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# Canonical Generalized Inversion Form of Kane's Equations of Motion for Constrained Mechanical Systems

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Additional information is available at the end of the chapter

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## Abstract

The canonical generalized inversion dynamical equations of motion for ideally constrained discrete mechanical systems are introduced in the framework of Kane's method. The canonical equations of motion employ the acceleration form of constraints and the Moore-Penrose generalized inversion-based Greville formula for general solutions of linear systems of algebraic equations. Moreover, the canonical equations of motion are explicit and nonminimal (full order) in the acceleration variables, and their derivation is made without appealing to the principle of virtual work or to Lagrange multipliers. The geometry of constrained motion is revealed by the canonical equations of motion in a clear and intuitive manner by partitioning the canonical accelerations' column matrix into two portions: a portion that drives the mechanical system to abide by the constraints and a portion that generates the momentum balance dynamics of the mechanical system. Some geometrical perspectives of the canonical equations of motion are illustrated via vectorial geometric visualization, which leads to verifying the Gauss' principle of least constraints and its Udwadia-Kalaba interpretation.

**Keywords:** canonical equations of motion, discrete mechanical systems, Kane's method, Gauss' principle of least constraints, canonical generalized speeds, Greville formula

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## 1. Introduction

Deriving mathematical models for dynamical systems is in the core of the discipline of analytical dynamics, and it is the step that precedes dynamical system's analysis, design, and control synthesis. For discrete mechanical systems, i.e., those composed of particles and rigid bodies, the mathematical models are in the forms of differential equations or differential/algebraic

equations that are derived by using fundamental laws of motion or energy principles. Because many mechanical systems nowadays are multi-bodied with numerous degrees of freedom and large numbers of *holonomic* and *nonholonomic* constraints, simplicity of the derived equations of motion is important for facilitating studying the mechanical system's characteristics and for extracting useful information out of its mathematical model. Hence, deriving the simplest possible form of the equations of motion that govern the dynamics of the mechanical system is crucial. Moreover, because the mechanical system's equations of motion are simulated on digital computers, computational efficiency of the derived differential equations of motion when numerically integrated is another factor by which the quality of the mathematical model is judged on.

It has been a general trend for over two centuries to employ d'Alembert's *principle of virtual work* [1] to derive equations of motion that involve no constraint forces. The principle was implemented by Lagrange [2] for deriving the first set of such equations, which constituted the first paradigm shift from the Newton-Euler's approach. The only other alternative to employing d'Alembert's principle has been to augment the equations with undetermined multipliers, an approach that was initiated by Lagrange himself. Other formulations that followed the trend include the Maggi [3] and Boltzmann-Hamel [4] formulations. A remarkable contribution of the Lagrangian approach to analytical dynamics is utilizing the concept of *generalized coordinates* instead of the Cartesian coordinate concept. The choice of generalized coordinates greatly affects simplicity of the derived equations of motion.

Another paradigm shift in the subject took place when Gibbs [5] and Appell [6] independently derived their equations of motion. For the first time, formulating the dynamical equations involved neither invoking d'Alembert's principle nor augmenting undetermined multipliers. Because d'Alembert's principle was to many analytical dynamics practitioners, "an ill-defined, nebulous, and hence objectionable principle," [7] the Gibbs-Appell model was widely accepted within the analytical dynamics community. Moreover, the absence of undetermined multipliers from the Gibbs-Appell equations contributed to maintaining simplicity and practicality of the equations for large constrained mechanical systems. Another feature of the Gibbs-Appell approach was initiating the concept of *quasi-velocities*, which equal in their number to the number of the degrees of freedom of the mechanical system. Similar to the advantage of generalized coordinates, carefully chosen quasi-velocities can lead to dramatic simplifications of the dynamical equations of motion.

One feature that is associated with the Gibbs-Appell's approach is that it is based on the differential *Gauss' principle of least constraints* [8] as was shown by Appell, [9] in contrast to the Lagrange's approach that is based on the variational *Hamilton's principle of least action* [10] as opposed. Another feature is adopting the *acceleration form of constraints* to model a mechanical system's constraints. Although easy by itself, employing the acceleration form eased the historical hurdle of modeling nonholonomic constraints that used to obstruct variational-based formulations, and it is a consequence of the differential theme that is based on Gauss' principle. In particular, the acceleration form bypassed d'Alembert's principle and the undetermined multiplier augmentation practices that produce false equations of nonholonomically constrained motion, and it unified the treatments of holonomic and nonholonomic constraints.

A key developments in the arena of analytical dynamics is the Kane's method for modeling constrained discrete mechanical systems [11–13]. Kane's method adopts a vector approach that

inspired useful geometric features of the derived equations of motion [14]. The *generalized active forces* and *generalized inertia forces* are obtained by scalar (dot) multiplications of the active and inertia forces, respectively, with the vector entities *partial angular velocities* and *partial velocities*. This process delicately eliminates the contribution of constraint forces without invoking the principle of virtual work. The resulting equations are simple and effective in describing the motion of nonconservative and nonholonomic systems within the same framework, requiring neither energy methods nor Lagrange multipliers.

The standard Kane's equations of motion for nonholonomic systems are minimal in generalized speeds, i.e., their number is equal to the number of degrees of freedom of the dynamical system, and only the independent portion of generalized speeds and their time derivatives appear in the equations. Nevertheless, information about dependent generalized speeds can be practically important, e.g., for the purpose of obtaining stability information about a dependent dynamics or when it desired to target a dependent dynamics with a control system design by using state space control methodologies.

On the other hand, generalized inversion and the Greville formula for general solutions of linear systems of algebraic equations were introduced to the subject of analytical dynamics by Udwadia and Kalaba [15, 16] as tools for deriving equations of constrained motion for discrete mechanical systems. The success that the formula met in modeling ideally constrained motion is due to its geometrical structure that captures orthogonality of ideal constraint forces on active and inertia forces, which is the essence of the principle of virtual work.

