Stochastic Leader-Follower Differential Game with Asymmetric Information

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Abstract

In this chapter, we discuss a leader-follower (also called Stackelberg) stochastic differential game with asymmetric information. Here the word "asymmetric" means that the available information of the follower is some sub-σ-algebra of that available to the leader, though they play as different roles in the classical literatures. Stackelberg equilibrium is represented by the stochastic versions of Pontryagin's maximum principle and verification theorem with partial information. A linear-quadratic (LQ) leader-follower stochastic differential game with asymmetric information is studied as applications. If some system of Riccati equations is solvable, the Stackelberg equilibrium admits a state feedback representation.

Keywords: backward stochastic differential equation (BSDE), leader-follower stochastic differential game, asymmetric information, stochastic filtering, linear-quadratic control, Stackelberg equilibrium

1. Introduction

Throughout this chapter, we denote by $\mathbb{R}^n$ the Euclidean space of n-dimensional vectors, by $\mathbb{R}^{n\times d}$ the space of $n \times d$ matrices, by $\mathcal{S}^n$ the space of $n \times n$ symmetric matrices. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. T appearing in the superscripts denotes the transpose of a matrix. f_x, f_{xx} denote the partial derivative and twice partial derivative with respect to x for a differentiable function f.

1.1. Motivation

In practice, there are many problems which motivate us to study the leader-follower stochastic differential games with asymmetric information. Here we present two examples.

Example 1.1: (Continuous time principal-agent problem) The principal contracts with the agent to manage a production process, whose cumulative proceeds (or output) Y_t evolve on $[0, T]$ as follows:

$$dY_t = Be_t dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t, \quad Y_0 = Y_0 \in \mathbb{R}, \tag{1}$$

where $e_t \in \mathbb{R}$ is the agent's effort choice, B represents the productivity of effort, and there are two additive shocks (due to the two independent Brownian motions $W, \tilde{W}$) to the output. The proceeds of the production add to the principal's asset y_t, which earns a risk free return r, and out of which he pays the agent $s_t \in \mathbb{R}$ and withdraws his own consumption $d_t \in \mathbb{R}$. Thus the principal's asset evolves as

$$dy_t = \left[ry_t + Be_t - s_t - d_t\right] dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t, \quad y_0 = y_0 \in \mathbb{R}, \tag{2}$$

where y_0 is the initial asset. The agent has his own wealth m_t, out of which he consumes c_t, thus

$$dm_t = [rm_t + s_t - c_t] dt + \overline{\sigma} dW_t + \tilde{\overline{\sigma}} d\tilde{W}_t, \quad m_0 = m_0 \in \mathbb{R}, \tag{3}$$

Thus, the agent earns the same rate of return r on his savings, gets income flows due to his payment s_t, and draws down wealth to consume. In the above $\sigma, \tilde{\sigma}, \overline{\sigma}, \tilde{\overline{\sigma}}$ are all constants. At the terminal time T, the principal makes a final payment s_T and the agent chooses consumption based on this payment and his terminal wealth m_T. In the above, we restrict y_t, s_t, d_t to be nonnegative.

We consider an optimal implementable contract problem in the so-called "hidden savings" information structure (Williams [1], also in Williams [2]). In this problem, the principal can observe his asset y_t and the agent's initial wealth m_0 but cannot monitor the agent's effort e_t, consumption c_t, and wealth m_t for $t > 0$. The principal must provide incentives for the agent to put forth the desired amount of the effort. For any s_t, d_t, the agent first chooses his effort e_t^* and consumption c_t^*, such that his exponential preference

$$J_1(e, c, s, d) = \mathbb{E}\left[-\int_0^T e^{-\rho t} \exp\left[-\lambda\left(c_t - \frac{1}{2}e_t^2\right)\right] dt + e^{-\rho T}(s_T + m_T)\right] \tag{4}$$

is maximized. Here $\rho > 0$ is the discount rate and $\lambda > 0$ denotes the risk aversion parameter. The above $\left(e_t^*, c_t^*\right)$ is called an implementable contract if it meets the recommended actions of the principal's, which is based on the principal's observable wealth y_t. Then, the principal selects his payment s_t^* and consumption d_t^* to maximize his exponential preference

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$$J_2(e^*, c^*, s, d) = \mathbb{E}\left[-\int_0^T e^{-\rho t} \exp(-\lambda d_t)\, dt + e^{-\rho T}\left(y_T - s_T\right)\right]. \tag{5}$$

Let $\mathcal{F}_t$ denote the σ-algebra generated by Brownian motions $W_s, \tilde{W}_s, 0 \le s \le t$. Intuitively, $\mathcal{F}_t$ contains all the information up to time t. Let $\mathcal{G}_{1,t}$ contains the information available to the agent, and $\mathcal{G}_{2,t}$ contains the information available to the principal, up to time t respectively. Moreover, $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. In the game problem, first the agent solves the following optimization problem:

$$J_1(e^*, c^*, s, d) = \max_{e,\, c} J_1(e, c, s, d), \tag{6}$$

where (e^*, c^*) is a $\mathcal{G}_{1,t}$-adapted process pair. And then the principal solves the following optimization problem:

$$J_2(e^*, c^*, s^*, d^*) = \max_{s,\, d} J_2(e^*, c^*, s, d), \tag{7}$$

where (s^*, d^*) is a $\mathcal{G}_{2,t}$-adapted process pair. This formulates a stochastic Stackelberg differential game with asymmetric information. In this setting, the agent is the follower and the principal is the leader. Any process quadruple (e^*, c^*, s^*, d^*) satisfying the above two equalities is called a Stackelberg equilibrium. In Williams [1], a solvable continuous time principal-agent model is considered under three information structures (full information, hidden actions, and hidden savings) and the corresponding optimal contract problems are solved explicitly. But it can not cover our model.

Example 1.2: (Continuous time manufacturer-newsvendor problem) Let $D(\cdot)$ be the demand rate for a product in the market, which satisfies

$$dD(t) = a\big(\mu - D(t)\big)dt + \sigma dW(t) + \tilde{\sigma}\tilde{W}(t), \quad D(0) = d_0 \in \mathbb{R}, \tag{8}$$

where $a, \mu, \sigma, \tilde{\sigma}$ are constants. Suppose that the market is consisted with a manufacturer selling the product to end users through a retailer. At time t, the retailer chooses an order rate $q(t)$ for the product and decides its retail price $R(t)$, and is offered a wholesale price $w(t)$ by the manufacturer. We assume that items can be salvaged at unit price $S \ge 0$, and that items cannot be stored, that is, they must be sold instantly or salvaged. The retailer will obtain an expected profit

$$J_1(q(\cdot), R(\cdot), w(\cdot)) = \mathbb{E}\int_0^T [(R(t) - S)\min[D(t), q(t)] - (w(t) - S)q(t)]\, dt. \tag{9}$$

When the manufacturer has a fixed production cost per unit $M \ge 0$, he will get an expected profit

$$J_2(q(\cdot), R(\cdot), w(\cdot)) = \mathbb{E}\int_0^T \left[(w(t) - M)q(t) - S\max\left[q_t - D_t, 0\right]\right] dt. \tag{10}$$

In the above, we assume that $S < M \le w(t) \le R(t)$.

Let $\mathcal{F}_t$ denote the σ-algebra generated by $W(s), \tilde{W}(s), 0 \le s \le t$, which contains all the information up to time t. At time t, let the information $\mathcal{G}_{1,t}$, $\mathcal{G}_{2,t}$ available to the retailer and the manufacturer, respectively, are both sub-σ-algebras of $\mathcal{F}_t$. Moreover, $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. This can be explained from the practical application's aspect. Specifically, the manufacturer chooses a wholesale price $w(t)$ at time t, which is a $\mathcal{G}_{2,t}$-adapted stochastic process. And the retailer chooses an order rate $q(t)$ and a retail price $R(t)$ at time t, which are $\mathcal{G}_{1,t}$-adapted stochastic processes. For any $w(\cdot)$, to select a $\mathcal{G}_{1,t}$-adapted process pair $(q^*(\cdot), R^*(\cdot))$ for the retailer such that

$$J_1(q^*(\cdot), R^*(\cdot), w(\cdot)) \equiv J_1(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)) = \max_{q(\cdot),\, R(\cdot)} J_1(q(\cdot), R(\cdot), w(\cdot)), \tag{11}$$

and then to select a $\mathcal{G}_{2,t}$-adapted process $w^*(\cdot)$ for the manufacturer such that

$$J_2(q^*(\cdot), R^*(\cdot), w^*(\cdot)) \equiv J_2(q^*(\cdot; w^*(\cdot)), R^*(\cdot; w^*(\cdot)), w^*(\cdot)) = \max_{w(\cdot)} J_2(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)), \tag{12}$$

formulates a leader-follower stochastic differential game with asymmetric information. In this setting, the manufacturer is the leader and the retailer is the follower. Any process triple $(q^*(\cdot), R^*(\cdot), w^*(\cdot))$ satisfying the above is called a Stackelberg equilibrium. In Øksendal et al. [3], a time-dependent newsvendor problem with time-delayed information is solved, based on stochastic differential game (with jump-diffusion) approach. But it cannot cover our model.

1.2. Problem formulation

Motivated by the examples earlier, in this chapter we study the leader-follower stochastic differential games with asymmetric information. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. $(W(\cdot), \tilde{W}(\cdot))$ is a standard $\mathbb{R}^2$-valued Brownian motion and $\{\mathcal{F}_t\}_{0 \le t \le T}$ be its natural augmented filtration and $\mathcal{F}_T = \mathcal{F}$ where $T > 0$ is a finite time duration. Let the state satisfy the *stochastic differential equation* (SDE)

$$\begin{cases} dx^{u_1,u_2}(t) = b(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dt + \sigma(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dW(t) \\ \qquad\qquad + \tilde{\sigma}(t, x^{u_1,u_2}(t), u_1(t), u_2(t))d\tilde{W}(t), \quad x^{u_1,u_2}(0) = x_0, \end{cases} \tag{13}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are control processes taken by the two players in the game, labeled 1 (the follower) and 2 (the leader), with values in nonempty convex sets $U_1 \subseteq \mathbb{R}$, $U_2 \subseteq \mathbb{R}$, respectively. $x^{u_1,u_2}(\cdot)$, the solution to SDE Eq. (13) with values in $\mathbb{R}$, is the state process with initial state $x_0 \in \mathbb{R}^n$. Here $b(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$, $\sigma(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$, $\tilde{\sigma}(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$ are given $\mathcal{F}_t$-adapted processes, for each (x, u_1, u_2).

Let us now explain the asymmetric information character between the follower (player 1) and the leader (player 2) in this chapter. Player 1 is the follower, and the information available to him at time t is based on some sub-σ-algebra $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$, where $\mathcal{G}_{2,t}$ is the information available to the leader. We assume in this and next sections that $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t} \subseteq \mathcal{F}_t$. We define the admissible control sets of the follower and the leader, respectively, as follows.

$$\mathcal{U}_k := \left\{ u_k | u_k : \Omega \times [0,T] \to U_k \text{ is } \mathcal{G}_{k,t} - \text{adapted and } \sup_{0 \le t \le T} \mathbb{E}|u_k(t)|^i < \infty, \ i = 1,2,\cdots \right\}, \ k = 1,2. \tag{14}$$

The game initiates with the announcement of the leaders control $u_2(\cdot) \in \mathcal{U}_2$. Knowing this, the follower would like to choose a $\mathcal{G}_{1,t}$-adapted control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$ to minimize his cost functional

$$J_1(u_1(\cdot), u_2(\cdot)) = \mathbb{E}\left[\int_0^T g_1(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dt + G_1(x^{u_1,u_2}(T))\right]. \tag{15}$$

Here $g_1(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$ is an $\mathcal{F}_t$-adapted process, and $G_1(x) : \Omega \times \mathbb{R} \to \mathbb{R}$ is an $\mathcal{F}_T$-measurable random variable, for each (x, u_1, u_2). Now the follower encounters a stochastic optimal control problem with partial information.

