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Anisotropic Propagation of Electromagnetic Waves

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Abstract

This chapter will analyze the properties of electromagnetic wave propagation in aniso-tropic media. Of particular interest are positive index, anisotropic, and magneto-dielectric media. Engineered anisotropic media provide unique electromagnetic properties includ-ing a higher effective refractive index, high permeability with relatively low magnetic loss tangent at microwave frequencies, and lower density and weight than traditional media. This chapter presents research including plane wave solutions to propagation in aniso-tropic media, a mathematical derivation of birefringence in anisotropic media, modal decomposition of rectangular waveguides filled with anisotropic media, and the full derivation of anisotropic transverse resonance in a partially loaded waveguide. These are fundamental theories in the area of electromagnetic wave propagation that have been reformulated for fully anisotropic magneto-dielectric media. The ensuing results will aide interested parties in understanding wave behavior for anisotropic media to enhance designs for radio frequency devices based on anisotropic and magnetic media.

Keywords: anisotropic, wave propagation, dispersion, birefringence, waveguides, transverse resonance

1. Introduction

Recently engineered materials have come to play a dominant role in the design and implemen-tation of electromagnetic devices and especially antennas. Metamaterials, ferrites, and magneto-dielectrics have all come to play a crucial role in advances made both in the function-ality and characterization of such devices. In fact, a movement towards utilizing customized material properties to replace the functionality of traditional radio frequency (RF) components such as broadband matching circuitry, ground planes, and directive elements is apparent in the literature and not just replacement of traditional substrates and superstrates with engineered

structures. A firm theoretical understanding of the electromagnetic properties of these materials is necessary for both design and simulation of new and improved RF devices.

Inherently, many of these engineered materials have anisotropic properties. Previously, the study of anisotropy had been limited mostly to the realm of optical frequencies where the phenomenon occurs naturally in substances such as liquid crystals and plasmas. However, the recent development of the aforementioned engineered materials has encouraged the study of electromagnetic anisotropy for applications at megahertz (MHz) and gigahertz (GHz) frequencies.

For the purposes of this chapter, an anisotropic electromagnetic medium defines permittivity ($\underline{\underline{\varepsilon_r}}$) and permeability $\underline{\underline{\mu_r}}$ as separate tensors where the values differ in all three Cartesian directions ($\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$ and $\mu_x \neq \mu_y \neq \mu_z$). This is known as the biaxial definition of anisotropic material which is more encompassing than the uniaxial definition which makes the simplifying assumption that $\varepsilon_x = \varepsilon_y = \varepsilon_t$ and $\mu_x = \mu_y = \mu_t$. The anisotropic definition also differs from the traditional isotropic definition where ε_r and μ_r are the same in all three Cartesian directions defining each by a single value. For the definition of the tensor equations see Section 3.1. Anisotropic media yield characteristics such as conformal surfaces, focusing and refraction of electromagnetic waves as they propagate through a material, high impedance surfaces for artificial magnetic conductors as well as high index, low loss, and lightweight ferrite materials. The following sections aim to discuss in more detail some RF applications directly impacted by the incorporation of anisotropic media and also give a firm understanding of electromagnetic wave propagation as it applies to anisotropic media for different RF applications.

2. Applications of anisotropy in radio frequency devices

Traditionally, the study of anisotropic properties was limited to a narrow application space where traditional ferrites, which exhibit natural anisotropy were the enabling technology. These types of applications included isolators, absorbers, circulators and phase shifters [1]. Traditional ferrites are generally very heavy and very lossy at microwave frequencies which are the two main limiting factors narrowing their use in RF devices; however, propagation loss is an important asset to devices such as absorbers. Anisotropy itself leads to propagation of an RF signal in different directions, which is important in devices such as circulators and isolators [1]. For phase shifters and other control devices the microwave signal is controlled by changing the bias field across the ferrite [1, 2]. However, newer versions of some of these devices, utilizing FETs and diodes in the case of phase shifters, rely on isotropic media to enable higher efficiency devices.