Inspired by the Udwadia-Kalaba equations of motion and the Greville formula, this chapter introduces a new form of Kane's equations of motion. The introduced equations of motion employ the acceleration form of constraints, and therefore holonomic and nonholonomic constraints are augmented within the momentum balance formulation in a unified manner and irrespective of being linear or nonlinear in generalized coordinates and generalized speeds. The equations of motion are nonminimal, i.e., no reduction of generalized speed's space dimensionality takes place from the number of generalized coordinates to the number of degrees of freedom. Furthermore, the new equations of motion are explicit, i.e., are separated in the generalized acceleration variables, and only one generalized acceleration variable appears in each equation.

The main feature of the derived equations of motion is the explicit algebraic and geometric partitioning of the generalized acceleration vector at every instant of time into two portions: one portion drives the mechanical system to abide by the constraint dynamics, and the other portion generates the momentum balance of the mechanical system as to follow Newton-Euler's laws of motion.

## 2. Kane's equations of motion for holonomic systems

Consider a set of  $\nu$  particles and  $\mu$  rigid bodies that form a *holonomic system*  $S_h$  possessing  $n$  degrees of freedom in an inertial reference frame  $J$ . Assume that  $n$  *generalized coordinates*  $q_1, \dots, q_n$  are used to describe the configuration of the system. Then a corresponding set of  $n$

*holonomic generalized speeds*  $u_{h_1}, \dots, u_{h_n}$  is used to model the kinematics of the system. The two sets are related by the kinematical differential equations [12, 13]:

$$\dot{q} = C(q, t)u_h + D(q, t), \quad (1)$$

where  $q \in \mathbb{R}^n$  is a column matrix containing the generalized coordinates;  $u_h \in \mathbb{R}^n$  is a column matrix containing the generalized speeds,  $\dot{q} = dq/dt$ ,  $C \in \mathbb{R}^{n \times n}$ ,  $D \in \mathbb{R}^n$ ; and  $C^{-1}$  exists for all  $q \in \mathbb{R}^n$  and all  $t \in \mathbb{R}$  [12, 13]. Kane's dynamical equations of motion for  $S_h$  are given by [12, 13]

$$F_r(q, u_h, t) + F_r^*(q, u_h, \dot{u}_h, t) = 0, \quad r = 1, \dots, n, \quad (2)$$

where  $F_r$  and  $F_r^*$  are the  $r^{\text{th}}$  holonomic generalized active force and the  $r^{\text{th}}$  holonomic generalized inertia force on the system, respectively, and  $\dot{u}_h = du_h/dt \in \mathbb{R}^n$  is a column matrix containing the *generalized accelerations*. Furthermore, the velocities and angular velocities of the particles and bodies comprising a mechanical system are linear in the generalized speeds  $u_{h_r}$ . Hence, the accelerations, angular accelerations, and consequently the generalized inertia forces are linear in the generalized accelerations  $\dot{u}_{h_r}$ . Therefore, a column matrix  $F^* \in \mathbb{R}^n$  containing  $F_r^*$ ,  $r = 1, \dots, n$  can be written in the following form [17]:

$$F^*(q, u_h, \dot{u}_h, t) = -Q(q, t)\dot{u}_h - L(q, u_h, t), \quad (3)$$

where the generalized inertia matrix  $Q \in \mathbb{R}^{n \times n}$  is assumed symmetric and positive definite and  $L \in \mathbb{R}^n$ . Hence, a matrix form of (2) is written as [17]

$$Q(q, t)\dot{u}_h = -L(q, u_h, t) + F(q, u_h, t). \quad (4)$$

### 3. Kane's equations of motion for nonholonomic systems

Let us now consider a modification of the kinematics of  $S_h$  that is made by imposing the following simple nonholonomic constraints on the generalized speeds [12, 13]:

$$u_{p+r} = \sum_{s=1}^p A_{rs}(q, t)u_s + B_r(q, t), \quad r = 1, \dots, m, \quad (5)$$

where  $u_1, \dots, u_n$  are the generalized speeds of the *nonholonomic system*  $S$  that is resulting from constraining  $S_h$  according to (5),  $m = n - p$ , and  $A_{rs}$  and  $B_r$  are scalar functions of the generalized coordinates  $q_1, \dots, q_n$ , and  $t$ . The *nonholonomic generalized speeds* are considered to satisfy the same kinematical relations with generalized coordinates as their holonomic counterparts, i.e.,

$$\dot{q} = C(q, t)u + D(q, t). \quad (6)$$

The system dynamics of  $S$  changes from that given by (2) accordingly. Let the generalized speed column matrix be partitioned as

$$u = [u_I^T \ u_D^T]^T, \quad (7)$$

where  $u_I = [u_1 \ \cdots \ u_p]^T$  and  $u_D = [u_{p+1} \ \cdots \ u_n]^T$ . Kane's dynamical equations of motion for  $S$  are given by [12, 13]

$$\tilde{F}_r(q, u_I, t) + \tilde{F}_r^*(q, u_I, \dot{u}_I, t) = 0, \quad r = 1, \dots, p, \quad (8)$$

where  $\tilde{F}_r$  and  $\tilde{F}_r^*$  are the  $r^{th}$  nonholonomic generalized active force and the  $r^{th}$  nonholonomic generalized inertia force on  $S$ , respectively. The relationships between holonomic generalized active forces on  $S_h$  and nonholonomic generalized active forces on  $S$  are given by [12, 13]

$$\tilde{F}_r(q, u_I, t) = F_r(q, u, t) + \sum_{s=1}^m F_{p+s}(q, u, t) A_{sr}, \quad r = 1, \dots, p. \quad (9)$$

