi(x,t,u) + O(\epsilon^2) \\ u^* &= u + \varepsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{3}$$

which leaves the Eq. (2) invariant. The infinitesimal transformations (3) are such that if u is solution of Eq. (2), then u^* is also a solution.

Herein, on invoking the invariance criterion as mentioned in [18], the following relation is deduced:

$$\eta^{ttt} - \eta^{txxxx} - \alpha\left(\eta^{xxx}u_t + \eta^t u_{xxx}\right) - (2\alpha+\beta)\left(\eta^{xx}u_{xt} + \eta^{xt}u_{xx}\right) - (\alpha+\beta)\left(\eta^x u_{xxt} + \eta^{xxt}u_x\right) = 0, \tag{4}$$

where $\eta^x, \eta^t, \eta^{xt}, \eta^{xx}, \eta^{xxx}, \eta^{ttt}, \eta^{txxxx}$ and η^{xxt} are extended (prolonged) infinitesimals acting on an enlarged space corresponding to $u_x, u_t, u_{xt}, u_{xx}, u_{xxx}, u_{ttt}, u_{txxxx}$ and u_{xxt}, respectively, given by:

$$\begin{aligned} \eta^x &= D_x\eta - u_x D_x\xi - u_t D_x\tau, \\ \eta^t &= D_t\eta - u_x D_t\xi - u_t D_t\tau, \\ \eta^{xx} &= D_x\eta^x - u_{xx}D_x\xi - u_{xt}D_x\tau, \\ \eta^{xt} &= D_t\eta^x - u_{xx}D_t\xi_- u_{xt}D_t\tau, \\ \eta^{xxx} &= D_x\eta^{xx} - u_{xxx}D_x\xi - u_{xxt}D_x\tau, \\ \eta^{ttt} &= D_t\eta^{tt} - u_{xtt}D_t\xi - u_{ttt}D_t\tau, \\ \eta^{xxxxt} &= D_t\eta^{xxxx} - u_{xxxxx}D_t\xi - u_{xxxxt}D_t\tau, \end{aligned} \tag{5}$$

where D_x and D_t are total derivative operators with respect to x and t respectively given as:

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots, \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + \cdots. \end{aligned}$$

Now, after computing (5) we get:

$$
\begin{aligned}
\eta^x &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,\\
\eta^t &= \eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t,\\
\eta^{xx} &= \eta_{xx} + u_x(2\eta_{xu} - \xi xx) - u_t\tau_{xx} +_x^2 (\eta_{uu} - 2\xi_{xu}) + u_{xx}(\eta_u - 2\xi_x) - 2u_{xt}\tau_x - 2u_t u_x \tau_{xu}\\
&\quad - u_x^3\xi_{uu} - u_x^2 u_t \tau_{uu} - 2u_x u_{xt\tau_u} - u_{xx}u_t\tau_u - 3u_x u_{xx}\xi_u,\\
\eta^{xt} &= \eta_{xt} + u_x(\eta_{tu} - \xi_{xt}) + \eta_t(\eta_{xu} - \tau_{xt}) - u_x^2\xi_{tu} - u_t^2\tau_{xu} - u_{xx}\xi_t - u_{xt}(\tau_t + \xi_x - \eta_u) - u_{tt}\tau_x\\
&\quad + u_x u_t(\eta_{uu} - \xi_{xu}) - u_x u_{xt}\xi_u - u_{xt}u_x\xi_u - u_{xt}u_t\tau_u - u_{tt}u_x\tau_u - u_x^2 u_t\xi_{uu} - u_t^2 u_x\tau_{uu}\\
&\quad - u_t u_x \tau_{tu} - u_t u_{xt}\tau_u - u_{xx}u_t\xi_u,\\
\eta^{xxx} &= \eta_{xxx} +_x (3\eta_{xxu} - \xi_{xxx}) - u_t\tau_{xxx} + u_{xx}(3\eta_{xu} - 3\xi_{xx}) - 3u_{xt}\tau_{xx} - 3u_x u_t\tau_{xxu} - 3u_{xxt}\tau_x\\
&\quad + u_x^2(3\eta_{xuu} - 3\xi_{xxu}) + u_x u_{xx}(3\eta_{uu} - 9\xi_{xu}) + u_x^3(\eta_{uuu} - 3\xi_{xuu}) + u_{xxx}(\eta_u - 3\xi_x)\\
&\quad - 2u_x u_{xt}\tau_{xu} - u_x^4\xi_{uuu} - 6u_x^2 u_{xx}\xi_{uu} - 3{u_{xx}}^2\xi_u - 4u_x u_{xxx}\xi_u - 3u_t u_x^2\tau_{xuu} - 3u_t u_{xx}\tau_{xu}\\
&\quad - 4u_x u_{tx}\tau_{xu} - u_x^3 u_t\tau_{uuu} - 3u_x u_t u_{xx}\tau_{uu} - 3u_x^2 u_{xt}\tau_{uu} - 3u_{tx}u_{xx}\tau_u - 3u_x u_{xxt}\tau_u\\
&\quad - u_t u_{xxx}\tau_u,\\
\eta^{ttt} &= \eta_{ttt} - u_x\xi_{ttt} + u_t(3\eta_{ttu} - \tau_{ttt}) + u_t^2(3\eta_{tuu} - 3\tau_{ttu}) + u_t^3(\eta_{uuu} - 3\tau_{tuu}) - u_t^4\tau_{uuu} - u_{tt}^2 3\tau_u\\
&\quad - 3u_x u_t\xi_{ttu} - 3u_x^2 u_t\xi_{tuu} - 3u_x u_{tt}\xi_{ut} - 6u_{xt}u_t\xi_{tu} - 3u_x u_{tt}\xi_{tu} + u_{xt}(4\eta_{xxxu} - 3\xi_{tt} - \xi_{xxxx})\\
&\quad + 3u_{tt}(\eta_{tu} - \tau_{tt}) - u_{ttt}\tau_t - 2u_{xtt}(\xi + 2\tau_{xxx})_t + u_{xxx}(\eta_u - 2\tau_t) - u_{xtt}\xi_t - u_{xtt}u_t\xi_u\\