As early as 1958, Collin showed that at microwave frequencies, where the wavelength is larger, it is possible to fabricate artificial dielectric media having anisotropic properties [3]. This has led some to investigate known theoretical solutions to typical RF problems, such as a microstrip patch antenna, and extend them utilizing anisotropic wave propagation in dielectric media [4, 5]. The anisotropic dielectric antenna shows interesting features of basic antenna applications featuring anisotropic substrates. While these solutions establish a framework for electromagnetic wave propagation in anisotropic media, they simplify the problem by necessarily setting μ_r to 1 and only focusing on dielectric phenomena of anisotropy.

The concept of artificial media is also exemplified by the proliferation of metamaterials research over the last few decades. Metamaterials incorporate the use of artificial microstructures made of subwavelength inclusions that are usually implemented with periodic and/or multilayered structures known as unit cells [6]. These devices operate where the wavelength is much larger than the characteristic dimensions of the unit cell elements. One characteristic feature of some types of metamaterials is wave propagation anisotropy [7]. Anisotropic metamaterials are used in applications such as directive lensing [8, 9], cloaking [10], electronic beam steering [11], and metasurfaces [12] among others.

Finally, a class of engineered materials exists that exhibits positive refractive index, anisotropy, and magneto-dielectric properties with reduced propagation loss at microwave frequencies compared to traditional ferrites. These materials show the unique ability to provide broadband impedance matches for very low profile antennas by exploiting the inherent anisotropy to redirect surface waves thus improving the impedance match of the antenna when very close to a ground plane. Antenna profile on the orders of a twentieth and a fortieth of a wavelength have been demonstrated using these materials with over an octave of bandwidth and positive realized gain [13, 14].

3. Plane wave solutions in an anisotropic medium

The recent development of low loss anisotropic magneto-dielectrics greatly expands the current antenna design space. Here we present a rigorous derivation of the wave equation and dispersion relationships for anisotropic magneto-dielectric media. All results agree with those presented by Meng et al. [15, 16]. Furthermore, setting $\underline{\underline{\mu_r}} = \underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix, yields results that agree with those presented by Pozar and Graham for anisotropic dielectric media [4, 5]. This section and the following section expand on the results presented by Meng et al., Pozar and Graham. Incorporating a fully developed derivation of anisotropic properties of both $\underline{\underline{\varepsilon_r}}$ and $\underline{\underline{\mu_r}}$ expands upon the simplification imposed by both Pozar and Graham that uses an isotropic value of $\mu_r = 1$. An expansion on the results of Meng et al. given in Section 4 develops the waveguide theory including a full modal decomposition utilizing the biaxial definition of anisotropy versus their simplified uniaxial definition. The derivation of anisotropic cavity resonance in Section 4 differs from that of Meng et al. by addressing the separate issue of how the direct relationship of an arbitrary volume of anisotropic material will distort the geometry of a cavity to maintain resonance at a given frequency. This property is especially important for the design of conformal cavity backed antennas for ground and air-based vehicle mobile vehicular platforms. Furthermore, the analysis of anisotropic properties is not restricted to double negative (DNG) materials, which is the case for both of the Meng et al. studies.

3.1. Source free anisotropic wave equation

In order to solve for the propagation constants, we will need to formulate the dispersion relationship from the anisotropic wave equation. This allows us to solve for the propagation constant in the normal direction of the anisotropic medium. We start with the anisotropic, time harmonic form of Maxwell's source free equations for the electric and magnetic fields $\underline{E}$ and $\underline{H}$

$$\nabla x \underline{E} = j\omega\mu_o\underline{\underline{\mu}}_r \cdot \underline{H}, \tag{1}$$

$$\nabla x \underline{H} = -j\omega\varepsilon_o\underline{\underline{\varepsilon}}_r \cdot \underline{E}, \tag{2}$$

where ω is the frequency in radians, ε_o is the permittivity of free space, μ_o is the permeability of free space, $\underline{E} = \underline{x}_oE_x + \underline{y}_oE_y + \underline{z}_oE_z$ and $\underline{H} = \underline{x}_oH_x + \underline{y}_oH_y + \underline{z}_oH_z$. We define $\underline{\underline{\mu}}_r$ and $\underline{\underline{\varepsilon}}_r$ as

$$\underline{\underline{\varepsilon}}_r = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \tag{3}$$

$$\underline{\underline{\mu}}_r = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix}. \tag{4}$$