In a similar manner, the relationships between holonomic generalized inertia forces on  $S_h$  and nonholonomic generalized inertia forces on  $S$  are given by [12, 13]

$$\tilde{F}_r^*(q, u_I, \dot{u}_I, t) = F_r^*(q, u, \dot{u}, t) + \sum_{s=1}^m F_{p+s}^*(q, u, \dot{u}, t) A_{sr}(q, t), \quad r = 1, \dots, p. \quad (10)$$

Substituting (9) and (10) in (8) yields the unreduced form of Kane's equations of motion for  $S$  [12, 13, 17]:

$$F_r(q, u, t) + F_r^*(q, u, \dot{u}, t) + \sum_{s=1}^m \left( F_{p+s}(q, u, t) + F_{p+s}^*(q, u, \dot{u}, t) \right) A_{sr}(q, t) = 0, \quad r = 1, \dots, p. \quad (11)$$

The simple nonholonomic constraint equations given by (5) can be rewritten in the following matrix representation [17]:

$$u_D = A(q, t)u_I + B(q, t), \quad (12)$$

where  $A \in \mathbb{R}^{m \times p}$  and  $B \in \mathbb{R}^m$ . Furthermore, (12) can be rewritten as [17]

$$A_1(q, t)u = B(q, t), \quad (13)$$

where  $A_1 \in \mathbb{R}^{m \times n}$  is given by

$$A_1(q, t) = [-A(q, t) \ I_{m \times m}]. \quad (14)$$

Also, (11) can be rewritten in the matrix form [17]:

$$A_2(q, t)F^*(q, u, \dot{u}, t) = -A_2(q, t)F(q, u, t), \quad (15)$$

where  $A_2 \in \mathbb{R}^{p \times n}$  is given by



$$A_2(q, t) = \begin{bmatrix} I_{p \times p} & A^T(q, t) \end{bmatrix}. \quad (16)$$

Hence, (15) becomes [17]

$$A_2(q, t)Q(q, t)\dot{u} = -A_2(q, t)L(q, u, t) + A_2(q, t)F(q, u, t). \quad (17)$$

Notice that (17) is obtained by multiplying both sides of (4) by  $A_2(q, t)$ . Therefore, the unique holonomic generalized acceleration vector  $\dot{u}_h$  that solves the fully determined system given by (4) solves the underdetermined system given by (17) also, among an infinite number of generalized acceleration vectors that satisfy (17), each of which preserves a constrained momentum balance dynamics of the mechanical system.

#### 4. Canonical generalized speeds

Choosing the set of generalized speeds is a crucial step in formulating Kane's dynamical Eqs. (2) and (8) because the extent of how complex these equations appear is affected by this choice. For every choice of nonholonomic generalized speeds  $u_1, \dots, u_n$ , we define the *canonical set of nonholonomic generalized speeds*  $w_1, \dots, w_n$  such that

$$w = Q^{1/2}(q, t)u, \quad (18)$$

where  $w$  is the column matrix containing  $w_1, \dots, w_n$  and  $Q^{1/2}$  is the square root matrix of  $Q$ . With this choice of generalized speeds, (13) becomes

$$\mathcal{A}_1(q, t)w = B(q, t), \quad (19)$$

where  $\mathcal{A}_1(q, t) = A_1(q, t)Q^{-1/2}(q, t)$ . The time derivative of (18) is

$$\dot{w} = \dot{Q}^{1/2}(q, u, t)u + Q^{1/2}(q, t)\dot{u} \quad (20)$$

where  $\dot{Q}^{1/2}$  is the element-wise time derivative of  $Q^{1/2}$  along the trajectory solutions of the kinematical differential Eqs. (6). Therefore, (17) becomes

$$A_2(q, t)Q^{1/2}(q, t)\dot{w} = -A_2(q, t)L(q, u, t) + A_2(q, t)F(q, u, t) + A_2(q, t)Q^{1/2}(q, t)\dot{Q}^{1/2}(q, u, t)u \quad (21)$$

and can be simplified further to the following form:

$$\mathcal{A}_2(q, t)\dot{w} = \mathcal{A}_2(q, t)Q^{-1/2}(q, t)(F(q, u, t) - L(q, u, t)) + \mathcal{A}_2(q, t)\dot{Q}^{1/2}(q, u, t)u \quad (22)$$

where  $\mathcal{A}_2(q, t) = A_2(q, t)Q^{1/2}(q, t)$ . We view the nonholonomic mechanical system dynamics as being composed of two parts: a constraint dynamics that is modeled by (19) and a momentum balance dynamics that is modeled by (22). Scaling velocity variables and constraint matrices by square roots of the inertia matrices for the purpose of characterizing constrained motion is implicit in Gauss' principle of least constraints [8] as will be shown later in this paper.

Moreover, deriving explicit equations of motion for constrained mechanical systems by utilizing this type of scaling is first due to Udwadia and Kalaba. [15, 16] The arguments of the functions are omitted in the remaining sections for brevity, unless necessary to clarify concepts.