SOCPF. For any chosen $u_2(\cdot) \in \mathcal{U}_2$ by the leader, choose a $\mathcal{G}_{1,t}$-adapted control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot)) \in \mathcal{U}_1$, such that

$$J_1(u_1^*(\cdot), u_2(\cdot)) \equiv J_1(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)) = \inf_{u_1 \in \mathcal{U}_1} J_1(u_1(\cdot), u_2(\cdot)), \tag{16}$$

subject to Eqs. (13) and (15). Such a $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$ is called an optimal control, and the corresponding solution $x^{u_1^*, u_2}(\cdot)$ to Eq. (13) is called an optimal state.

In the following step, once knowing that the follower will take such an optimal control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$, the leader would like to choose a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot)$ to minimize his cost functional

$$J_2(u_1^*(\cdot), u_2(\cdot)) = \mathbb{E}\left[\int_0^T g_2\big(t, x^{u_1^*,u_2}(t), u_1^*(t; u_2(t)), u_2(t)\big)dt + G_2\big(x^{u_1^*,u_2}(T)\big)\right]. \tag{17}$$

Here $g_2(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}, G_2(x) : \Omega \times \mathbb{R} \to \mathbb{R}$ are given $\mathcal{F}_t$-adapted processes, for each (x, u_1, u_2). Now the leader encounters a stochastic optimal control problem with partial information.

SOCPL. Find a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot) \in \mathcal{U}_2$, such that

$$J_2(u_1^*(\cdot), u_2^*(\cdot)) = J_2(u_1^*(\cdot; u_2^*(\cdot)), u_2^*(\cdot)) = \inf_{u_2 \in \mathcal{U}_2} J_2(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)), \tag{18}$$

subject to Eqs. (13) and (17). Such a $u_2^*(\cdot)$ is called an optimal control, and the corresponding solution $x^*(\cdot) \equiv x^{u_1^*, u_2^*}(\cdot)$ to Eq. (13) is called an optimal state. We will rewrite the problem for the leader in more detail in the next section. We refer to the problem mentioned above as a *leader-follower stochastic differential game with asymmetric information*. If there exists a control process pair $\left(u_1^*(\cdot), u_2^*(\cdot)\right) = \left(u_1^*(\cdot; u_2^*(\cdot)), u_2^*(\cdot)\right)$ satisfying Eqs. (16) and (18), we refer to it as a *Stackelberg equilibrium*.

In this chapter, we impose the following assumptions.

(A1.1) *For each* $\omega \in \Omega$, *the functions* $b, \sigma, \tilde{\sigma}, g_1$ *are twice continuously differentiable in* (x, u_1, u_2). *For each* $\omega \in \Omega$, *functions* g_2 *and* G_1, G_2 *are continuously differentiable in* (x, u_1, u_2) *and* x, *respectively. Moreover, for each* $\omega \in \Omega$ *and any* $(t, x, u_1, u_2) \in [0, T] \times \mathbb{R} \times U_1 \times U_2$, *there exists* $C > 0$ *such that*

$$\begin{aligned}&(1+|x|+|u_1|+|u_2|)^{-1}|\phi(t,x,u_1,u_2)| + |\phi_x(t,x,u_1,u_2)| + |\phi_{u_1}(t,x,u_1,u_2)| \\ &+|\phi_{u_2}(t,x,u_1,u_2)| + |\phi_{xx}(t,x,u_1,u_2)| + |\phi_{u_1u_1}(t,x,u_1,u_2)| + |\phi_{u_2u_2}(t,x,u_1,u_2)| \le C,\end{aligned} \tag{19}$$

for $\phi = b, \sigma, \tilde{\sigma}$, and

$$\begin{aligned}&\left(1+|x|^2\right)^{-1}|G_1(x)| + (1+|x|)^{-1}|G_{1x}(x)| + \left(1+|x|^2\right)^{-1}|G_2(x)| + (1+|x|)^{-1}|G_{2x}(x)| \le C,\\ &\left(1+|x|^2+|u_1|^2+|u_2|^2\right)^{-1}|g_1(t,x,u_1,u_2)| + (1+|x|+|u_1|+|u_2|)^{-1}(|g_{1x}(t,x,u_1,u_2)| + |g_{1u_1}(t,x,u_1,u_2)| \\ &\quad + |g_{1u_2}(t,x,u_1,u_2)|) + |g_{1xx}(t,x,u_1,u_2)| + |g_{1u_1u_1}(t,x,u_1,u_2)| + |g_{1u_2u_2}(t,x,u_1,u_2)| \le C,\\ &\left(1+|x|^2+|u_1|^2+|u_2|^2\right)^{-1}|g_2(t,x,u_1,u_2)| + (1+|x|+|u_1|+|u_2|)^{-1}(|g_{2x}(t,x,u_1,u_2)| \\ &\quad + |g_{2u_1}(t,x,u_1,u_2)| + |g_{2u_2}(t,x,u_1,u_2)|) \le C.\end{aligned} \tag{20}$$

1.3. Literature review and contributions of this chapter

Differential games are initiated by Issacs [4], which are powerful in modeling dynamic systems where more than one decision-makers are involved. Differential games have been researched by many scholars and have been applied in biology, economics, and finance. Stochastic differential games are differential games for stochastic systems involving noise terms. See Basar and Olsder [5] for more information about differential games. Recent developments for stochastic differential games can be seen in Hamadène [6], Wu [7], An and Øksendal [8], Wang and Yu [9, 10], and the references therein.

Leader-follower stochastic differential game is the stochastic and dynamic formulation of the Stackelberg game, which was introduced by Stackelberg [11] in 1934, when the concept of a hierarchical solution for markets where some firms have power of domination over others, is defined. This solution concept is now known as the Stackelberg equilibrium, which in the context of two-person nonzero-sum games, involves players with asymmetric roles, one leader and one follower. Pioneer study for stochastic Stackelberg differential games can be seen in Basar [12]. Specifically, a leader-follower stochastic differential game begins with the follower aims at minimizing his cost functional in response to the leader's decision on the whole duration of the game. Anticipating the follower's optimal decision depending on his entire strategy, the leader selects an optimal strategy in advance to minimize his cost functional, based on the stochastic Hamiltonian system satisfied by the follower's optimal decision. The pair of the leader's optimal strategy and the follower's optimal response is known as the Stackelberg equilibrium.

A linear-quadratic (LQ) leader-follower stochastic differential game was studied by Yong [13] in 2002. The coefficients of the the cost functionals and system are random, the diffusion term of the state equation contain the controls, and the weight matrices for the controls in the cost functionals are not necessarily positive definite. The related Riccati equations are derived to give a state feedback representation of the Stackelberg equilibrium in a nonanticipating way. Bensoussan et al. [14] obtained the global maximum principles for both open-loop and closed-loop stochastic Stackelberg differential games, whereas the diffusion term does not contain the controls.

In this chapter, we study a leader-follower stochastic differential game with asymmetric information. Our work distinguishes itself from these mentioned above in the following aspects. (1) In our framework, the information available to the follower is based on some sub-σ-algebra of that available to the leader. Moreover, both information filtration available to the leader and the follower could be sub-σ-algebras of the complete information filtration naturally generated by the random noise source. This gives a new explanation for the asymmetric information feature between the follower and the leader, and endows our problem formulation more practical meanings in realty. (2) Our work is established in the context of partial information, which is different from that of partial observation (see e.g., Wang et al. [15]) but related to An and Øksendal [8], Huang et al. [16], Wang and Yu [10]. (3) An important class of LQ leader-follower stochastic differential game with asymmetric information is proposed and then completely solved, which is a natural generalization of that in Yong [13]. It consists of a stochastic optimal control problem of SDE with partial information for the follower, and followed by a stochastic optimal control problem of *forward-backward stochastic differential equation* (FBSDE) with complete information for the leader. This problem is new in differential game theory and have considerable impacts in both theoretical analysis and practical meaning with future application prospect, although it has intrinsic mathematical difficulties. (4) The Stackelberg equilibrium of this LQ problem is characterized in terms of the *forward-backward stochastic differential filtering equations* (FBSDFEs) which arises naturally in our setup. These FBSDFEs are new and different from those in [10, 16]. (5) The Stackelberg equilibrium of this LQ problem is explicitly given, with the help of some new Riccati equations.

The rest of this chapter is organized as follows. In Section 2, we solve our problem to find the Stackelberg equilibrium. In Section 3, we apply our theoretical results to an LQ problem. Finally, Section 4 gives some concluding remarks.

2. Stackelberg equilibrium

2.1. The Follower's problem

In this subsection, we first solve **SOCPF**. For any chosen $u_2(\cdot) \in \mathcal{U}_2$, let $u_1^*(\cdot)$ be an optimal control for the follower and the corresponding optimal state be $x^{u_1^*, u_2}(\cdot)$. Define the Hamiltonian function $H_1 : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$H_1\left(t,x,u_1,u_2;q,k,\tilde{k}\right)=qb(t,x,u_1,u_2)+k\sigma(t,x,u_1,u_2)+\tilde{k}\tilde{\sigma}(t,x,u_1,u_2)-g_1(t,x,u_1,u_2).\tag{21}$$

Let an $\mathcal{F}_t$-adapted process triple $\left(q(\cdot),k(\cdot),\tilde{k}(\cdot)\right)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfies the adjoint BSDE

$$\begin{cases}-dq(t)=\Big\{b_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))q(t)+\sigma_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))k(t)\\ \qquad +\tilde{\sigma}_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))\tilde{k}(t)-g_{1x}(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))\Big\}dt\\ \qquad -k(t)dW(t)-\tilde{k}(t)d\tilde{W}(t),\quad q(T)=-G_{1x}\left(x^{u_1^*,u_2}(T)\right).\end{cases}\tag{22}$$

Proposition 2.1 *Let* **(A1.1)** *hold. For any given $u_2(\cdot)\in\mathcal{U}_2$, let $u_1^*(\cdot)$ be the optimal control for* **SOCPF***, and $x^{u_1^*,u_2}(\cdot)$ be the corresponding optimal state. Let $\left(q(\cdot),k(\cdot),\tilde{k}(\cdot)\right)$ be the adjoint process triple. Then we have*

$$\mathbb{E}\left[\left\langle H_{1u_1}\left(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t);q(t),k(t),\tilde{k}(t)\right),u_1-u_1^*(t)\right\rangle\Big|\mathcal{G}_{1,t}\right]\geq 0,\quad a.e.t\in[0,T],a.s.,\tag{23}$$

holds, for any $u_1\in U_1$.

Proof Similar to the proof of Theorem 2.1 of [10], we can get the result.

Proposition 2.2 *Let* **(A1.1)** *hold. For any given $u_2(\cdot)$, let $u_1^*(\cdot)\in\mathcal{U}_1$ and $x^{u_1^*,u_2}(\cdot)$ be the corresponding state. Let $\left(q(\cdot),k(t),\tilde{k}(\cdot)\right)$ be the adjoint process triple. For each $(t,\omega)\in[0,T]\times\Omega$, $H_1\left(t,\cdot,\cdot,u_2(t);q(t),k(t),\tilde{k}(t)\right)$ is concave, $G_1(\cdot)$ is convex, and*

$$\mathbb{E}\left[H_1(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t);q(t),k(t),\tilde{k}(t))|\mathcal{G}_{1,t}\right]=\max_{u_1\in U_1}\mathbb{E}\left[H_1(t,x^{u_1^*,u_2}(t),u_1,u_2(t);q(t),k(t),\tilde{k}(t))|\mathcal{G}_{1,t}\right],\tag{24}$$

holds for $a.e.t\in[0,T]$, a.s. Then $u_1^*(\cdot)$ is an optimal control for **SOCPF**.