&\quad - u_t^3 u_x\xi_{uuu} - 3u_t^2 u_{xt}\xi_{uu} - 3u_{xt}utt\xi_u - 2u_{xtt}u_t\xi_u - u_t u_{ttt}(\tau_u + \xi_u) - u_t u_{tt}(9\tau_{tu} - 3\eta_{uu})\\
&\quad - 6u_t^2 u_{tt}\tau_{uu} - 3u_t u_{ttt}\xi_u,\\
\eta^{xxxxt} &= \eta_{xxxxt} + u_x(4\eta_{xxxtu} - \xi_{xxxxt}) + u_t\tau_{xxxxt} - u_x^2\tau_{xxxxu} + u_x^3(6\eta_{xxtuu} - 4\xi_{xxxtu})\\
&\quad + u_x^3(4\eta_{xtuuu} - 6\xi_{xxtuu}) + u_x^4(\eta_{tuuuu} - 4\xi_{xtuuuu}) - u_x^5\xi_{tuuuu} - 4u_{xt}\tau_{xxxt}\\
&\quad + u_{xx}(6\eta_{xxtu} - 4\xi_{xxxt}) + 2u_{xxxt}(2\eta_{xu} - 3\xi_{xx} - 2\tau_{xt}) + u_{xxx}(4\eta_{xtu} - 6\xi_{xxt})\\
&\quad + u_{xxxx}(\eta_{tu} - 4\xi_{xt}) + u_{xxt}(6\eta_{xxu} - 6\tau xxt - 4\xi_{xxx}) + u_{xxx}u_{xt}(4\eta_{uu} - 16\xi_{xu})\\
&\quad + 6u_{xx}u_{xxt}(\eta_{uu} - 4\xi_{xu}) + u_{xxxx}u_t(\eta_{uu} - 4\xi_{xu}) + 4u_x u_{xxxt}(\eta_{uu} - 4\xi_{xu})\\
&\quad - 6u_x^2 u_{tt}\tau_{xxuu} - 24u_x u_t u_{xt}\tau_{xxuu} - 6u_{xx}u_t^2\tau_{xxuu} - 4u_x u_{tt}\tau_{xxxu} - 8u_{xt}u_t\tau_{xxxu}\\
&\quad - 10u_x^2 u_{xxxt}\xi_{uu} - 5u_x u_t u_{xxxx}\xi_{uu} - 30u_{xx}u_{xxt}u_x\xi_{uu} - 20u_{xxx}u_{xt}u_x\xi_{uu}\\
&\quad - 15u_{xx}^2 u_{xt}\xi_{uu} - 5u_x u_{xxxxt}\xi_u - 10u_{xx}u_{xxxt}\xi_u - 5u_{xt}u_{xxxx}\xi_u - 10u_{xxt}u_{xxx}\xi_u\\
&\quad - 5u_x u_{xxxx}\xi_{tu} - 10u_{xx}u_{xxx}\xi_{tu} - 6u_x^2 u_{xxt}\tau_{tuu} - 12u_{xx}u_{xt}u_x\tau_{tuu} - 4u_x u_t u_{xxx}\tau_{tuu}\\
&\quad - 3u_{xx}^2 u_t\tau_{tuu} - 12u_x u_{xxt}\tau_{xtu} - 4u_t u_{xxx}\tau_{xtu} - u_{xx}u_{xt}(12\tau_{xtu} + 18\xi_{xxu}) - 18u_x u_{xxt}\xi_{xxu}\\
&\quad - 6u_t u_{xxx}\xi_{xxu} - 4u_{xxx}u_{xtt}\tau_u - 6u_{xxt}^2\tau u - u_{tt}u_{xxxx}\tau u - 8u_{xt}u_{xxxt}\tau u - 6u_{xx}u_{xxtt}\tau u\\
&\quad - u_t u_{xxxxt}\tau u - 4u_x u_{xxxtt}\tau u - 12u_x^2 u_{xtt}\tau xuu - 24u_x u_t u_{xxt}\tau xuu - 12u_x u_{xx}u_{tt}\tau xuu\\
&\quad - 24u_{xt}^2 u_x\tau xuu - 4u_t^2 u_{xxx}\tau xuu - 24u_t u_{xx}u_{xt}\tau xuu - 4u_x^3 u_{xtt}\tau uuu - 12u_t u_x^2 u_{xxt}\tau uuu
\end{aligned}
$$