## 5. Generalized accelerations from the acceleration form of constraints

Time differentiating the constraint dynamics given by (19) yields [17]

$$\mathcal{A}_1 \dot{w} = V_1, \quad (23)$$

where  $V_1 \in \mathbb{R}^m$  is given by

$$V_1 = \dot{B}(q, u, t) - \dot{\mathcal{A}}_1(q, u, t)w, \quad (24)$$

where  $\dot{B}$  and  $\dot{\mathcal{A}}_1$  are the element-wise time derivatives of  $B$  and  $\mathcal{A}_1$  along the trajectory solutions of the kinematical differential Eqs. (6). The general solution of the above-written acceleration form of constraint equations for  $\dot{w}$  is given by the Greville formula as [18–20]

$$\dot{w} = \mathcal{A}_1^+ V_1 + \mathcal{P}_1 y_1, \quad (25)$$

where  $\mathcal{A}_1^+$  is the Moore-Penrose generalized inverse (MPGI) [21, 22] of  $\mathcal{A}_1$  and

$$\mathcal{P}_1 = I_{n \times n} - \mathcal{A}_1^+ \mathcal{A}_1 \quad (26)$$

is the projection matrix on the nullspace of  $\mathcal{A}_1$  and  $y_1 \in \mathbb{R}^n$  is an arbitrary vector as for satisfying the acceleration form given by (25) but is yet to be determined to obtain the unique natural generalized acceleration. Because  $Q^{-1/2}$  is of full rank, it follows that  $\mathcal{A}_1$  retains the full row rank of  $A_1$  and hence that  $\mathcal{A}_1^+ \in \mathbb{R}^{n \times m}$  is given by the closed form expression:

$$\mathcal{A}_1^+ = \mathcal{A}_1^T (\mathcal{A}_1 \mathcal{A}_1^T)^{-1}. \quad (27)$$

In (25), the following holds

$$\mathcal{A}_1^+ V_1 \in \mathcal{R}(\mathcal{A}_1^T), \quad \mathcal{P}_1 y_1 \in \mathcal{N}(\mathcal{A}_1) \quad (28)$$

where  $\mathcal{R}(\cdot)$  and  $\mathcal{N}(\cdot)$  refer to the range space and the nullspace, respectively. The term  $\mathcal{A}_1^+ V_1$  in (25) is the minimum norm solution of (23) for  $\dot{w}$  among infinitely many solutions that are parameterized by  $y_1$ .

## 6. Generalized accelerations from the momentum balance dynamics

Let  $V_2 \in \mathbb{R}^n$  be given by



$$V_2 = Q^{-1/2}(F - L) + \dot{Q}^{1/2}u. \quad (29)$$

Then the momentum balance Eq. (22) takes the following compact form:

$$\mathcal{A}_2 \dot{w} = \mathcal{A}_2 V_2 \quad (30)$$

where  $A_2$  retains the full row rank of  $A_2$  because  $Q^{1/2}$  is of full row rank. Hence, another expression for the general solution of  $\dot{w}$  is obtained by utilizing the Greville formula to solve (30) and is given by

$$\dot{w} = \mathcal{A}_2^+ \mathcal{A}_2 V_2 + \mathcal{P}_2 y_2 \quad (31)$$

where  $\mathcal{A}_2^+ \in \mathbb{R}^{n \times p}$  is given by the closed form expression:

$$\mathcal{A}_2^+ = \mathcal{A}_2^T (\mathcal{A}_2 \mathcal{A}_2^T)^{-1} \quad (32)$$

and

$$\mathcal{P}_2 = I_{n \times n} - \mathcal{A}_2^+ \mathcal{A}_2 \quad (33)$$

and  $y_2 \in \mathbb{R}^n$  is an arbitrary vector as for satisfying the momentum balance dynamics given by (30), but its unique value that solves for the natural generalized acceleration vector  $\dot{w}$  is yet to be determined, and

$$\mathcal{A}_2^+ \mathcal{A}_2 V_2 \in \mathcal{R}(\mathcal{A}_2^+ \mathcal{A}_2) = \mathcal{R}(\mathcal{A}_2^T), \quad \mathcal{P}_2 y_2 \in \mathcal{N}(\mathcal{A}_2). \quad (34)$$

The term  $\mathcal{A}_2^+ \mathcal{A}_2 V_2$  in (31) is the minimum norm solution of (30) for  $\dot{w}$  among infinitely many solutions that are parameterized by  $y_2$ .

## 7. Canonical generalized inversion Kane's equations of motion

Since  $A_1$  and  $A_2$  are full row rank matrices and their numbers of rows  $m$  and  $p$  sum up to the full space dimension  $n$  and since

$$\begin{aligned} \mathcal{A}_1 \mathcal{A}_2^T &= (A_1 Q^{-1/2}) (A_2 Q^{1/2})^T = (A_1 Q^{-1/2}) (A_2 Q^{1/2})^T \\ &= A_1 Q^{-1/2} Q^{1/2} A_2^T = A_1 A_2^T = -A + A = \mathbf{0}_{m \times m} \end{aligned} \quad (35)$$

it follows that the row spaces of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are orthogonal complements, i.e.,

$$\mathcal{R}(\mathcal{A}_1^T) = [\mathcal{R}(\mathcal{A}_2^T)]^\perp. \quad (36)$$

Nevertheless, since [23]

$$\mathcal{R}(\mathcal{A}_1^T) = [\mathcal{N}(\mathcal{A}_1)]^\perp \quad (37)$$

then it follows from (36) that

$$\mathcal{R}(\mathcal{A}_2^T) = \mathcal{N}(\mathcal{A}_1). \quad (38)$$

Since the only part in the expression of  $\dot{w}$  given by (25) that is in  $\mathcal{N}(\mathcal{A}_1)$  is the second term  $\mathcal{P}_1 y_1$ , and since the only part in the equivalent expression of  $\dot{w}$  given by (31) that is in  $\mathcal{R}(\mathcal{A}_2^T)$  is the first term  $\mathcal{A}_2^+ \mathcal{A}_2 V_2$ , it follows from (38) that

$$\mathcal{P}_1 y_1 = \mathcal{A}_2^+ \mathcal{A}_2 V_2. \quad (39)$$

Substituting (39) in (25) yields the canonical generalized inversion form of Kane's equations for nonholonomic systems:

$$\dot{w} = \mathcal{A}_1^+ V_1 + \mathcal{A}_2^+ \mathcal{A}_2 V_2. \quad (40)$$