Proof Similar to the proof of Theorem 2.3 of [10], we can obtain the result.

2.2. The Leader's problem

In this subsection, we first state the **SOCPL**. Then, we give the maximum principle and verification theorem. For any $u_2(\cdot)\in\mathcal{U}_2$, by Eq. (23), we assume that a functional $u_1^*(t)=u_1^*\left(t;\widehat{x}^{u_1^*,\hat{u}_2}(t),\widehat{u}_2(t),\widehat{q}(t),\widehat{k}(t),\widehat{\tilde{k}}(t)\right)$ is uniquely defined, where

$$\widehat{x}^{u_1^*,\hat{u}_2}(t):=\mathbb{E}\left[x^{u_1^*,u_2}(t)|\mathcal{G}_{1,t}\right],\widehat{u}_2(t):=\mathbb{E}[u_2(t)|\mathcal{G}_{1,t}],\widehat{q}(t):=\mathbb{E}[q(t)|\mathcal{G}_{1,t}],\widehat{k}(t):=\mathbb{E}[k(t)|\mathcal{G}_{1,t}],\widehat{\tilde{k}}(t):=\mathbb{E}\left[\tilde{k}(t)|\mathcal{G}_{1,t}\right].\tag{25}$$

For the simplicity of notations, we denote $x^{u_2}(\cdot)\equiv x^{u_1^*,u_2}(\cdot)$ and define ϕ^L on $\Omega\times[0,T]\times\mathbb{R}\times U_2$ as $\phi^L(t,x^{u_2}(t),u_2(t)):=\phi\left(t,x^{u_1^*,u_2}(t),u_1^*\left(t;\widehat{x}_2^{u_1^*,\hat{u}}(t),\widehat{u}_2(t),\widehat{q}(t),\widehat{k}(t),\tilde{k}(t)\right),u_2(t)\right)$, for $\phi=b,\sigma,\tilde{\sigma},\ g_1$, respectively. Then after substituting the above control process $u_1^*(\cdot)$ into Eq. (22), the leader encounters the controlled FBSDE system

$$\begin{cases} dx^{u_2}(t) = b^L(t,x^{u_2}(t),u_2(t))dt + \sigma^L(t,x^{u_2}(t),u_2(t))dW(t) + \tilde{\sigma}^L(t,x^{u_2}(t),u_2(t))d\tilde{W}(t), \\ -dq(t) = \Big\{ b_x^L(t,x^{u_2}(t),u_2(t))q(t) + \sigma_x^L(t,x^{u_2}(t),u_2(t))k(t) + \tilde{\sigma}_x^L(t,x^{u_2}(t),u_2(t))\tilde{k}(t) \\ \qquad -g_{1x}^L(t,x^{u_2}(t),u_2(t)) \Big\} dt - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \quad x^{u_2}(0) = x_0, \quad q(T) = -G_{1x}(x^{u_2}(T)). \end{cases} \tag{26}$$

Note that Eq. (26) is a controlled *conditional mean-field FBSDE*, which now is regarded as the "state" equation of the leader. That is to say, the state for the leader is the quadruple $\left(x^{u_2}(\cdot), q(\cdot), k(\cdot), \tilde{k}(\cdot)\right)$.

Remark 2.1 The equality $u_1^*(t) = u_1^*\left(t; \widehat{x}^{u_1^*,\hat{u}_2}(t), \widehat{u}_2(t), \widehat{q}(t), \widehat{k}(t), \widehat{\tilde{k}}(t)\right)$ does not hold in general. However, for LQ case, it is satisfied and we will make this point clear in the next section.

Define

$$\begin{aligned} &J_2^L(u_2(\cdot)) := J_2\left(u_1^*(\cdot), u_2(\cdot)\right) = \mathbb{E}\left[\int_0^T g_2(t, x^{u_1^*,u_2}(t), u_1^*(t), u_2(t))dt + G_2\left(x^{u_1^*,u_2}(T)\right)\right] \\ &\equiv \mathbb{E}\left[\int_0^T g_2(t, x^{u_1^*,u_2}(t), u_1^*(t; \widehat{x}^{u_1^*,\hat{u}_2}(t), \widehat{u}_2(t), \widehat{q}(t), \widehat{\tilde{k}}(t), \tilde{k}(t)), u_2(t))dt + G_2\left(x^{u_1^*,u_2}(T)\right)\right] \\ &:= \mathbb{E}\left[\int_0^T g_2^L(t, x^{u_2}(t), u_2(t))dt + G_2(x^{u_2}(T))\right], \end{aligned} \tag{27}$$

where $g_2^L : \Omega \times [0,T] \times \mathbb{R} \times U_2 \to \mathbb{R}$. Note the cost functional of the leader is also conditional mean-field's type. We propose the stochastic optimal control problem with partial information of the leader as follows.

SOCPL. Find a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot) \in \mathcal{U}_2$, such that

$$J_2^L\left(u_2^*(\cdot)\right) = \inf_{u_2 \in \mathcal{U}_2} J_2^L(u_2(\cdot)), \tag{28}$$

subject to Eqs. (26) and (27). Such a $u_2^*(\cdot)$ is called an optimal control, and the corresponding solution $x^*(\cdot) \equiv x^{u_2^*}(\cdot)$ to Eq. (26) is called an optimal state process for the leader.

Let $u_2^*(\cdot)$ be an optimal control for the leader, and the corresponding state $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ is the solution to Eq. (26). Define the Hamiltonian function of the leader $H_2 : \Omega \times [0,T] \times \mathbb{R}^n \times U_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} &H_2\left(t, x^{u_2}, u_2, q, k, \tilde{k}; y, z, \tilde{z}, p\right) = yb^L(t, x^{u_2}, u_2) + z\sigma^L(t, x^{u_2}, u_2) + \tilde{z}\tilde{\sigma}^L(t, x^{u_2}, u_2) \\ &+g_2^L(t, x^{u_2}, u_2) - p\left[b_x^L(t, x^{u_2}, u_2)q + \sigma_x^L(t, x^{u_2}, u_2)k + \tilde{\sigma}_x^L(t, x^{u_2}, u_2)\tilde{k} - g_{1x}^L(t, x^{u_2}, u_2)\right]. \end{aligned} \tag{29}$$

Let $\phi^{L*}(t) \equiv \phi^L\left(t, x^*(t), \widehat{x}^*(t), u_2^*(t), \widehat{u}_2^*(t)\right)$ for $\phi = b, \sigma, \tilde{\sigma}, g_1, g_2$ and all their derivatives. Suppose that $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the unique $\mathcal{F}_t$-adapted solution to the adjoint conditional mean-field FBSDE of the leader

$$\begin{cases} dp(t) = \Big\{ b_x^{L*}(t)p(t) + \mathbb{E}[b_{\hat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}] \Big\} dt + \Big\{ \sigma_x^{L*}(t)p(t) + \mathbb{E}[\sigma_{\hat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}] \Big\} dW(t) \\ \qquad + \Big\{ \tilde{\sigma}_x^{L*}(t)p(t) + \mathbb{E}[\tilde{\sigma}_{\hat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}] \Big\} d\tilde{W}(t), \quad p(0) = 0, \\ -dy(t) = \Big\{ b_x^{L*}(t)y(t) + \mathbb{E}[b_{\hat{x}}^{L*}(t)y(t)|\mathcal{G}_{1,t}] + \sigma_x^{L*}(t)z(t) + \mathbb{E}[\sigma_{\hat{x}}^{L*}(t)z(t)|\mathcal{G}_{1,t}] \\ \qquad + \tilde{\sigma}_x^{L*}(t)\tilde{z}(t) + \mathbb{E}\Big[\tilde{\sigma}_{\hat{x}}^{L*}(t)\tilde{z}(t)|\mathcal{G}_{1,t}\Big] - b_{xx}^{L*}(t)q^*(t)p(t) - \mathbb{E}\Big[b_{x\hat{x}}^{L*}q^*(t)p(t)|\mathcal{G}_{1,t}\Big] \\ \qquad - \sigma_{xx}^{L*}(t)k(t)p(t) - \mathbb{E}\Big[\sigma_{x\hat{x}}^{L*}(t)k(t)p(t)|\mathcal{G}_{1,t}\Big] - \tilde{\sigma}_{xx}^{L*}(t)\tilde{k}(t)p(t) - \mathbb{E}\Big[\sigma_{x\hat{x}}^{L*}(t)\tilde{k}(t)p(t)|\mathcal{G}_{1,t}\Big] \\ \qquad + g_{1xx}^{L*}(t)p(t) + \mathbb{E}[g_{1x\hat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}] + g_{2x}^{L*}(t) + \mathbb{E}[g_{2\hat{x}}^{L*}(t)|\mathcal{G}_{1,t}] \Big\} dt - z(t)dW(t) - \tilde{z}(t)d\tilde{W}(t), \\ y(T) = G_{1xx}(x^*(T))p(T) + G_{2x}(x^*(T)). \end{cases} \tag{30}$$

Now, we have the following two results.

Proposition 2.3 *Let* **(A1.1)** *hold. Let* $u_2^*(\cdot) \in \mathcal{U}_2$ *be an optimal control for* **SOCPL** *and* $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ *be the optimal state. Let* $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))$ *be the adjoint quadruple, then*

$$\begin{aligned} &\mathbb{E}\Big[\Big\langle H_{2u_2}(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)), u_2 - u_2^*(t)\Big\rangle \\ &\quad + \Big\langle \mathbb{E}[H_{2\hat{u}_2}(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))|\mathcal{G}_{1,t}], \hat{u}_2 - \hat{u}_2^*(t)\Big\rangle \Big| \mathcal{G}_{2,t}\Big] \\ &\quad \geq 0, \quad a.e. t \in [0,T], a.s., \quad \text{for any} \quad u_2 \in U_2. \end{aligned} \tag{31}$$

Proof The maximum condition Eq. (31) can be derived by convex variation and adjoint technique, as Anderson and Djehiche [17]. We omit the details for saving space. See also Li [18], Yong [19] and the references therein for mean-field stochastic optimal control problems. □

Proposition 2.4 *Let* **(A1.1)** *hold. Let* $u_2^*(\cdot) \in \mathcal{U}_2$ *and* $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ *be the corresponding state, with* $G_{1xx}(x) \equiv G_1 \in \mathcal{S}^n$. *Let* $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))$ *be the adjoint quadruple. For each* $(t, \omega) \in [0,T] \times \Omega$, *suppose that* $H_2(t, \cdot, \cdot, \cdot, \cdot, \cdot; y(t), z(t), \tilde{z}(t), p(t))$ *and* $G_2(\cdot)$ *are convex, and*

$$\begin{aligned} &\mathbb{E}\Big[H_2(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)) \\ &\quad + \mathbb{E}[H_2(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))|\mathcal{G}_{1,t}]\Big|\mathcal{G}_{2,t}\Big] \\ &\quad = \max_{u_2 \in U_2} \mathbb{E}\Big[H_2(t, x^*(t), u_2, q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)) \\ &\quad + \mathbb{E}[H_2(t, x^*(t), u_2, q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))\Big|\mathcal{G}_{1,t}]\Big|\mathcal{G}_{2,t}\Big], \quad a.e. t \in [0,T], a.s. \end{aligned} \tag{32}$$

Then $u_2^*(\cdot)$ is an optimal control for **SOCPL**.