Applying Eqs. (3) and (4) to Eqs. (1) and (2) yields the following

$$\underline{x}_o\left(\frac{dE_Z}{dy} - \frac{dE_Y}{dz}\right) + \underline{y}_o\left(\frac{dE_X}{dz} - \frac{dE_Z}{dx}\right) + \underline{z}_o\left(\frac{dE_Y}{dx} - \frac{dE_X}{dy}\right) = -j\omega\mu_o\left(\mu_xH_X\underline{x}_o + \mu_yH_Y\underline{y}_o + \mu_zH_Z\underline{z}_o\right), \tag{5}$$

$$\underline{x}_o\left(\frac{dH_Z}{dy} - \frac{dH_Y}{dz}\right) + \underline{y}_o\left(\frac{dH_X}{dz} - \frac{dH_Z}{dx}\right) + \underline{z}_o\left(\frac{dH_Y}{dx} - \frac{dH_X}{dy}\right) = j\omega\varepsilon_o\left(\varepsilon_xE_X\underline{x}_o + \varepsilon_yE_Y\underline{y}_o + \varepsilon_zE_Z\underline{z}_o\right). \tag{6}$$

Using the radiation condition, we assume a solution of $\underline{E}(x, y, z) = \underline{E}(x, y)e^{-jk_zz}$ [17]. Now isolate the individual components of (5) by taking the dot product with $\underline{x}_o$, $\underline{y}_o$, and $\underline{z}_o$ respectively. This operation yields the following equations

$$(d/dy)E_z - jk_zE_y = -j\omega\mu_o\mu_xH_x, \tag{7}$$

$$jk_zE_x - (d/dx)E_z = -j\omega\mu_o\mu_yH_y, \tag{8}$$

$$(d/dx)E_y - (d/dy)E_x = -j\omega\mu_o\mu_zH_z. \tag{9}$$

Assuming a solution of $\underline{H}(x, y, z) = \underline{H}(x, y)e^{-jk_zz}$ for (6), the same procedure yields [17]

$$(d/dy)H_z - jk_zH_y = j\omega\varepsilon_o\varepsilon_xE_x, \tag{10}$$

$$jk_zH_x - (d/dx)H_z = j\omega\varepsilon_o\varepsilon_yE_y, \tag{11}$$

$$(d/dx)H_y - (d/dy)H_x = -j\omega\mu_o\mu_zH_z. \tag{12}$$

Using (7)–(12) allows for the transverse field components of the electric and magnetic fields in terms of the derivatives of H_z and E_z as

$$E_x = -\frac{j}{k_o^2 \mu_y \varepsilon_x - k_z^2}\left(\omega\mu_o\mu_y(d/dy)H_z + k_z(d/dx)E_z\right), \tag{13}$$

$$E_y = \frac{j}{k_o^2 \mu_x \varepsilon_y - k_z^2}\left(\omega\mu_o\mu_x(d/dx)H_z - k_z(d/dy)E_z\right), \tag{14}$$

$$H_x = \frac{j}{k_o^2 \mu_x \varepsilon_y - k_z^2}\left(\omega\varepsilon_o\varepsilon_y(d/dy)E_z - k_z(d/dx)H_z\right), \tag{15}$$

$$H_y = -\frac{j}{k_o^2 \mu_y \varepsilon_x - k_z^2}\left(\omega\varepsilon_o\varepsilon_x(d/dx)E_z + k_z(d/dy)H_z\right). \tag{16}$$

The relationships for the transverse field components, applied to (1) and (2), yield the following solutions for $\underline{H}$ and $\underline{E}$, respectively

$$\underline{H} = -\left(\underline{\underline{\mu_r}}^{-1}/j\omega\varepsilon_o\right) \cdot (\nabla x\underline{E}), \tag{17}$$

$$\underline{E} = \left(\underline{\underline{\varepsilon_r}}^{-1}/j\omega\varepsilon_o\right) \cdot (\nabla x\underline{H}). \tag{18}$$