The same result is obtained by using the fact:

$$\mathcal{N}(\mathcal{A}_2) = [\mathcal{R}(\mathcal{A}_2^T)]^\perp \quad (41)$$

which implies by using (36) that

$$\mathcal{R}(\mathcal{A}_1^T) = \mathcal{N}(\mathcal{A}_2) \quad (42)$$

Since the only part in the expression of  $\dot{w}$  given by (25) that is in  $\mathcal{R}(\mathcal{A}_1^T)$  is the first term  $\mathcal{A}_1^+ V_1$ , and since the only part in the equivalent expression of  $\dot{w}$  given by (31) that is in  $\mathcal{N}(\mathcal{A}_2)$  is the second term  $\mathcal{P}_2 y_2$ , it follows from (42) that

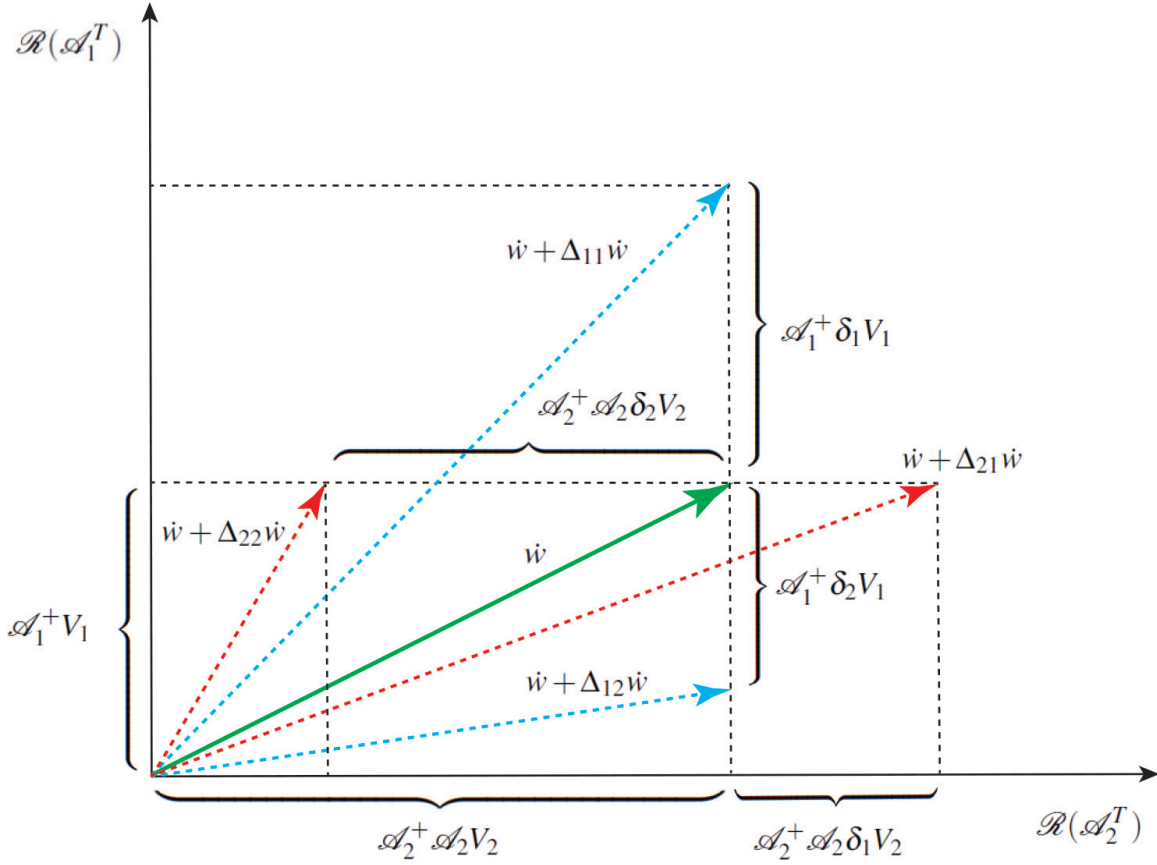
$$\mathcal{P}_2 y_2 = \mathcal{A}_1^+ V_1. \quad (43)$$

(Substituting (43) in (31) yields Eq. (40). Eq. (20) can be used to express (40) in terms of the original generalized acceleration vector  $\ddot{u}$ , resulting in

$$\ddot{u} = Q^{-1/2} \left( \mathcal{A}_1^+ V_1 + \mathcal{A}_2^+ \mathcal{A}_2 V_2 - \dot{Q}^{1/2} u \right). \quad (44)$$

## 8. Geometric interpretation of the canonical generalized inversion form

Adopting the canonical set  $w_1, \dots, w_n$  of generalized speeds in deriving the dynamical equations for a mechanical system reveals the geometry of its constrained motion. **Figure 1** depicts a geometrical visualization of the  $n$  dimensional Euclidian space at an arbitrary time instant  $t$ . The vertical and the horizontal axes resemble the orthogonally complements  $m$  dimensional and  $p$  dimensional subspaces  $\mathcal{R}(\mathcal{A}_1^T)$  and  $\mathcal{R}(\mathcal{A}_2^T)$ , respectively.



**Figure 1.** Geometric visualization of the constrained generalized acceleration vector  $\dot{w}$ .

In viewing the canonical generalized acceleration  $\dot{w}$  given by (40) as the geometrical vector shown in **Figure 1**, it is shown to be composed of two components that are orthogonal to each other: The vertical component  $\mathcal{A}_1^+ V_1$  resides in  $\mathcal{R}(\mathcal{A}_1^T)$ , and it enforces the constraint dynamics given by (23), and the horizontal component  $\mathcal{A}_2^+ \mathcal{A}_2 V_2$  resides in  $\mathcal{R}(\mathcal{A}_2^T)$ , and it generates the momentum balance dynamics given by (30).