Proof This follows similar to Shi [20]. We omit the details for simplicity. □

3. Applications to LQ case

In order to illustrate the theoretical results in Section 2, we study an LQ leader-follower stochastic differential game with asymmetric information. In this section, we let $\mathcal{G}_{1,t} := \sigma\{\tilde{W}_s; 0 \le s \le t\}$ and $\mathcal{G}_{2,t} = \mathcal{F}_t$. This game is a special case of the one in Section 2, but the resulting deduction is very technically demanding. We split this section into two subsections, to deal with the problems of the follower and the leader, respectively.

3.1. Problem of the follower

Suppose that the state $x^{u_1,u_2} \in \mathbb{R}$ satisfies a linear SDE

$$\begin{cases} dx^{u_1,u_2}(t) = [Ax^{u_1,u_2}(t) + B_1u_1(t) + B_2u_2(t)]dt + [Cx^{u_1,u_2}(t) + D_1u_1(t) + D_2u_2(t)]dW(t) \\ \qquad\qquad + \tilde{C}x^{u_1,u_2}(t) + \tilde{D}_1u_1(t) + \tilde{D}_2u_2(t)]d\tilde{W}(t), \\ x^{u_1,u_2}(0) = x_0. \end{cases} \tag{33}$$

Here, u_1 is the follower's control process and u_2 is the leader's control process, which take values both in $\mathbb{R}$; $A, C, \tilde{C}, B_1, D_1, \tilde{D}_1, B_2, D_2, \tilde{D}_2$ are constants. In the first step, for announced u_2, the follower would like to choose a $\mathcal{G}_{1,t}$-adapted, square-integrable control u_1^* to minimize the cost functional

$$J_1(u_1, u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(Q_1|x^{u_1,u_2}(t)|^2 + N_1|u_1(t)|^2\right)dt + G_1|x^{u_1,u_2}(T)|^2\right]. \tag{34}$$

In the second step, knowing that the follower would take u_1^*, the leader wishes to choose an $\mathcal{F}_t$-adapted, square-integrable control u_2^* to minimize

$$J_2(u_1^*, u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(Q_2\left|x^{u_1^*,u_2}(t)\right|^2 + N_2|u_2(t)|^2\right)dt + G_2\left|x^{u_1^*,u_2}(T)\right|^2\right], \tag{35}$$

where $Q_1, Q_2, G_1, G_2 \ge 0, N_1 \ge 0, N_2 > 0$ are constants. This is an LQ leader-follower stochastic differential game with asymmetric information. We wish to find its Stackelberg equilibrium (u_1^*, u_2^*).

Define the Hamiltonian function of the follower as

$$\begin{aligned} &H_1\left(t, x, u_1, u_2, q, k, \tilde{k}\right) \\ &= q(Ax + B_1u_1 + B_2u_2) + k(Cx + D_1u_1 + D_2u_2) + \tilde{k}\left(\tilde{C}x + \tilde{D}_1u_1 + \tilde{D}_2u_2\right) - \frac{1}{2}Q_1x^2 - \frac{1}{2}N_1u_1^2. \end{aligned} \tag{36}$$

For given control u_2, suppose that there exists a $\mathcal{G}_{1,t}$-adapted optimal control u_1^* of the follower, and the corresponding optimal state is $x^{u_1^*,u_2}$. By Proposition 2.1, Eq. (36) yields that

$$0 = N_1 u_1^*(t) - B_1 \widehat{q}(t) - D_1 \widehat{k}(t) - \tilde{D}_1 \widehat{\tilde{k}}(t), \tag{37}$$

where the $\mathcal{F}_t$-adapted process triple $\left(q, k, \tilde{k}\right) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfies the BSDE

$$\begin{cases} -dq(t) = \left[Aq(t) + Ck(t) + \tilde{C}\tilde{k}(t) - Q_1 x^{u_1^*, u_2}(t)\right]dt - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \\ q(T) = -G_1 x^{u_1^*, u_2}(T). \end{cases} \tag{38}$$

We wish to obtain the state feedback form of u_1^*. Noting the terminal condition of Eq. (38) and the appearance of u_2, we set

$$q(t) = -P(t)x^{u_1^*, u_2}(t) - \varphi(t), \quad t \in [0, T], \tag{39}$$

for some deterministic and differentiable $\mathbb{R}$-valued function $P(t)$, and $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted process φ which admits the BSDE

$$d\varphi(t) = \alpha(t)dt + \beta(t)d\tilde{W}(t), \quad \varphi(T) = 0. \tag{40}$$

In the above equation, $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ are $\mathcal{F}_t$-adapted processes, which are to be determined later. Now, applying Itô's formula to Eq. (39), we have

$$\begin{aligned} -dq(t) = {} & \left[\dot{P}(t)x^{u_1^*, u_2}(t) + P(t)Ax^{u_1^*, u_2}(t) + \alpha(t) + P(t)B_1 u_1^*(t) + P(t)B_2 u_2(t)\right]dt \\ & + P(t)\left[Cx^{u_1^*, u_2}(t) + D_1 u_1^*(t) + D_2 u_2(t)\right]dW(t) \\ & + \left\{P(t)\left[\tilde{C}x^{u_1^*, u_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 u_2(t)\right] + \beta(t)\right\}d\tilde{W}(t). \end{aligned} \tag{41}$$

Comparing Eq. (41) with Eq. (38), we arrive at

$$\begin{aligned} k(t) &= -P(t)\left[Cx^{u_1^*, u_2}(t) + D_1 u_1^*(t) + D_2 u_2(t)\right], \\ \tilde{k}(t) &= -P(t)\left[\tilde{C}x^{u_1^*, u_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 u_2(t)\right] - \beta(t), \end{aligned} \tag{42}$$

and

$$\alpha(t) = -\left[\dot{P}(t) + 2AP(t) + Q_1\right]x^{u_1^*, u_2}(t) - A\varphi(t) - P(t)B_1 u_1^*(t) - P(t)B_2 u_2(t) + Ck(t) + \tilde{C}\tilde{k}(t), \tag{43}$$

respectively. Taking $\mathbb{E}[\cdot|\mathcal{G}_{1,t}]$ on both sides of Eqs. (39) and (42), we get

$$\widehat{q}(t) = -P(t)\widehat{x}^{u_1^*, \hat{u}_2}(t) - \widehat{\varphi}(t), \tag{44}$$

and

$$\begin{aligned} \widehat{k}(t) &= -P(t)\left[C\widehat{x}^{u_1^*, \hat{u}_2}(t) + D_1 u_1^*(t) + D_2 \widehat{u}_2(t)\right], \\ \widehat{\tilde{k}}(t) &= -P(t)\left[\tilde{C}\widehat{x}^{u_1^*, \hat{u}_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 \widehat{u}_2(t)\right] - \widehat{\beta}(t), \end{aligned} \tag{45}$$

respectively. Applying Lemma 5.4 in [21] to Eqs. (33) and (38) corresponding to u_1^*, we derive the optimal filtering equation

$$\begin{cases} d\widehat{x}^{u_1^*,\hat{u}_2}(t) = \left[A\widehat{x}^{u_1^*,\hat{u}_2}(t) + B_1u_1^*(t) + B_2\widehat{u}_2(t)\right]dt + \left[\tilde{C}\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{D}_1u_1^*(t) + \tilde{D}_2\widehat{u}_2(t)\right]d\tilde{W}(t), \\ -d\widehat{q}(t) = \left[A\widehat{q}(t) + C\widehat{k}(t) + \tilde{C}\hat{\tilde{k}}(t) - Q_1\widehat{x}^{u_1^*,\hat{u}_2}(t)\right]dt - \tilde{k}(t)d\tilde{W}(t), \\ \widehat{x}^{u_1^*,\hat{u}_2}(0) = x_0, \quad \widehat{q}(T) = -G_1\widehat{x}^{u_1^*,\hat{u}_2}(T). \end{cases} \tag{46}$$

Note that Eq. (46) is not a classical FBSDFE, since the generator of the BSDE depends on an additional process $\widehat{k}$. For given u_2, it is important if Eq. (46) admits a unique $\mathcal{G}_{1,t}$-adapted solution $\left(\widehat{x}^{u_1^*,\hat{u}_2}, \widehat{q}, \widehat{k}, \widehat{\tilde{k}}\right)$. We will make it clear soon. For this target, first, by Eq. (37) and supposing that.

(A2.1) $\tilde{N}_1(t) := N_1 + D_1^2P(t) + \tilde{D}_1^2P(t) > 0$, $\forall t \in [0, T]$,

we immediately arrive at

$$u_1^*(t) = -\tilde{N}_1^{-1}(t)\left[\tilde{S}_1(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{S}(t)\widehat{u}_2(t) + B_1\widehat{\varphi}(t) + \tilde{D}_1\widehat{\beta}(t)\right], \tag{47}$$

where $\tilde{S}_1(t) := \left(B_1 + CD_1 + \tilde{C}\tilde{D}_1\right)P(t)$, $\tilde{S}(t) := \left(D_1D_2 + \tilde{D}_1\tilde{D}_2\right)P(t)$. Substituting Eq. (47) into Eq. (43), we can obtain that if

$$\begin{cases} \dot{P}(t) + \left(2A + C^2 + \tilde{C}^2\right)P(t) - \left(B_1 + CD_1 + \tilde{C}\tilde{D}_1\right)^2\left[N_1 + D_1^2P(t) + \tilde{D}_1^2P(t)\right]^{-1}P(t)^2 + Q_1 = 0, \\ P(T) = G_1, \end{cases} \tag{48}$$

admits a unique differentiable solution $P(t)$, then

$$\begin{aligned} \alpha(t) = &-\tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)x^{u_1^*,u_2}(t) + \tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)\hat{x}^{u_1^*,\hat{u}_2}(t) - A\varphi(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)B_1\widehat{\varphi}(t) \\ &-\tilde{S}_2(t)u_2(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)\widehat{u}_2(t) - \tilde{C}\beta(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{D}_1\widehat{\beta}(t), \end{aligned} \tag{49}$$

where $\tilde{S}_2(t) := \left(B_2 + CD_2 + \tilde{C}\tilde{D}_2\right)P(t)$. By **(A2.1)**, we know that Eq. (48) admits a unique solution $P(t) > 0$ from standard Riccati equation theory [22]. In particular, if $\tilde{C} = \tilde{D}_1 = 0$, Eq. (48) reduces to

$$\begin{cases} \dot{P}(t) + \left(2A + C^2\right)P(t) - (B_1 + CD_1)^2\left[N_1 + D_1^2P(t)\right]^{-1}P(t)^2 + Q_1 = 0, \\ P(T) = G_1, \quad N_1 + D_1^2P(t) > 0, \end{cases} \tag{50}$$

which recovers the standard one in [22]. With Eq. (49), the BSDE Eq. (40) takes the form

$$\begin{cases} -d\varphi(t) = [\tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)x^{u_1^*,u_2}(t) - \tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + A\varphi(t) - \tilde{S}_1(t)\tilde{N}^{1-1}(t)B_1\widehat{\varphi}(t) \\ \qquad +\left(\tilde{C} - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{D}_1\right)\beta(t) + \tilde{S}_2(t)u_2(t) - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)\widehat{u}_2(t)]dt - \beta(t)d\tilde{W}(t), \\ \varphi(T) = 0. \end{cases} \tag{51}$$