Taking the cross product of both sides and substituting (1) and (2) for the right hand side of (17) and (18) yields

$$\nabla x\underline{\underline{\varepsilon_r}}^{-1} \cdot (\nabla x\underline{H}) = k_o^2 \underline{\underline{\mu_r}} \cdot \underline{H}, \tag{19}$$

$$\nabla x\underline{\underline{\mu_r}}^{-1} \cdot (\nabla x\underline{E}) = -k_o^2 \underline{\underline{\varepsilon_r}} \cdot \underline{E}. \tag{20}$$

Equations (19) and (20) represent the vector wave equations in an anisotropic medium [12].

3.2. Dispersion equation for H_z

We expand (19) in terms of (13)–(16)

$$\nabla x\left\{\frac{x_o}{\varepsilon_x}\left[(d/dy)H_Z - (d/dz)H_y\right] + \frac{y_o}{\varepsilon_y}\left[(d/dz)H_x - (d/dx)H_z\right]+ \frac{z_o}{\varepsilon_z}\left[(d/dx)H_y - (d/dy)H_x\right]\right\} = k_o^2 \underline{\underline{\mu_r}} \cdot \underline{H}, \tag{21}$$

Evaluating the remaining cross product of (21) yields the final form of the expanded wave equation

$$\underline{x_o}\Pi_x + \underline{y_o}\Pi_y + \underline{z_o}\Pi_z = k_o^2 \underline{\underline{\mu_r}} \cdot \underline{H}, \tag{22}$$

$$\Pi_x = \left[(d^2/dxdy)H_y - (d^2/dy^2)H_x\right]/\varepsilon_z - \left[(d^2/dz^2)H_x - (d^2/dxdz)H_z\right]/\varepsilon_y, \tag{23}$$

$$\Pi_y = \left[(d^2/dydz)H_z - (d^2/dz^2)H_y\right]/\varepsilon_x - \left[(d^2/dx^2)H_y - (d^2/dxdy)H_x\right]/\varepsilon_z, \tag{24}$$

$$\Pi_z = \left[(d^2/dxdz)H_x - (d^2/dx^2)H_z\right]/\varepsilon_y - \left[(d^2/dy^2)H_z - (d^2/dxdy)H_x\right]/\varepsilon_x. \tag{25}$$

Taking the dot product of (22) with $\underline{z}_o$ allows the isolation of H_z on the right hand side of the equation in terms of (265 on the left hand side

$$\left[(d^2/dydz)H_y - (d^2/dy^2)H_z\right]/\varepsilon_x - \left[(d^2/dx^2)H_z + (d^2/dxdz)H_x\right]/\varepsilon_y = k_o^2\mu_z H_z, \tag{26}$$

By keeping in mind that $d/dz = -jk_z$, setting $E_z = 0$, and differentiating (15) and (16) by $d^2/dxdz$ and $d^2/dydz$, produces the following result

$$\left[\frac{k_z^2}{\varepsilon_y\left(k_z^2 - k_o^2\varepsilon_y\mu_x\right)} - \frac{1}{\varepsilon_y}\right](d^2/dx^2)H_z + \left[\frac{k_z^2}{\varepsilon_x\left(k_z^2 - k_o^2\varepsilon_x\mu_y\right)} - \frac{1}{\varepsilon_x}\right](d^2/dy^2)H_z = k_o^2\mu_z H_z. \tag{27}$$

Combining the d^2H_z/dx^2 and d^2H_z/dy^2 terms in (27) gives the following second order differential dispersion equation for H_z

$$\frac{k_o^2\mu_x}{k_o^2\mu_x\varepsilon_y - k_z^2}(d^2/dx^2)H_z + \frac{k_o^2\mu_y}{k_o^2\mu_y\varepsilon_x - k_z^2}(d^2/dy^2)H_z + k_o^2\mu_z H_z = 0. \tag{28}$$