Moreover, the vertical component of  $\dot{w}$  is the shortest in length “minimum norm” solution among infinitely many solutions of (23) that are parameterized by  $y_1$  according to (25). These solutions can also be represented by arbitrary horizontal deviation vectors:  $\Delta_{2i}\dot{w} = \mathcal{A}_2^+ \mathcal{A}_2 \delta_i V_2 \in \mathcal{R}(\mathcal{A}_2^T)$ ,  $i = 1, 2, \dots$  as

$$\dot{w} + \Delta_{2i}\dot{w} = \mathcal{A}_1^+ V_1 + \mathcal{A}_2^+ \mathcal{A}_2 (V_2 + \delta_i V_2) \quad (45)$$

and are shown to solve (23) by direct substitution and noticing that  $\mathcal{A}_1 \mathcal{A}_1^+ = I_{m \times m}$  and  $\mathcal{A}_1 \mathcal{A}_2^+ = \mathbf{0}_{m \times p}$ . Two of these solutions are plotted (in dotted red) in **Figure 1** for arbitrary vectors  $\delta_1 V_2$  and  $\delta_2 V_2$  in  $\mathbb{R}^n$ , in addition to the natural generalized acceleration vector  $\dot{w}$  that is obtained by setting  $\delta_i V_2 = \mathbf{0}_n$ .

Similarly, the horizontal component  $\mathcal{A}_2 \mathcal{A}_2^+ V_2$  of  $\dot{w}$  is the shortest solution among infinitely many solutions of (30) that are parameterized by  $y_2$  according to (31). These solutions can also be represented by arbitrary vertical deviation vectors:  $\Delta_{1i}\dot{w} = \mathcal{A}_1^+ \delta_i V_1 \in \mathcal{R}(\mathcal{A}_1^T)$ ,  $i = 1, 2, \dots$  as

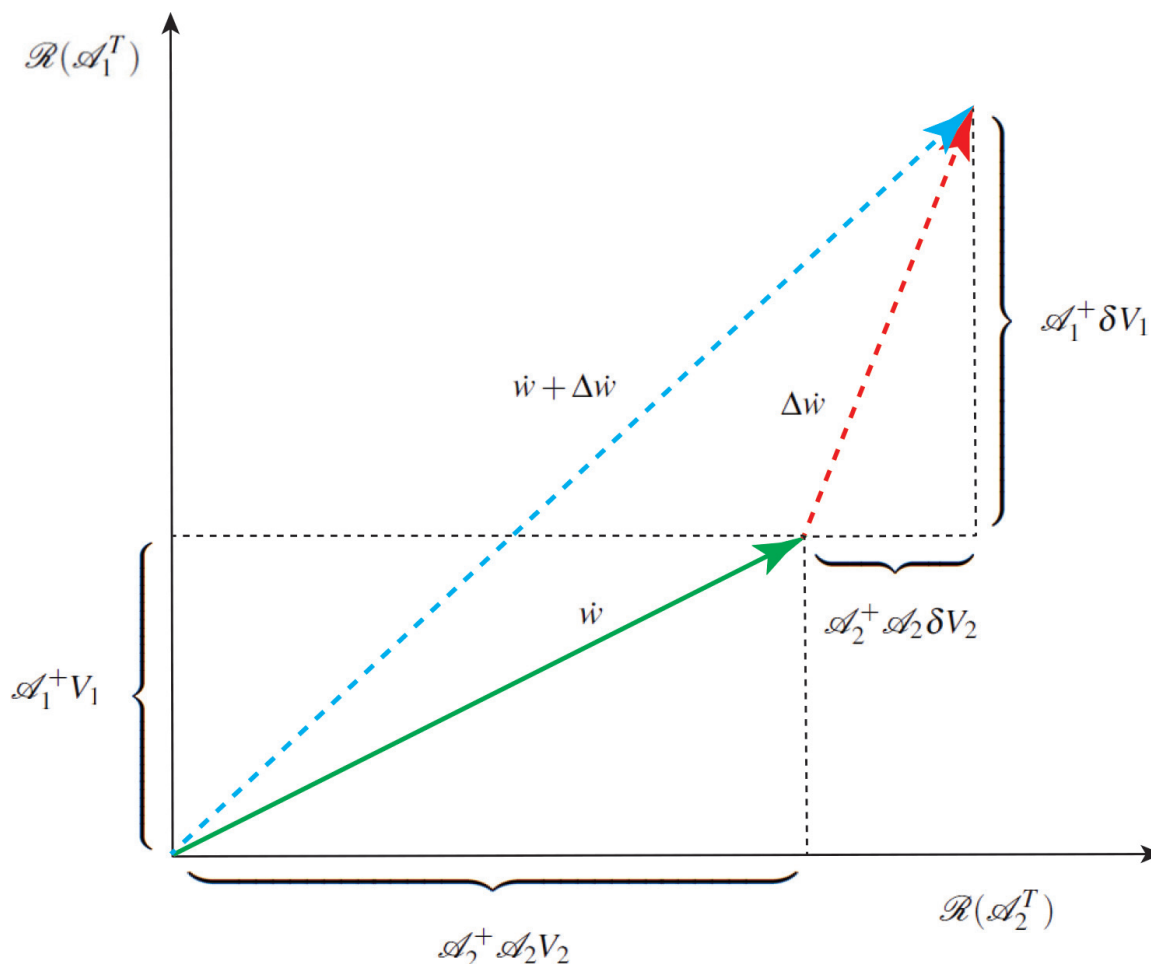
$$\dot{w} + \Delta_1 \dot{w} = \mathcal{A}_2^+ \mathcal{A}_2 V_2 + \mathcal{A}_1^+ (V_1 + \delta_i V_1) \quad (46)$$

and are shown to solve (30) by direct substitution and noticing that  $\mathcal{A}_2 \mathcal{A}_2^+ = I_{p \times p}$  and  $\mathcal{A}_2 \mathcal{A}_1^+ = \mathbf{0}_{p \times m}$ . Two of these solutions are plotted (in dotted blue) in **Figure 1** for arbitrary vectors  $\delta_1 V_1$  and  $\delta_2 V_1$  in  $\mathbb{R}^m$ , in addition to the natural generalized acceleration vector  $\dot{w}$  that is obtained by setting  $\delta_i V_1 = \mathbf{0}_m$ . Notice that the canonical generalized acceleration vector  $\dot{w}$  is the only solution that solves (45, 46) simultaneously and is obtained by setting  $\delta_i V_2 = \mathbf{0}_n$  and  $\delta_i V_1 = \mathbf{0}_m$ .