Moreover, for given u_2, plugging Eq. (47) into the forward equation of Eq. (46), and letting

$$\begin{cases} \tilde{A}(t) := A - B_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \tilde{\tilde{C}}(t) := \tilde{C} - \tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \tilde{B}_2(t) := B_2 - B_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \\ \tilde{F}_1(t) := -B_1\tilde{N}_1^{-1}(t)B_1, \ \ \tilde{B}_1(t) := -B_1\tilde{N}_1^{-1}(t)\tilde{D}_1, \tilde{F}_3(t) := -\tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{D}_1, \ \ \tilde{\tilde{D}}_2(t) := \tilde{D}_2 - \tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{S}(t), \end{cases} \tag{52}$$

we have

$$\begin{cases} d\widehat{x}^{u_1^*,\hat{u}_2}(t) = \left[\tilde{A}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + \tilde{B}_2(t)\widehat{u}_2(t)\right]dt \\ \qquad + \left[\tilde{\tilde{C}}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{\tilde{D}}_2(t)\widehat{u}_2(t)\right]d\tilde{W}(t), \ \ \widehat{x}^{u_1^*,\hat{u}_2}(0) = x_0, \end{cases} \tag{53}$$

which admits a unique $\mathcal{G}_{1,t}$-adapted solution $\widehat{x}^{u_1^*,\hat{u}_2}$, for given $\left(\widehat{\varphi}, \widehat{\beta}\right)$. Applying Lemma 5.4 in [21] to Eq. (51) again, we have

$$-d\widehat{\varphi}(t) = \left[\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t) + \tilde{F}_4(t)\widehat{u}_2(t)\right]dt - \widehat{\beta}(t)d\tilde{W}(t), \ \ \widehat{\varphi}(T) = 0, \tag{54}$$

where $\tilde{F}_4(t) := \tilde{S}_2(t) - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)$. For given $\widehat{u}_2$, Eq. (54) admits a unique solution $\left(\widehat{\varphi}, \widehat{\beta}\right)$ from standard BSDE theory. Putting Eqs. (53) and (54) together, we get

$$\begin{cases} d\widehat{x}^{u_1^*,\hat{u}_2}(t) = \left[\tilde{A}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + \tilde{B}_2(t)\widehat{u}_2(t)\right]dt \\ \qquad + \left[\tilde{\tilde{C}}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{\tilde{D}}_2(t)\widehat{u}_2(t)\right]d\tilde{W}(t), \\ -d\widehat{\varphi}(t) = \left[\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\beta(t) + \tilde{F}_4(t)\widehat{u}_2(t)\right]dt - \widehat{\beta}(t)d\tilde{W}(t), \ \ \widehat{x}^{u_1^*,\hat{u}_2}(0) = x_0, \ \ \widehat{\varphi}(T) = 0, \end{cases} \tag{55}$$

which admits a unique $\mathcal{G}_{1,t}$-adapted solution $\left(\widehat{x}^{u_1^*,\hat{u}_2}, \widehat{\varphi}, \widehat{\beta}\right)$. By Eqs. (55), (44), (45), and (47), we can uniquely obtain the solvability of Eq. (46). Moreover, we can check that the convexity/concavity conditions in Proposition 2.2 hold, and u_1^* given by Eq. (47) is really optimal. We summarize the above procedure in the following theorem.

Theorem 3.1 *Let* **(A2.1)** *hold,* $P(t)$ *satisfy Eq.* (48). *For chosen* u_2 *of the leader,* u_1^* *given by Eq.* (47) *is the optimal control of the follower, where* $\left(\widehat{x}^{u_1^*,\hat{u}_2}, \widehat{\varphi}, \widehat{\beta}\right)$ *is the unique* $\mathcal{G}_{1,t}$*-adapted solution to Eq.* (55).

3.2. Problem of the leader

Since the leader knows that the follower will take u_1^* by Eq. (47), the state equation of the leader writes

$$\begin{cases} dx^{u_2}(t) = \Big[Ax^{u_2}(t) + \big(\tilde{A}(t) - A\big)\widehat{x}^{\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + B_2u_2(t) + \big(\tilde{B}_2(t) - B_2\big)\widehat{u}_2(t)\Big]dt \\ \qquad + \Big[Cx^{u_2}(t) + \tilde{F}_5(t)\widehat{x}^{\hat{u}_2}(t) + \tilde{\tilde{B}}_1(t)\widehat{\varphi}(t) + \tilde{\tilde{D}}_1(t)\widehat{\beta}(t) + D_2u_2(t) + \tilde{F}_2(t)\widehat{u}_2(t)\Big]dW(t) \\ \qquad + \Big[\tilde{C}x^{u_2}(t) + \big(\tilde{\tilde{C}}(t) - \tilde{C}\big)\widehat{x}^{\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{D}_2u_2(t) + \big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\big)\widehat{u}_2(t)\Big]d\tilde{W}(t), \\ -d\widehat{\varphi}(t) = \big(\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t) + \tilde{F}_4(t)\widehat{u}_2(t)\big)dt - \widehat{\beta}(t)d\tilde{W}(t), \quad x^{u_2}(0) = x_0, \quad \widehat{\varphi}(T) = 0, \end{cases} \tag{56}$$

where $x^{u_2} \equiv x^{u_1^*, u_2}$, $\widehat{x}^{\hat{u}_2} \equiv \widehat{x}^{u_1^*, \hat{u}_2}$ and $\tilde{\tilde{B}}_1(t) := -B_1\tilde{N}_1^{-1}(t)D_1$, $\tilde{\tilde{D}}_1(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{D}_1$, $\tilde{F}_5(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t)$, $\tilde{F}_2(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{S}(t)$. Noting that Eq. (56) is a decoupled conditional mean-field FBSDE, its solvability for $\mathcal{F}_t$-adapted solution $\big(x^{u_2}, \widehat{\varphi}, \widehat{\beta}\big)$ can be easily guaranteed.

The problem of the leader is to choose an $\mathcal{F}_t$-adapted optimal control u_2^* such that the cost functional

$$J_2(u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \Big(Q_2|x^{u_2}(t)|^2 + N_2|u_2(t)|^2\Big)dt + G_2|x^{u_2}(T)|^2\right] \tag{57}$$

is minimized. Define the Hamiltonian function of the leader as

$$\begin{aligned} &H_2\Big(t, x^{u_2}, u_2, \widehat{\varphi}, \widehat{\beta}; y, z, \tilde{z}, p\Big) = \frac{1}{2}\Big(Q_2|x^{u_2}|^2 + N_2|u_2|^2\Big) \\ &+ y\Big[Ax^{u_2} + \big(\tilde{A}(t) - A\big)\widehat{x}^{\hat{u}_2} + \tilde{F}_1(t)\widehat{\varphi} + \tilde{B}_1(t)\widehat{\beta} + B_2u_2 + \big(\tilde{B}_2(t) - B_2\big)\widehat{u}_2\Big] \\ &+ p\Big(\tilde{A}(t)\widehat{\varphi} + \tilde{\tilde{C}}(t)\widehat{\beta} + \tilde{F}_4(t)\widehat{u}_2\Big) + z\Big(Cx^{u_2} + \tilde{F}_5(t)\widehat{x}^{\hat{u}_2} + \tilde{\tilde{B}}_1(t)\widehat{\varphi} + \tilde{\tilde{D}}_1(t)\widehat{\beta} + D_2u_2 + \tilde{F}_2(t)\widehat{u}_2\Big) \\ &+ \tilde{z}\Big[\tilde{C}x^{u_2} + \big(\tilde{\tilde{C}}(t) - \tilde{C}\big)\widehat{x}^{\hat{u}_2} + \tilde{B}_1(t)\widehat{\varphi} + \tilde{F}_3(t)\widehat{\beta} + \tilde{D}_2u_2 + \big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\big)\widehat{u}_2\Big]. \end{aligned} \tag{58}$$

Suppose that there exists an $\mathcal{F}_t$-adapted optimal control u_2^* of the leader, and the corresponding optimal state is $\big(x^*, \widehat{\varphi}^*, \widehat{\beta}^*\big) \equiv \big(x^{u_2^*}, \widehat{\varphi}^*, \widehat{\beta}^*\big)$. Then by Propositions 2.3, 2.4, Eq. (58) yields that

$$0 = N_2u_2^*(t) + \tilde{F}_4(t)\widehat{p}(t) + B_2y(t) + \big(\tilde{B}_2(t) - B_2\big)\widehat{y}(t) + D_2z(t) + \tilde{F}_2(t)\widehat{z}(t) + \tilde{D}_2(t)\tilde{z}(t) + \big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\big)\widehat{\tilde{z}}(t), \tag{59}$$

where the $\mathcal{F}_t$-adapted process $(p, y, z, \tilde{z})$ satisfies

$$
\begin{cases}
dp(t) = \Big[\tilde{A}(t)p(t) + \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t)\Big]dt \\
\qquad\qquad + \Big[\tilde{\tilde{C}}(t)p(t) + \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t)\Big]d\tilde{W}(t), \\
-dy(t) = \Big[Ay(t) + \Big(\tilde{A}(t) - A\Big)\widehat{y}(t) + Cz(t) + \tilde{F}_5(t)\widehat{z}(t) + \tilde{C}\tilde{z}(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t) + Q_2x^*(t)\Big]dt \\
\qquad\qquad - z(t)dW(t) - \tilde{z}(t)d\tilde{W}(t), \ \ p(0) = 0, \ \ y(T) = G_2x^*(T).
\end{cases} \tag{60}
$$

In fact, the problem of the leader can also be solved by a direct calculation of the derivative of the cost functional. Without loss of generality, let $x_0 \equiv 0$, and set $u_2^* + \varepsilon u_2$ for $\epsilon > 0$ sufficiently small, with $u_2 \in \mathbb{R}$. Then it is easy to see from the linearity of Eqs. (56) and (60), that the solution to Eq. (56) is $x^* + \epsilon x^{u_2}$. We first have

$$
\begin{aligned}
\tilde{J}(\epsilon) &:= J_2\big(u_2^* + \epsilon u_2\big) = \frac{1}{2}\mathbb{E}\int_0^T [\langle Q_2(x^*(t) + \epsilon x^{u_2}(t)), x^*(t) + \epsilon x^{u_2}(t)\rangle \\
&+ \big\langle N_2\big(u_2^*(t) + \epsilon u_2(t)\big), u_2^*(t) + \epsilon u_2(t)\big\rangle]dt + \frac{1}{2}\mathbb{E}\langle G_2(x^*(T) + \epsilon x^{u_2}(T)), x^*(T) + \epsilon x^{u_2}(T)\rangle.
\end{aligned} \tag{61}
$$

Hence

$$
0 = \frac{\partial \tilde{J}(\epsilon)}{\partial \epsilon}\bigg|_{\varepsilon=0} = \mathbb{E}\int_0^T \big[\langle Q_2x^*(t), x^{u_2}(t)\rangle + \big\langle N_2u_2^*(t), u_2(t)\big\rangle\big]dt + \mathbb{E}\langle G_2x^*(T), x^{u_2}(T)\rangle. \tag{62}
$$

Let the $\mathcal{F}_t$-adapted process quadruple $(p, y, z, \tilde{z})$ satisfy Eq. (60). Then we have

$$
0 = \mathbb{E}\int_0^T \big[\langle Q_2x^*(t), x^{u_2}(t)\rangle + \big\langle N_2u_2^*(t), u_2(t)\big\rangle\big]dt + \mathbb{E}\langle y(T), x^{u_2}(T)\rangle. \tag{63}
$$

Applying Itô's formula to $\langle x^{u_2}(t), y(t)\rangle - \langle p(t), \widehat{\varphi}(t)\rangle$, noting Eqs. (56) and (60), we derive.