$$
\begin{aligned}
&- 6u_{xx}u_{tt}u_x^2\tau uuu - 12u_{xt}^2u_x^2\tau uuu - 4u_t^2u_{xxx}\tau uuu - 24u_xu_tu_{xx}u_{xt}\tau uuu - 3u_t^2u_{xx}^2\tau uuu\\
&- 12u_xu_{xxtt}\tau xu - 8u_tu_{xxxt}\tau xu - 12u_{xx}u_{xtt}\tau xu - 24u_{xxt}u_{xt}\tau xu - 4u_{xxx}u_{tt}\tau xu\\
&- u_{tt}\tau_{xxxx} + u_t\eta_{xxxxu} - 10u_x^3u_{xx}\xi_{tuuu} - 24u_x^2u_{xx}\xi_{xtuu} + u_xu_{xx}\left(12\eta_{xtuu} - 18\xi_{xxtu}\right)\\
&- 6u_x^2u_t^2\tau_{xxuuu} - u_x^2u_t(4\xi_{xxxuu} - 6\tau_{xxtuu}) + u_x^3u_t\left(4\eta_{xuuuu} - 6\xi_{xxuuu} - 4\tau_{xtuuu}\right)\\
&+ u_x^4u_t\left(\eta_{uuuuu} - 4\xi_{xuuuu} - \tau_{tuuuu}\right) + 6u_x^2u_t\eta_{xxuuu} - 4u_x^3u_t^2\tau_{xuuuu} - 10u_x^3u_{xx}u_t\xi_{uuuu}\\
&- 5u_x^4u_{xt}\xi_{uuuu} + u_x^2u_{xx}u_t\left(6\eta_{uuuu} - 24\xi_{xuuu} - 6\tau_{tuuu}\right) + u_x^3u_{xt}\left(4\eta_{uuuu} - 16\xi_{xuuu} - 4\tau_{tuuu}\right)\\
&- 4u_x^3u_{tt}\tau_{xuuu} - 24u_x^2u_tu_{xt}\tau_{xuuu} - 12u_{xx}u_t^2\tau_{xuuu} - 10u_x^2u_{xxx}\xi_{xuu} - 15u_xu_{xx}^2\xi_{xuu}\\
&+ 6u_x^2u_{xt}\left(2\eta_{xuuu} - 2\tau_{xtuu} - 3\xi_{xxuu}\right) + 6u_xu_tu_{xx}\left(2\eta_{xuuu} - 2\tau_{xtuu} - 3\xi_{xxuu}\right) - 10u_x^3u_{xxt}\xi_{uuu}\\
&- 10u_tu_x^2u_{xxx}\xi_{uuu} - 30u_{xx}u_{xt}uu_x^2\xi_{uuu} - 15u_{xx}^2u_xu_t\xi_{uuu} - u_x^4u_t^2\tau_{uuuuu} - 8u_x^3u_tu_{xt}\tau_{uuuu}\\
&+ u_xu_t\left(4\eta_{xxxuu} - \xi_{xxxxu} - 4\tau_{xxxtu}\right) - u_x^5u_t\xi_{uuuuu} - u_x^4u_{xt}\tau_{uuuu} - 6u_x^2u_t^2u_{xx}\tau_{uuuu}\\
&- 4u_xu_t^2\tau_{xxxuu} + 6u_x^2u_{tt}\eta_{tuuu} - 24u_x^2u_{xxt}\xi_{xuu} + 12u_{xx}u_{xt}u_x\left(\eta_{uuu} - 4\xi_{xuu}\right) - 16u_xu_tu_{xxx}\xi_{xuu}\\
&- 12u_{xx}^2u_t\xi_{xuu} + u_xu_{xxt}12\eta_{xuu} + u_{xxx}u_t4\eta_{xuu} + u_{xx}u_{xt}\eta_{xuu} - 6u_x^2u_{xxxtt}\tau_{uu} - 8u_xu_tu_{xxxt}\tau_{uu}\\
&- 12u_xu_{xx}u_{xtt}\tau_{uu} - 24u_xu_{xt}u_{xxt}\tau_{uu} - 4u_xu_{tt}u_{xxx}\tau_{uu} - u_t^2u_{xxxx}\tau_{uu} - 12u_{xx}u_{xxt}u_t\tau_{uu}\\
&- 8u_{xxx}u_{xt}u_t\tau_{uu} - 3u_{xx}^2u_{tt}\tau_{uu} - 12u_{xx}u_{xt}^2\tau_{uu} + 6u_x^2u_{xxt}\eta_{uuu} + 4u_xu_tu_{xxx}\eta_{uuu} + 3u_{xx}^2u_t\eta_{uuu}\\
&- 12u_xu_{xt}\tau_{xxtu} - 6u_{xx}u_t\tau_{xxtu} + u_xu_{xxx}\left(\eta_{tuu} - 16\xi_{xtu}\right) + u_{xx}^2\left(\eta_{tuu} - 12\xi_{xtu}\right) - 8u_xu_{xt}\xi_{xxxu}\\
&- 4u_{xx}u_t\xi_{xxxu} + 12u_xu_{xt}\eta_{xxuu} + 6u_{xx}u_t\eta_{xxuu} - 12u_{xxt}u_t\tau_{xxu} - 12u_xu_{xtt}\tau_{xxu} - 6u_{xx}u_{tt}\tau_{xxu}\\
&- 12u_{xt}^2\tau_{xxu} - 4u_{xxx}u_{xt}\tau_{xu} - 6u_{xx}u_{xxt}\tau_{tu} - u_tu_{xxxx}\tau_{tu} - 4u_xu_{xxxt}\tau_{tu} - 6u_{xxtt}\tau_{xx}\\
&- 4u_{xxxtt} - u_{xxxxx}\xi_t + u_{xxxxt}\left(\eta_u - \tau_t - 4\xi_x\right) - u_{xxxxt}u_t\tau_u - u_{xxxxx}u_t\xi_u.\tau_x
\end{aligned}
\tag{6}
$$