Now consider a general deviation vector  $\Delta \dot{w}$  that is composed of arbitrary vertical and horizontal deviation components from  $\dot{w}$  as shown in **Figure 2**. The vertical component  $\mathcal{A}_1^+ \delta V_1$  abides by (46) but violates (45), and the horizontal component  $\mathcal{A}_2^+ \mathcal{A}_2 \delta V_2$  abides by (45) but violates (46). Hence:

$$\Delta \dot{w} = \mathcal{A}_1^+ \delta V_1 + \mathcal{A}_2^+ \mathcal{A}_2 \delta V_2. \quad (47)$$

The deviated canonical generalized acceleration vector  $\dot{w} + \Delta \dot{w}$  is obtained by summing (40) and (47) as



**Figure 2.** Deviation from the constrained generalized acceleration vector  $\dot{w}$ .

$$\dot{w} + \Delta\dot{w} = \mathcal{A}_1^+(V_1 + \delta V_1) + \mathcal{A}_2^+\mathcal{A}_2(V_2 + \delta V_2) \quad (48)$$

and is shown in **Figure 2** in dotted blue. On the other hand, the canonical holonomic generalized acceleration vector in terms of the canonical generalized speeds is obtained from (4) and (20) as

$$\dot{w}_h = V_2 \quad (49)$$

where  $w_h = Q^{1/2}u$  and  $u$  solves (4). Decomposing the expression of  $\dot{w}_h$  along  $\mathcal{R}(\mathcal{A}_1)$  and  $\mathcal{R}(\mathcal{A}_2)$  yields

$$\dot{w}_h = V_2 = \mathcal{P}_1 V_2 + \mathcal{P}_2 V_2 \quad (50)$$

$$= (I_{n \times n} - \mathcal{A}_1^+ \mathcal{A}_1) V_2 + (I_{n \times n} - \mathcal{A}_2^+ \mathcal{A}_2) V_2 \quad (51)$$

$$= \mathcal{A}_2^+ \mathcal{A}_2 V_2 + \mathcal{A}_1^+ \mathcal{A}_1 V_2. \quad (52)$$

Let us now specify the deviated generalized acceleration vector  $\dot{w} + \Delta\dot{w}$  to be  $\dot{w}_h$  as shown in **Figure 3**. Equating the two expressions (48) and (52) and solving for  $\delta V_1$  and  $\delta V_2$  yield

$$\delta V_1 = \mathcal{A}_1 V_2 - V_1, \quad (53)$$

and

$$\delta V_2 = \mathbf{0}_n. \quad (54)$$

Substituting  $\delta V_1$  and  $\delta V_2$  in (47) yields

$$\Delta\dot{w} = \mathcal{A}_1^+(\mathcal{A}_1 V_2 - V_1) \quad (55)$$

which corresponds to the vertical solid red vector in **Fig. (3)**. Notice that  $\Delta\dot{w}$  is the shortest among all deviation vectors that end up at  $\dot{w}_h$  (two of which are shown in dotted red) by deviating from generalized acceleration vectors that abide by the constraint dynamics given by (23) (two of which are shown in dotted green), i.e.,

$$\|\Delta\dot{w}\| = \|\dot{w}_h - \dot{w}\| = \min_i \|\Delta\dot{w}_i\| = \min_i \|\dot{w}_h - \dot{w}_i\|, i = 1, 2, \dots \quad (56)$$