$$
\begin{aligned}
0 &= \mathbb{E}\int_0^T \langle Q_2x^*(t) + Ay(t) + Cz(t) + \tilde{C}\tilde{z}(t), x^{u_2}(t)\rangle dt \\
&+ \mathbb{E}\int_0^T \Big\langle \Big(\tilde{A}(t) - A\Big)y(t) + \tilde{F}_5(t)z(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t), \widehat{x}^{\hat{u}_2}(t)\Big\rangle dt \\
&+ \mathbb{E}\int_0^T \langle N_2u_2^*(t) + B_2y(t) + D_2z(t) + \tilde{D}_2\tilde{z}(t), u_2(t)\rangle dt \\
&+ \mathbb{E}\int_0^T \Big\langle \big(\tilde{B}_2 - B_2\big)y(t) + \tilde{F}_2(t)z(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), \widehat{u}_2(t)\Big\rangle dt \\
&- \mathbb{E}\int_0^T \Big\langle Q_2x^*(t) + Ay(t) + \Big(\tilde{A}(t) - A\Big)\widehat{y}(t) + Cz(t) + \tilde{C}\tilde{z}(t) + \tilde{F}_5(t)\widehat{z}(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t), x^{u_2}(t)\Big\rangle dt \\
&+ \mathbb{E}\int_0^T \Big\langle \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t), \widehat{\varphi}(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t), \widehat{\beta}(t)\Big\rangle dt \\
&+ \mathbb{E}\int_0^T \Big\langle p(t), \tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t)\Big\rangle dt - \mathbb{E}\int_0^T \Big\langle \widehat{\varphi}(t), \tilde{A}(t)p(t) + \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t)\Big\rangle dt \\
&- \mathbb{E}\int_0^T \Big\langle \widehat{\beta}(t), \tilde{\tilde{C}}(t)p(t) + \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle p(t), \tilde{F}_4(t)\widehat{u}_2(t)\Big\rangle dt \\
&= \mathbb{E}\int_0^T \langle N_2u_2^*(t) + B_2y(t) + D_2z(t) + \tilde{D}_2\tilde{z}(t), u_2(t)\rangle dt \\
&+ \mathbb{E}\int_0^T \Big\langle \big(\tilde{B}_2(t) - B_2\big)y(t) + \tilde{F}_2(t)z(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), \widehat{u}_2(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle p(t), \tilde{F}_4(t)\widehat{u}_2(t)\Big\rangle dt \\
&= \mathbb{E}\int_0^T \langle N_2u_2^*(t) + \tilde{F}_4(t)\widehat{p}(t) + B_2y(t) + \big(\tilde{B}_2(t) - B_2\big)\widehat{y}(t) + D_2z(t) + \tilde{F}_2(t)\widehat{z}(t) \\
&+ \tilde{D}_2(t)\tilde{z}(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), u_2(t)\rangle dt.
\end{aligned} \tag{64}
$$

This implies Eq. (59).

In the following, we wish to obtain a "nonanticipating" representation for the optimal controls u_2^* and u_1^*. For this target, let us regard $(x^*, p)^T$ as the optimal state, put

$$X = \begin{pmatrix} x^* \\ p \end{pmatrix}, \quad Y = \begin{pmatrix} y^* \\ \widehat{\varphi} \end{pmatrix}, Z = \begin{pmatrix} z \\ 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \tilde{z} \\ \widehat{\beta}^* \end{pmatrix}, \tag{65}$$

and (suppressing some t below)

$$\begin{cases} \mathcal{A}_1 := \begin{pmatrix} A & 0 \\ 0 & \tilde{A}(t) \end{pmatrix}, \mathcal{A}_2 := \begin{pmatrix} \tilde{A}(t) - A & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{B}}_1 := \begin{pmatrix} 0 & \tilde{B}_1(t) \\ \tilde{B}_1(t) & 0 \end{pmatrix}, \tilde{\tilde{\mathcal{B}}}_1 := \begin{pmatrix} 0 & 0 \\ \tilde{\tilde{B}}_1(t) & 0 \end{pmatrix}, \\ \mathcal{B}_2 := \begin{pmatrix} B_2 \\ 0 \end{pmatrix}, \tilde{\mathcal{B}}_2 := \begin{pmatrix} \tilde{B}_2(t) - B_2 \\ 0 \end{pmatrix}, \mathcal{C}_1 := \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{C}}_1 := \begin{pmatrix} \tilde{C} & 0 \\ 0 & \tilde{\tilde{C}}(t) \end{pmatrix}, \\ \tilde{\mathcal{C}}_2 := \begin{pmatrix} \tilde{\tilde{C}}(t) - \tilde{C} & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\tilde{\mathcal{D}}}_1 := \begin{pmatrix} 0 & \tilde{\tilde{D}}_1(t) \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{D}}_2 := \begin{pmatrix} \tilde{D}_2 \\ 0 \end{pmatrix}, \tilde{\tilde{\mathcal{D}}}_2 := \begin{pmatrix} \tilde{\tilde{D}}_2(t) - \tilde{D}_2 \\ 0 \end{pmatrix}, \\ \mathcal{D}_2 := \begin{pmatrix} D_2 \\ 0 \end{pmatrix}, \mathcal{G}_2 := \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{F}}_1 := \begin{pmatrix} 0 & \tilde{F}_1(t) \\ \tilde{F}_1(t) & 0 \end{pmatrix}, \tilde{\mathcal{F}}_2 := \begin{pmatrix} \tilde{F}_2(t) \\ 0 \end{pmatrix}, \quad X_0 := \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ \tilde{\mathcal{F}}_3 := \begin{pmatrix} 0 & \tilde{F}_3(t) \\ \tilde{F}_3(t) & 0 \end{pmatrix}, \tilde{\mathcal{F}}_4 := \begin{pmatrix} 0 \\ \tilde{F}_4(t) \end{pmatrix}, \tilde{\mathcal{F}}_5 := \begin{pmatrix} \tilde{F}_5(t) & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{Q}_2 := \begin{pmatrix} Q_2 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{66}$$

With the notations, Eq. (56) with Eq. (60) is rewritten as

$$\begin{cases} dX(t) = \Big(\mathcal{A}_1 X(t) + \mathcal{A}_2 \widehat{X}(t) + \tilde{\mathcal{F}}_1 Y(t) + \tilde{\tilde{\mathcal{B}}}_1 Z(t) + \tilde{\mathcal{B}}_1 \tilde{Z}(t) + \mathcal{B}_2 w^*(t) + \tilde{\mathcal{B}}_2 \widehat{u}_2^*(t)\Big)dt \\ \qquad + \Big(\mathcal{C}_1 X(t) + \tilde{\mathcal{F}}_5 \widehat{X}(t) + \tilde{\tilde{\mathcal{B}}}_1^T Y(t) + \tilde{\tilde{\mathcal{D}}}_1 \tilde{Z}(t) + \mathcal{D}_2 w^*(t) + \tilde{\mathcal{F}}_2 \widehat{u}_2^*(t)\Big)dW(t) \\ \qquad + \Big(\tilde{\mathcal{C}}_1 X(t) + \tilde{\mathcal{C}}_2 \widehat{X}(t) + \tilde{\mathcal{B}}_1^T Y(t) + \tilde{\tilde{\mathcal{D}}}_1^T Z(t) + \tilde{\mathcal{F}}_3 \tilde{Z}(t) + \tilde{\mathcal{D}}_2 w^*(t) + \tilde{\tilde{\mathcal{D}}}_2 \widehat{u}_2^*(t)\Big)d\tilde{W}(t), \\ -dY(t) = \Big(\mathcal{Q}_2 X(t) + \mathcal{A}_1^T Y(t) + \mathcal{A}_2^T \widehat{Y}(t) + \mathcal{C}_1^T Z(t) + \tilde{\mathcal{F}}_5^T \widehat{Z}(t) + \tilde{\mathcal{C}}_1^T \tilde{Z}(t) + \tilde{\mathcal{C}}_2^T \tilde{Z}(t) + \tilde{\mathcal{F}}_4 \widehat{u}_2^*(t)\Big)dt \\ \qquad - Z(t)dW(t) - \tilde{Z}(t)d\tilde{W}(t), \quad X(0) = X_0, \quad Y(T) = \mathcal{G}_2 X(T). \end{cases} \tag{67}$$

Noting Eq. (59), we have

$$\begin{aligned} u_2^*(t) &= -N_2^{-1}\Big[\tilde{\mathcal{F}}_4^T \widehat{X}(t) + \mathcal{B}_2^T Y(t) + \tilde{\mathcal{B}}_2^T \widehat{Y}(t) + \mathcal{D}_2^T Z(t) + \tilde{\mathcal{F}}_2^T \widehat{Z}(t) + \tilde{\mathcal{D}}_2^T \tilde{Z}(t) + \tilde{\tilde{\mathcal{D}}}_2^T \tilde{Z}(t)\Big], \\ \widehat{u}_2^*(t) &= -N_2^{-1}\Big[\tilde{\mathcal{F}}_4^T \widehat{X}(t) + \big(\mathcal{B}_2 + \tilde{\mathcal{B}}_2\big)^T \widehat{Y}(t) + \big(\mathcal{D}_2 + \tilde{\mathcal{F}}_2\big)^T \widehat{Z}(t) + \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T \tilde{Z}(t)\Big]. \end{aligned} \tag{68}$$

Inserting Eq. (68) into Eq. (67), we get

$$
\begin{cases}
dX(t) = \Big(\mathcal{A}_1 X(t) + \overline{\mathcal{A}}_2 \widehat{X}(t) + \overline{\tilde{\mathcal{F}}}_1 Y(t) + \overline{\mathcal{B}}_2 \widehat{Y}(t) + \mathcal{B}_3 Z(t) + \overline{\overline{\mathcal{B}}}_2 \widehat{Z}(t) + \overline{\tilde{\mathcal{B}}}_1 \tilde{Z}(t) + \overline{\tilde{\mathcal{B}}}_2 \tilde{Z}(t)\Big) dt \\
\qquad + \Big(\mathcal{C}_1 X(t) + \overline{\tilde{\mathcal{F}}}_5 \widehat{X}(t) + \tilde{\mathcal{B}}_3^T Y(t) + \overline{\mathcal{D}}_2 \widehat{Y}(t) + \tilde{\tilde{\mathcal{D}}}_2 Z(t) + \overline{\overline{\mathcal{D}}}_2 \widehat{Z}(t) + \mathcal{D}_3 \tilde{Z}(t) + \overline{\tilde{\mathcal{D}}}_2 \tilde{Z}(t)\Big) dW(t) \\
\qquad + \Big(\tilde{\mathcal{C}}_1 X(t) + \overline{\tilde{\mathcal{C}}}_2 \widehat{X}(t) + \overline{\tilde{\mathcal{B}}}_1^T Y(t) + \overline{\mathcal{D}}_3 \widehat{Y}(t) + \mathcal{D}_3^T Z(t) + \overline{\overline{\mathcal{D}}}_3^T \widehat{Z}(t) + \overline{\tilde{\mathcal{F}}}_3 \tilde{Z}(t) + \overline{\tilde{\mathcal{D}}}_3 \tilde{Z}(t)\Big) d\tilde{W}(t), \\
-dY(t) = \Big(\mathcal{Q}_2 X(t) + \overline{\tilde{\mathcal{F}}}_4 \widehat{X}(t) + \mathcal{A}_1^T Y(t) + \overline{\mathcal{A}}_2^T \widehat{Y}(t) + \tilde{\mathcal{C}}_1^T \tilde{Z}(t) + \mathcal{C}_1^T Z(t) + \overline{\tilde{\mathcal{F}}}_5^T \widehat{Z}(t) + \overline{\tilde{\mathcal{C}}}_2^T \tilde{Z}(t)\Big) dt \\
\qquad - Z(t) dW(t) - \tilde{Z}(t) d\tilde{W}(t), \quad X(0) = X_0, \quad Y(T) = \mathcal{G}_2 X(T),
\end{cases}
\tag{69}
$$