The Lie classical method for determining the similarity variables of (2) is mainly consists of finding the infinitesimals $\tau, \xi,$ and $\eta,$ which are functions of x, t, u. After substituting the values of $\eta^x, \eta^t, \eta^{xt}, \eta^{xx}, \eta^{xxx}, \eta^{ttt}, \eta^{txxxx}$ and η^{xxt} from (5) to (4) and equating the coefficients of different differentials of u to zero, we get a number of PDEs in τ, ξ, and η, that need to be satisfied. Solving these system of PDEs, we obtain the infinitesimals τ, ξ, and η as follows:

$$
\begin{aligned}
\tau &= C_1 + tC_4\\
\xi &= C_2 + \frac{x}{2}C_4\\
\eta &= C_3 - \frac{u}{2}C_4,
\end{aligned}
\tag{7}
$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Corresponding vector fields can be written as:

$$
V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}, V_3 = \frac{\partial}{\partial u}, V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}. \tag{8}
$$

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3. Symmetry reductions and invariant solutions

To obtain the symmetry reductions of Eq. (2), we have to solve the characteristic equation:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{9}$$

where ξ, τ and η are given by Eq. (7).

To solve Eq. (9), following cases will be considered: (*i*) $V_1 + \mu V_2 + \lambda V_3$ and (*ii*) V_4, where μ, λ are arbitrary constants.

Case (i) $V_1 + \mu V_2 + \lambda V_3$

On solving Eqs. (9) we have,

$$\begin{aligned} \rho &= x - \mu t \\ u &= \lambda t + F(\rho), \end{aligned} \tag{10}$$

where ρ is new independent variables and $F(\rho)$ is new dependent variable. Substituting (10) into Eq. (2), we obtain the reduced ODE which reads:

$$\left[\mu(2\alpha+\beta)F' - \mu^3 - \alpha\lambda\right]F''' + \mu\left[(2\alpha+\beta)F''^2 + F'''''\right] = 0, \tag{11}$$

where primes ($'$) denotes derivative with respect to ρ.

Let assume the solution of ODE (11) in following form:

$$F = a_0 + a_1\rho + \frac{a_2}{\rho}, \tag{12}$$

where a_0, a_1, and a_2 needs to be determined. Substituting (12) into ODE (11) and equating coefficients of the different powers of ρ equal to zero, we obtain:

$$\begin{aligned} a_0 &= arbitrary \\ a_1 &= \frac{\mu^3 + \alpha\lambda}{\mu(2\alpha+\beta)} \\ a_2 &= \frac{12}{2\alpha+\beta}. \end{aligned} \tag{13}$$

Corresponding solution of ODE (11) can be written as:

$$F = a_0 + \left(\frac{\mu^3 + \alpha\lambda}{\mu(2\alpha+\beta)}\right)\rho + \frac{12}{(2\alpha+\beta)\rho}, \tag{14}$$

where $\beta \neq -2\alpha$.

Corresponding solution of main Eq. (2) is given by:

$$u(x,t) = \lambda t + a_0 + \left(\frac{\mu^3 + \alpha\lambda}{\mu(2\alpha+\beta)}\right)(x - \mu t) + \frac{12}{(2\alpha+\beta)(x-\mu t)}, \tag{15}$$

with $\beta \neq -2\alpha$.

Some more solutions of ODE (11) are given by:

$$\begin{aligned}
&(i)\, F(\rho) = b_3 \pm \frac{6\sqrt{\mu(\alpha\lambda+\mu^3)}}{\mu(2\alpha+\beta)} \tanh\left(b_1 - \frac{\sqrt{\mu(\alpha\lambda+\mu^3)}\rho}{2\mu}\right) \text{with } \beta \neq -2\alpha,\\
&(ii)\, F(\rho) = b_4 + b_5 \cosh\left(b_2 \pm \frac{\sqrt{\mu(\alpha\lambda+\mu^3)}\rho}{\mu}\right) \text{with } \beta = -2\alpha,\\
&(iii)\, F(\rho) = b_3 + b_4 \coth\left(b_1 + \frac{1}{2}\frac{\sqrt{\mu(\alpha\lambda+\mu^3)}\rho}{\mu}\right) \text{with } \beta = \frac{2}{b_4}\left(-b_4\alpha + 3\frac{\sqrt{\mu(\alpha\lambda+\mu^3)}}{\mu}\right),
\end{aligned} \tag{16}$$

where b_1, b_2, b_3, b_4 and b_5 are arbitrary constants.