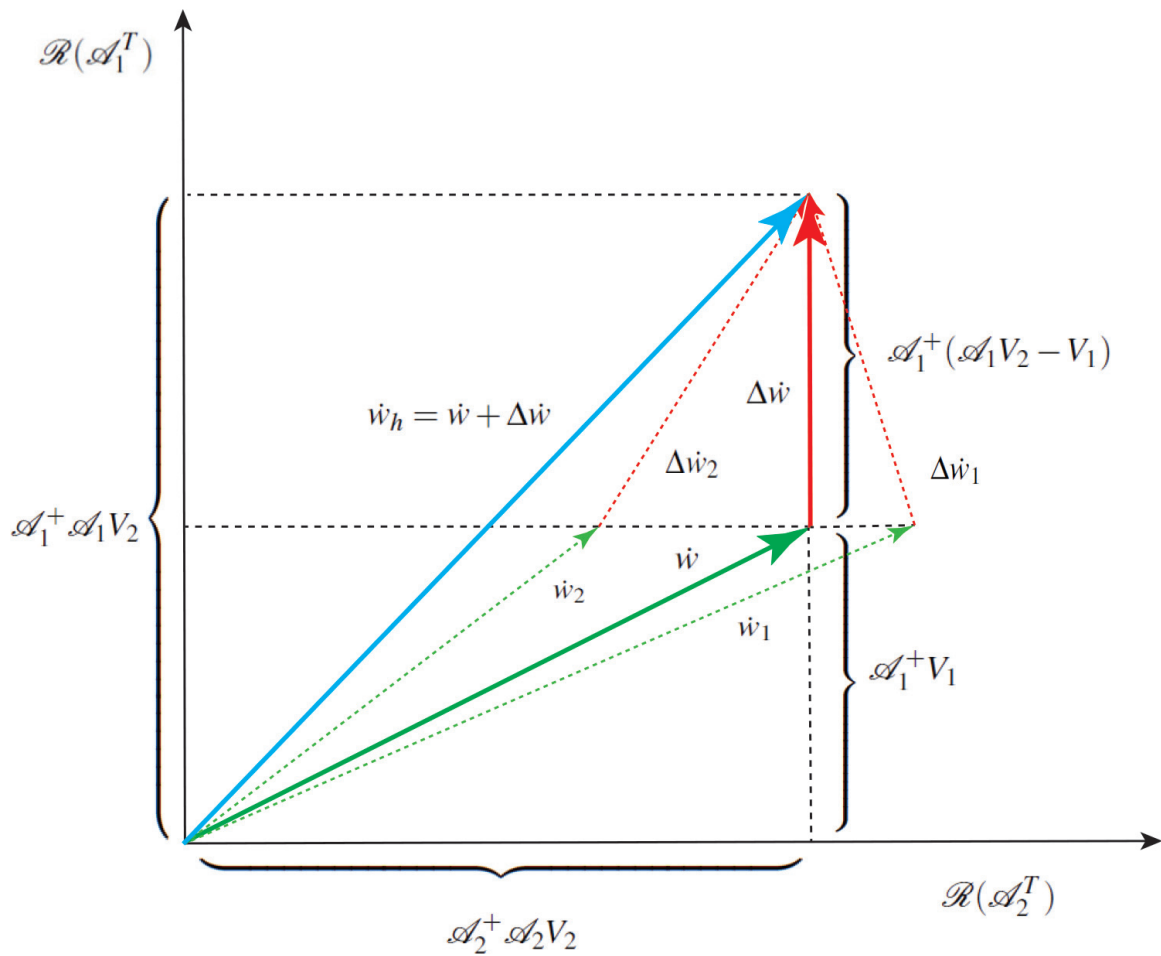
where  $\dot{w}_i$  satisfies

$$\mathcal{A}_1 \dot{w}_i = V_1 \quad (57)$$

and  $\|x\|$  denotes the Euclidean norm of  $x$  given by  $\|x\| = \sqrt{x^T x}$ . Moreover,  $\Delta w$  can be expressed in terms of the original set of generalized speeds as

$$\Delta w = Q^{1/2} \Delta u \quad (58)$$

where  $\Delta u = u_h - u$  is the difference between holonomic and nonholonomic generalized speeds. Therefore:



**Figure 3.** Deviation from the constrained generalized acceleration vector  $\dot{w}$ .

$$\|\Delta \dot{w}\| = \|Q^{1/2} \Delta \dot{u} + \dot{Q}^{1/2} \Delta u\| = \min_i \|Q^{1/2} \Delta \dot{u}_i + \dot{Q}^{1/2} \Delta u\|, i = 1, 2, \dots \quad (59)$$

where  $\Delta \dot{u}_i = \dot{u}_h - \dot{u}_i$  and  $\dot{u}_i$  satisfies

$$\mathcal{A}_1 \dot{w}_i = \mathcal{A}_1 (Q^{1/2} \dot{u}_i + \dot{Q}^{1/2} u) = V_1. \quad (60)$$

Nevertheless, (59) implies that

$$\|Q^{1/2} \Delta \dot{u}\| = \min_i \|Q^{1/2} \Delta \dot{u}_i\|, i = 1, 2, \dots, \quad (61)$$

which in terms of the square Euclidean norm implies that

$$\|Q^{1/2} \Delta \dot{u}\|^2 = \Delta \dot{u}^T Q^{1/2} Q^{1/2} \Delta \dot{u} = \min_i (\Delta \dot{u}_i^T Q^{1/2} Q^{1/2} \Delta \dot{u}_i) \quad (62)$$

$$= \Delta \dot{u}^T Q \Delta \dot{u} = \min_i (\Delta \dot{u}_i^T Q \Delta \dot{u}_i), i = 1, 2, \dots \quad (63)$$



Eq. (63) is exactly the statement of Gauss' principle of least constraints [8]. The present geometric interpretation of Gauss' principle was first introduced by Udwadia and Kalaba [24].

## 9. Conclusion

The chapter introduces the canonical generalized inversion dynamical equations of motion for nonholonomic mechanical systems in the framework of Kane's method. The introduced equations of motion use the Greville formula and utilize its geometric structure to produce a full order set of dynamical equations for the nonholonomic system. Moreover, the acceleration form of constraint equations is adopted in a similar manner as in the classical Gibbs-Appell, Udwadia-Kalaba, and Bajodah-Hodges-Chen formulations.

The philosophy on which the present formulation of the dynamical equations of motion is based views the constrained system dynamics of the mechanical system as being composed of a constraint dynamics and a momentum balance dynamics that is unaltered by augmenting the constraints. Inverting both dynamics by means of two Greville formulae and invoking the geometric relations between the resulting two expressions yield the unique natural canonical generalized acceleration vector.

Because the momentum balance dynamics and the acceleration form of constraint dynamics are linear in generalized accelerations, only linear geometric and algebraic mathematical tools are needed to analyze constrained motion of discrete mechanical systems. Also, the present linear analysis is valid in despite of dependencies among the constraint equations and changes in rank that the constraint matrix  $A$  may experience because the matrices  $A_1$  and  $A_2$  are always of full row ranks and their  $m$  and  $p$  rows span two orthogonally complement row spaces. Another advantage of maintaining full row ranks of  $A_1$  and  $A_2$  is that their generalized inverses have explicit and closed form expressions, which alleviate the need for employing numerical methods for computing generalized inverses.

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