where

$$
\begin{cases}
\overline{\mathcal{A}}_2 := \mathcal{A}_2 - (\mathcal{B}_2 + \tilde{\mathcal{B}}_2) N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{B}}}_1 := \tilde{\mathcal{B}}_1 - \mathcal{B}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \overline{\mathcal{B}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \\
\overline{\overline{\mathcal{B}}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \overline{\tilde{\mathcal{B}}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\mathcal{B}_3 := \tilde{\tilde{\mathcal{B}}}_1 - \mathcal{B}_2 N_2^{-1} \mathcal{D}_2^T, \overline{\tilde{\mathcal{C}}}_2 := \tilde{\mathcal{C}}_2 - \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big) N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\tilde{\tilde{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \mathcal{D}_2^T, \overline{\mathcal{D}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \overline{\overline{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \\
\mathcal{D}_3 := \tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \overline{\mathcal{D}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \\
\overline{\overline{\mathcal{D}}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \overline{\tilde{\mathcal{D}}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\overline{\tilde{\mathcal{F}}}_1 := \tilde{\mathcal{F}}_1 - \mathcal{B}_2 N_2^{-1} \mathcal{B}_2^T, \overline{\tilde{\mathcal{F}}}_3 := \tilde{\mathcal{F}}_3 - \tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \tilde{\tilde{\mathcal{F}}}_4 := -\tilde{\mathcal{F}}_4 N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{F}}}_5 := \tilde{\mathcal{F}}_5 - (\mathcal{D}_2 + \tilde{\mathcal{F}}_2) N_2^{-1} \tilde{\mathcal{F}}_4^T.
\end{cases}
\tag{70}
$$

We need to decouple Eq. (69). Similar to Eq. (39), put

$$
Y(t) = \mathcal{P}_1(t) X(t) + \mathcal{P}_2(t) \widehat{X}(t), \quad t \in [0, T], \tag{71}
$$

where $\mathcal{P}_1(t), \mathcal{P}_2(t)$ are differentiable, deterministic 2×2 matrix-valued functions with $\mathcal{P}_1(T) = \mathcal{G}_2, \mathcal{P}_2(T) = 0$. Applying Lemma 5.4 in [21] to the forward equation in Eq. (35), we obtain

$$
\begin{cases}
d\widehat{X}(t) = \Big[(\mathcal{A}_1 + \overline{\mathcal{A}}_2) \widehat{X}(t) + \Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big) \widehat{Y}(t) + \Big(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\Big) \widehat{Z}(t) + \Big(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\Big) \tilde{Z}(t)\Big] dt \\
\quad + \Big[\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) \widehat{X}(t) + \Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big) \widehat{Y}(t) + \Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big) \widehat{Z}(t) + \Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big) \tilde{Z}(t)\Big] d\tilde{W}(t), \\
\widehat{X}(0) = X_0.
\end{cases}
\tag{72}
$$

Applying Itô's formula to (3.31), we get

$$\begin{aligned}
dY(t) = &\Big\{\Big(\dot{\mathcal{P}}_1 + \mathcal{P}_1\mathcal{A}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1\Big)X(t) + \Big[\dot{\mathcal{P}}_2 + \mathcal{P}_1\overline{\mathcal{A}}_2 + \mathcal{P}_1\overline{\mathcal{B}}_2\mathcal{P}_1 + \mathcal{P}_2\big(\mathcal{A}_1 + \overline{\mathcal{A}}_2\big) \\
&+\mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_1, +\mathcal{P}_1\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_2, +\mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_2\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{B}_3Z(t) + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\tilde{Z}(t) \\
&+\Big[\mathcal{P}_1\overline{\overline{\mathcal{B}}}_2 + \mathcal{P}_2\Big(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\Big)\Big]\widehat{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{B}}}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\Big)\Big]\tilde{Z}(t)\Big\}dt \\
&+\Big\{\big(\mathcal{P}_1\mathcal{C}_1 + \mathcal{P}_1(t)\mathcal{B}_3^T\mathcal{P}_1\big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{F}}}_5 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2)\Big]\widehat{X}(t) + \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2Z(t) \\
&+\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\widehat{Z}(t) + \mathcal{P}_1\mathcal{D}_3\tilde{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{Z}(t)\Big\}dW(t) \\
&+\Big\{\Big(\mathcal{P}_1\tilde{\mathcal{C}}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\Big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1 + \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 \\
&+\mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big), +\mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{D}_3^TZ(t) + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\tilde{Z}(t) \\
&+\Big[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\Big]\widehat{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\Big]\tilde{Z}(t)\Big\}d\tilde{W}(t) \\
= &-\Big\{\big(\mathcal{Q}_2 + \mathcal{A}_1^T\mathcal{P}_1\big)X(t) + \Big(\overline{\tilde{\mathcal{F}}}_4 + \overline{\mathcal{A}}_2^T\mathcal{P}_1 + \mathcal{A}_1^T\mathcal{P}_2 + \overline{\mathcal{A}}_2^T\mathcal{P}_2\Big)\widehat{X}(t) + \mathcal{C}_1^TZ(t) + \overline{\tilde{\mathcal{F}}}_5^T\widehat{Z}(t) \\
&+\tilde{\mathcal{C}}_1^T\tilde{Z}(t) + \overline{\tilde{\mathcal{C}}}_2^T\tilde{Z}(t)\Big\}dt + Z(t)dW(t) + \tilde{Z}(t)d\tilde{W}(t).
\end{aligned} \tag{73}$$

Comparing $dW(t)$ and $d\tilde{W}(t)$ on both sides of Eq. (73), we have

$$\begin{aligned}
Z(t) = &\big(\mathcal{P}_1\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_1\big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{F}}}_5 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2)\Big]\widehat{X}(t) \\
&+ \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2Z(t) + \mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\widehat{Z}(t) + \mathcal{P}_1\mathcal{D}_3\tilde{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{Z}(t), \\
\tilde{Z}(t) = &\Big(\mathcal{P}_1\tilde{\mathcal{C}}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\Big)X(t) + \Big[\mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1, +\mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 \\
&+ \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2, +\mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{D}_3^TZ(t) \\
&+ \Big[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\Big]\widehat{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\tilde{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\Big]\tilde{Z}(t).
\end{aligned} \tag{74}$$

Taking $\mathbb{E}[\cdot|\mathcal{G}_{1,t}]$, we derive

$$\begin{aligned}
\widehat{Z}(t) = &\Big[\mathcal{P}_1\Big(\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5\Big) + \mathcal{P}_1\big(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\big)\mathcal{P}_1 + \mathcal{P}_1\big(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\big)\mathcal{P}_2\Big]\widehat{X}(t) \\
&+ \mathcal{P}_1\Big(\tilde{\tilde{\mathcal{D}}}_2 + \overline{\overline{\mathcal{D}}}_2\Big)\widehat{Z}(t) + \mathcal{P}_1\Big(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2\Big)\tilde{Z}(t), \\
\widehat{\tilde{Z}}(t) = &\Big[\mathcal{P}_1\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) + \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1, +\mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big), +\mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1 \\
&+ \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2\Big]\widehat{X}(t) \\
&+ (\mathcal{P}_1 + \mathcal{P}_2)\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\widehat{Z}(t) + (\mathcal{P}_1 + \mathcal{P}_2)\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\tilde{Z}(t).
\end{aligned} \tag{75}$$

Supposing that (I_2 denotes the 2×2 unit matrix)

$$\begin{aligned}
&(\mathbf{A2.2}) \quad \tilde{\mathcal{N}}_2^{-1} := \left[I_2 - \mathcal{P}_1 \left(\tilde{\tilde{\mathcal{D}}}_2 + \overline{\overline{\mathcal{D}}}_2 \right) \right]^{-1} \text{ and} \\
&\tilde{\tilde{\mathcal{N}}}_2^{-1} := \left[I_2 - (\mathcal{P}_1 + \mathcal{P}_2) \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3 \right) \tilde{\mathcal{N}}_2^{-1} \mathcal{P}_1 \left(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2 \right) - (\mathcal{P}_1 + \mathcal{P}_2) \left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3 \right) \right]^{-1} \text{ exist,}
\end{aligned} \tag{76}$$

we get

$$\widehat{Z}(t) = \Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \widehat{X}(t), \; \tilde{Z}(t) = \tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \widehat{X}(t), \tag{77}$$

where

$$\begin{cases}
\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) := \tilde{\mathcal{N}}_2^{-1} \Big\{ \mathcal{P}_1 \left(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2 \right) \tilde{\tilde{\mathcal{N}}}_2^{-1} (\mathcal{P}_1 + \mathcal{P}_2) \Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3 \right) (\mathcal{P}_1 + \mathcal{P}_2) + \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3 \right) \tilde{\mathcal{N}}_2^{-1} \\
\qquad \mathcal{P}_1 \left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + (\mathcal{B}_3^T + \overline{\mathcal{D}}_2)(\mathcal{P}_1 + \mathcal{P}_2) \right] \Big] + \mathcal{P}_1 \left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + (\mathcal{B}_3^T + \overline{\mathcal{D}}_2)(\mathcal{P}_1 + \mathcal{P}_2) \right] \Big\}, \\
\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) := \tilde{\tilde{\mathcal{N}}}_2^{-1} (\mathcal{P}_1 + \mathcal{P}_2) \Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3 \right) (\mathcal{P}_1 + \mathcal{P}_2) \\
\qquad + \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3 \right) \tilde{\mathcal{N}}_2^{-1} \mathcal{P}_1 \left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + (\mathcal{B}_3^T + \overline{\mathcal{D}}_2)(\mathcal{P}_1 + \mathcal{P}_2) \right] \Big].
\end{cases} \tag{78}$$

Inserting Eq. (77) into Eq. (74), we have

$$\begin{aligned}
Z(t) = {} & \left(\mathcal{P}_1 \mathcal{C}_1 + \mathcal{P}_1 \mathcal{B}_3^T \mathcal{P}_1 \right) X(t) + \mathcal{P}_1 \Big[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T \mathcal{P}_2, + \overline{\mathcal{D}}_2 (\mathcal{P}_1 + \mathcal{P}_2), + \overline{\overline{\mathcal{D}}}_2 \Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\
& + \overline{\tilde{\mathcal{D}}}_2 \tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \Big] \widehat{X}(t) + \mathcal{P}_1 \tilde{\tilde{\mathcal{D}}}_2 Z(t) + \mathcal{P}_1 \mathcal{D}_3 \tilde{Z}(t), \\
\tilde{Z}(t) = {} & \left(\mathcal{P}_1 \tilde{\mathcal{C}}_1 + \mathcal{P}_1 \overline{\tilde{\mathcal{B}}}_1^T \mathcal{P}_1 \right) X(t) + \Big[\mathcal{P}_2 \left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 \right) + \mathcal{P}_1 \overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_2 \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3 \right) \mathcal{P}_1 + \mathcal{P}_1 \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3 \right) \mathcal{P}_2 \\
& + \mathcal{P}_1 \overline{\mathcal{D}}_3 \mathcal{P}_1 + \mathcal{P}_2 \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3 \right) \mathcal{P}_2 + \left[\mathcal{P}_1 \overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2 \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3 \right) \right] \mathcal{P}_1 \overline{\overline{\mathcal{D}}}_2 \Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\
& + \left[\mathcal{P}_1 \overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2 \left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3 \right) \right] \mathcal{P}_1 \overline{\tilde{\mathcal{D}}}_2 \tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \Big] \widehat{X}(t) + \mathcal{P}_1 \mathcal{D}_3^T Z(t) + \mathcal{P}_1 \overline{\tilde{\mathcal{F}}}_3 \tilde{Z}(t).
\end{aligned} \tag{79}$$