Corresponding solutions of main Eq. (2) are given by:

$$(i)\, u(x,t) = \lambda t + b_3 \pm \frac{6\sqrt{\mu(\alpha\lambda+\mu^3)}}{\mu(2\alpha+\beta)} \tanh\left(b_1 - \frac{\sqrt{\mu(\alpha\lambda+\mu^3)}(x-\mu t)}{2\mu}\right) \text{with } \beta \neq -2\alpha, \tag{17}$$

$$(ii)\, u(x,t) = \lambda t + b_4 + b_5 \cosh\left(b_2 \pm \frac{\sqrt{\mu(\alpha\lambda+\mu^3)}(x-\mu t)}{\mu}\right) \text{with } \beta = -2\alpha, \tag{18}$$

$$\begin{aligned}
(iii)\, u(x,t) &= \lambda t + b_3 + b_4 \coth\left(b_1 + \frac{1}{2}\frac{\sqrt{\mu(\alpha\lambda+\mu^3)}(x-\mu t)}{\mu}\right)\\
&\text{with } \beta = \frac{2}{b_4}\left(-b_4\alpha + 3\frac{\sqrt{\mu(\alpha\lambda+\mu^3)}}{\mu}\right),
\end{aligned} \tag{19}$$

where b_1, b_2, b_3, b_4, and b_5 are arbitrary constants.

Case (ii) V_4

On solving Eq. (9) for vector field V_4, we have:

$$\begin{aligned}
\phi &= \frac{t}{x^2}\\
u &= \frac{G(\phi)}{x},
\end{aligned} \tag{20}$$

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where ϕ is new independent variables and $G(\phi)$ is new dependent variable. Substituting (20) into Eq. (2), we obtain the reduced ODE which reads

$$
\begin{aligned}
&(138\alpha + 54\beta)\phi G'^2 + (8(beta + 2\alpha)\phi^3 G''' + (30\alpha + 18\beta)G + (148\alpha + 68\beta)\phi^2 G'' - 360)G' \\
&+8(\beta + 2\alpha)\phi^3 G''^2 + 4(\alpha + \beta)\phi^2 GG''' + (26\alpha + 22\beta)\phi GG'' - 16\phi^4 G''''' + (1 - 1020\phi^2)G''' \\
&-1320\phi G'' - 240\phi^3 G'''' = 0,
\end{aligned}
\tag{21}
$$

where primes ($'$) denotes derivatives with respect to ϕ.

Let assume the solution of ODE (21) in following form:

$$G = \frac{b_2}{\phi^2} + \frac{b_1}{\phi} + a_0 + a_1\phi + a_2\phi^2, \tag{22}$$

where b_1, b_2, a_0, a_1 and a_2 needs to be determined.

Substituting (22) into ODE (21) and equating coefficients of the different powers of ϕ equal to zero, we obtain:

$$
\begin{aligned}
&(i)\ a_0 = arbitrary, a_1 = a_2 = b_1 = 0, b_2 = \frac{1}{5\alpha + 3\beta} \\
&(ii)\ a_0 = a_1 = a_2 = 0, b_1 = arbitrary, b_2 = \frac{1}{5\alpha + 3\beta}
\end{aligned}
\tag{23}
$$

Corresponding solution of ODE (21) can be written as:

$$
\begin{aligned}
&(i)\ G = \frac{1}{(5\alpha + 3\beta)\phi^2} + a_0, \\
&(ii)\ G = \frac{1}{(5\alpha + 3\beta)\phi^2} + \frac{b_1}{\phi},
\end{aligned}
\tag{24}
$$

where b_1 is arbitrary constant.

Corresponding solution of main Eq. (2) can be written as:

$$
\begin{aligned}
&(i)\ u(x,t) = \frac{1}{x}\left(\frac{x^4}{(5\alpha + 3\beta)t^2} + a_0\right), \\
&(ii)\ u(x,t) = \frac{x^3}{(5\alpha + 3\beta)t^2} + \frac{b_1 x}{t},
\end{aligned}
\tag{25}
$$

where b_1 is arbitrary constant.

4. Conclusion

In this chapter, we derived the symmetry variables and symmetry transformations of the generalized fifth order non-linear partial differential equation. We applied Lie symmetry analysis for investigating considered nonlinear partial differential equation and using similarity variables, given equation is reduced into ordinary differential equations. We derived explicit exact solutions of considered partial differential equation corresponding to each ordinary differential equation obtained by reduction.

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