Supposing that

$$\begin{aligned}
&(\mathbf{A2.3}) \; \overline{\mathcal{N}}_2^{-1} := \left(I_2 - \mathcal{P}_1 \tilde{\tilde{\mathcal{D}}}_2 \right)^{-1} := \left[I_{2n} + \mathcal{P}_1 \mathcal{D}_2 N_2^{-1} \mathcal{D}_2^T \right]^{-1} \\
&\text{and} \quad \overline{\overline{\mathcal{N}}}_2^{-1} := \left(I_2 - \mathcal{P}_1 \mathcal{D}_3^T \overline{\mathcal{N}}_2^{-1} \mathcal{P}_1 \mathcal{D}_3 - \mathcal{P}_1 \overline{\tilde{\mathcal{F}}}_3 \right)^{-1} := \Big[I_{2n} - \mathcal{P}_1 \left(\tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T \right)^T \\
&\qquad \times \left[I_{2n} + \mathcal{P}_1 \mathcal{D}_2 N_2^{-1} \mathcal{D}_2^T \right]^{-1} \mathcal{P}_1 \left(\tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T \right) - \mathcal{P}_1 \left(\tilde{\mathcal{F}}_3 - \tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T \right) \Big]^{-1} \text{ exist,}
\end{aligned} \tag{80}$$

we get

$$Z(t) = \Sigma_1(\mathcal{P}_1, \mathcal{P}_2)X(t) + \Sigma_2(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \quad \tilde{Z}(t) = \tilde{\Sigma}_1(\mathcal{P}_1, \mathcal{P}_2)X(t) + \tilde{\Sigma}_2(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \tag{81}$$

where.

$$\begin{cases} \Sigma_1(\mathcal{P}_1,\mathcal{P}_2) := \overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1 + \mathcal{D}_3\mathcal{P}_1\left(\mathcal{C}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\right) + \mathcal{D}_3\overline{\overline{\mathcal{N}}}^{2-1}\mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1(\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1)\right], \\ \tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2) := \overline{\overline{\mathcal{N}}}_2^{-1}\mathcal{P}_1\left[\mathcal{C}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1 + \mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1(\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1)\right], \\ \Sigma_2(\mathcal{P}_1,\mathcal{P}_2) := \overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\Big\{\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2) \\ \quad + \mathcal{D}_3\overline{\overline{\mathcal{N}}}_2^{-1}\left[\mathcal{P}_2\left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\right) + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_1 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2\right. \\ \quad + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \left[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \left[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2) \\ \quad \left. + \mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\right]\right\}, \\ \tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) := \overline{\overline{\mathcal{N}}}_2^{-1}\Big\{\mathcal{P}_2\left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\right) + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_1 + \mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 \\ \quad + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 + \left[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \left[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2) \\ \quad + \mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\right]\Big\}. \end{cases} \tag{82}$$

Comparing the coefficients of dt in Eq. (73) and putting Eqs. (77) and (81) into them, we get

$$\begin{cases} 0 = \dot{\mathcal{P}}_1 + \mathcal{P}_1\mathcal{A}_1 + \mathcal{A}_1^T\mathcal{P}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1 + \mathcal{Q}_2 + (\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3)\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \left(\tilde{\mathcal{C}}_1^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\right)\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2), \\ 0 = \dot{\mathcal{P}}_2 + \mathcal{P}_2(\mathcal{A}_1 + \overline{\mathcal{A}}_2) + (\mathcal{A}_1 + \overline{\mathcal{A}}_2)^T\mathcal{P}_2 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_1 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_2 \\ \qquad + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{A}}_2 + \overline{\mathcal{A}}_2^T\mathcal{P}_1 + \mathcal{P}_1\overline{\mathcal{B}}_2\mathcal{P}_1 + \overline{\tilde{\mathcal{F}}}_4 + (\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3)\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) \\ \qquad + \left(\tilde{\mathcal{C}}_1^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\right)\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \left[\overline{\tilde{\mathcal{F}}}_5^T + \mathcal{P}_1\overline{\overline{\mathcal{B}}}_2 + \mathcal{P}_2\left(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\right)\right]\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) \\ \qquad + \left[\overline{\tilde{\mathcal{C}}}_2^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_2 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\right)\right]\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2), \;\; \mathcal{P}_1(T) = \mathcal{G}_2, \;\; \mathcal{P}_2(T) = 0. \end{cases} \tag{83}$$

Note that the system of Riccati equations (83) is not standard, and its solvability is open. Due to some technical reason, we can not obtain the solvability of it now. However, in some special case, $\mathcal{P}_1(t)$ and $\mathcal{P}_2(t)$ are not coupled. Then we can first solve the first equation of $\mathcal{P}_1(t)$, then that of $\mathcal{P}_2(t)$ by standard Riccati equation theory. We will not discuss for the space limit. And we will consider the general solvability of Eq. (83) in the future.

Instituting Eqs. (77) and (81) into Eq. (68), we obtain

$$\begin{aligned} u_2^*(t) = &-N_2^{-1}\Big[\mathcal{B}_2^T\mathcal{P}_1 + \mathcal{D}_2^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{D}}_2^T\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\tilde{\mathcal{F}}_4^T + \mathcal{B}_2^T\mathcal{P}_2 + \tilde{\mathcal{B}}_2^T(\mathcal{P}_1+\mathcal{P}_2) \\ &+ \mathcal{D}_2^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{F}}_2^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{D}}_2^T\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\tilde{\mathcal{D}}}_2^T\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t), \end{aligned} \tag{84}$$

and the optimal "state" $X = (x^*, p)^T$ of the leader satisfies

$$\begin{cases} dX(t) = \Big\{\Big[\mathcal{A}_1 + \overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1 + \mathcal{B}_3\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\mathcal{A}}_2 + \overline{\tilde{\mathcal{F}}}_1\mathcal{P}_2 + \overline{\mathcal{B}}_2(\mathcal{P}_1+\mathcal{P}_2) \\ \qquad +\mathcal{B}_3\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{B}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}dt \\ \qquad + \Big\{\Big[\mathcal{C}_1 + \tilde{\mathcal{B}}_3^T\mathcal{P}_1 + \tilde{\tilde{\mathcal{D}}}_2\Sigma_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\tilde{\mathcal{F}}}_5 + \tilde{\mathcal{B}}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1+\mathcal{P}_2) \\ \qquad +\tilde{\tilde{\mathcal{D}}}_2\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}dW(t) \\ \qquad + \Big\{\Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1 + \mathcal{D}_3^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\tilde{\mathcal{C}}}_2 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_2 + \overline{\mathcal{D}}_3(\mathcal{P}_1+\mathcal{P}_2) \\ \qquad +\mathcal{D}_3^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_3^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_3\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}d\tilde{W}(t), \\ X(0) = X_0, \end{cases} \tag{85}$$

where $\widehat{X}$ is governed by

$$\begin{cases} d\widehat{X}(t) = \Big[\mathcal{A}_1 + \overline{\mathcal{A}}_2 + \Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)(\mathcal{P}_1+\mathcal{P}_2) + \mathcal{B}_3\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2) \\ \quad +\mathcal{B}_3\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{B}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)dt \\ \qquad + \Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)(\mathcal{P}_1+\mathcal{P}_2) + \mathcal{D}_3^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2) \\ \quad +\mathcal{D}_3^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_3^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_3\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)d\tilde{W}(t), \\ \widehat{X}(0) = X_0. \end{cases} \tag{86}$$

We summarize the above analysis in the following theorem.

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Theorem 3.2 *Let* **(A2.1)**∼**(A2.3)** *hold,* $(\mathcal{P}_1(t), \mathcal{P}_2(t))$ *satisfy Eq.* (83), $\widehat{X}$ *be the* $\mathcal{G}_{1,t}$*-adapted solution to Eq.* (86), *and* X *be the* $\mathcal{F}_t$*-adapted solution to Eq.* (85). *Define* $\left(Y, Z, \tilde{Z}\right)$ *by Eqs.* (71) *and* (81), *respectively. Then Eq.* (69) *holds, and* u_2^* *given by Eq.* (84) *is a feedback optimal control of the leader.*

Finally, the optimal control u_1^* of the follower can also be represented in a nonanticipating way. In fact, by Eq. (47), noting Eqs. (68), (71), and (77), we have

$$\begin{aligned} u_1^*(t) &= -\tilde{N}_1^{-1}(t)\left(\tilde{S}_1^T(t)\widehat{x}^*(t) + \tilde{S}(t)\widehat{u}_2^*(t) + B_1^T\widehat{\varphi}^*(t) + \tilde{D}_1^T\beta^*(t)\right) \\ &= -\tilde{N}_1^{-1}(t)\left[\left(\tilde{S}_1^T(t) \ \ 0\right)\widehat{X}(t) + \tilde{S}(t)\widehat{u}_2^*(t) + \left(0 \quad B_1^T\right)\widehat{Y}(t) + \left(0 \quad \tilde{D}_1^T\right)\tilde{Z}(t)\right] \\ &= -\tilde{N}_1^{-1}(t)\Big[\left(\tilde{S}_1^T(t) \ \ 0\right) - \tilde{S}(t)N_2^{-1}[\tilde{\mathcal{F}}_4^T + \left(\mathcal{B}_2 + \tilde{\mathcal{B}}_2\right)^T(\mathcal{P}_1 + \mathcal{P}_2) + \left(\mathcal{D}_2 + \tilde{\mathcal{F}}_2\right)^T\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\ &\quad + \left(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\right)^T\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)] + \left(0 \quad B_1^T\right)(\mathcal{P}_1 + \mathcal{P}_2) + \left(0 \quad \tilde{D}_1^T\right)\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\Big]\widehat{X}(t), \end{aligned} \tag{87}$$

which is observable for the follower.

Remark 3.3 When we consider the complete information case, that is, $\tilde{W}(\cdot)$ disappears and $\mathcal{G}_{1,t} = \mathcal{F}_t$, Theorems 3.1 and 3.2 coincide with Theorems 2.3 and 3.3 in Yong [13].

4. Concluding remarks

In this chapter, we have studied a leader-follower stochastic differential game with asymmetric information. This kind of game problem possesses several attractive features. First, the game problem has the Stackelberg feature, which means the two players play as different roles during the game. Thus the usual approach to deal with game problems, such as [6–8, 10], where the two players act as equivalent roles, does not apply. Second, the game problem has the asymmetric information between the two players, which was not considered in [3, 13, 14]. In detail, the information available to the follower is based on some sub-σ-algebra of that available to the leader. Stochastic filtering technique is introduced to compute the optimal filtering estimates for the corresponding adjoint processes, which act as the solution to some FBSDFE. Third, the Stackelberg equilibrium is represented in its state feedback form for the LQ problem under some appropriate assumptions. Some new conditional mean-field FBSDEs and system of Riccati equations are introduced to deal with the leader's LQ problem.

In principle, Theorems 3.1 and 3.2 provide a useful tool to seek Stackelberg equilibrium. As a first step in this direction, we apply our results to the LQ problem to obtain explicit solutions. We hope to return to the more general case in our future research. It is worthy to study the closed-loop Stackelberg equilibrium for our problem, as well as the solvability of the system of Riccati equations. These challenging topics will be considered in our future work.

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Notes

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