3.3. Dispersion equation for E_z

Expanding the $\nabla x\underline{E}$ term of (18) in terms of (13)–(16) yields

$$\nabla x\Big\{\underline{x}_o[(d/dy)E_Z - (d/dz)E_Y]/\mu_x + \underline{y}_o[(d/dz)E_X - (d/dx)E_Z]/\mu_y$$
$$+\underline{z}_o[(d/dx)E_Y - (d/dy)E_X]/\mu_z\Big\} = k_o^2\underline{\varepsilon}_r \cdot \underline{E}. \tag{29}$$

Evaluating the remaining cross product of (29) gives the final form of the expanded wave equation

$$\underline{x}_o\xi_x + \underline{y}_o\xi_y + \underline{z}_o\xi_z = k_o^2\underline{\varepsilon}_r \cdot \underline{E}, \tag{30}$$

$$\xi_x = \left[(d^2/dxdy)E_Y - (d^2/dy^2)E_x\right]/\mu_z - \left[(d^2/dz^2)E_x - (d^2/dxdz)E_z\right]/\mu_y, \tag{31}$$

$$\xi_y = \left[(d^2/dydz)E_z - (d^2/dz^2)E_y\right]/\mu_x - \left[(d^2/dx^2)E_y - (d^2/dxdy)E_x\right]/\mu_z, \tag{32}$$

$$\xi_z = \left[(d^2/dxdz)E_x - (d^2/dx^2)E_z\right]/\mu_y - \left[(d^2/dy^2)E_z - (d^2/dydz)E_y\right]/\mu_x. \tag{33}$$

Taking the dot product of (30) with $\underline{z}_o$ allows isolation of the E_z component on the right hand side of the equation in terms of (33) on the left hand side

$$\left[(d^2/dxdz)E_x - (d^2/dx^2)E_z\right]/\mu_y + \left[(d^2/dydz)E_y - (d^2/dy^2)E_z\right]/\mu_x = k_o^2\varepsilon_z E_z, \tag{34}$$

Keeping in mind that $d/dz = -jk_z$, setting $H_z = 0$, and differentiating (15) and (16) by $d^2/dxdz$ and $d^2/dydz$ produces the following result

$$\left[\frac{k_z^2}{\mu_y\left(k_z^2 - k_o^2\mu_y\varepsilon_x\right)} - \frac{1}{\mu_y}\right](d^2/dx^2)E_z + \left[\frac{k_z^2}{\mu_x\left(k_z^2 - k_o^2\mu_x\varepsilon_y\right)} - \frac{1}{\mu_x}\right](d^2/dy^2)E_z = k_o^2\varepsilon_z E_z. \qquad (35)$$

Combining the d^2E_z/dx^2 and d^2E_z/dy^2 terms in (35) gives the following second order differential dispersion equation for E_z

$$\frac{k_o^2\varepsilon_x}{k_o^2\mu_y\varepsilon_x - k_z^2}(d^2/dx^2)E_z + \frac{k_o^2\varepsilon_y}{k_o^2\mu_x\varepsilon_y - k_z^2}(d^2/dy^2)E_z + k_o^2\varepsilon_z E_z = 0. \qquad (36)$$

3.4. Transmission and reflection from an anisotropic half-space

Birefringence is a characteristic of anisotropic media where a single incident wave entering the boundary of an anisotropic medium gives rise to two refracted waves as shown in **Figure 1** or a single incident wave leaving gives rise to two reflected waves as shown in **Figure 2**. We call these two waves the ordinary wave and the extraordinary wave. To see how the anisotropy of a medium gives rise to the birefringence phenomenon, Eqs. (28) and (36) will yield a solution for k_z in the medium.

Equations (28) and (36) yield the following solutions in unbounded anisotropic media restricted by the radiation condition in all three dimensions

$$E_z(x, y, z) = E_o e^{-j\left(k_x x + k_y y + k_z z\right)}, \qquad (37)$$

$$H_z(x, y, z) = H_o e^{-j\left(k_x x + k_y y + k_z z\right)}. \qquad (38)$$
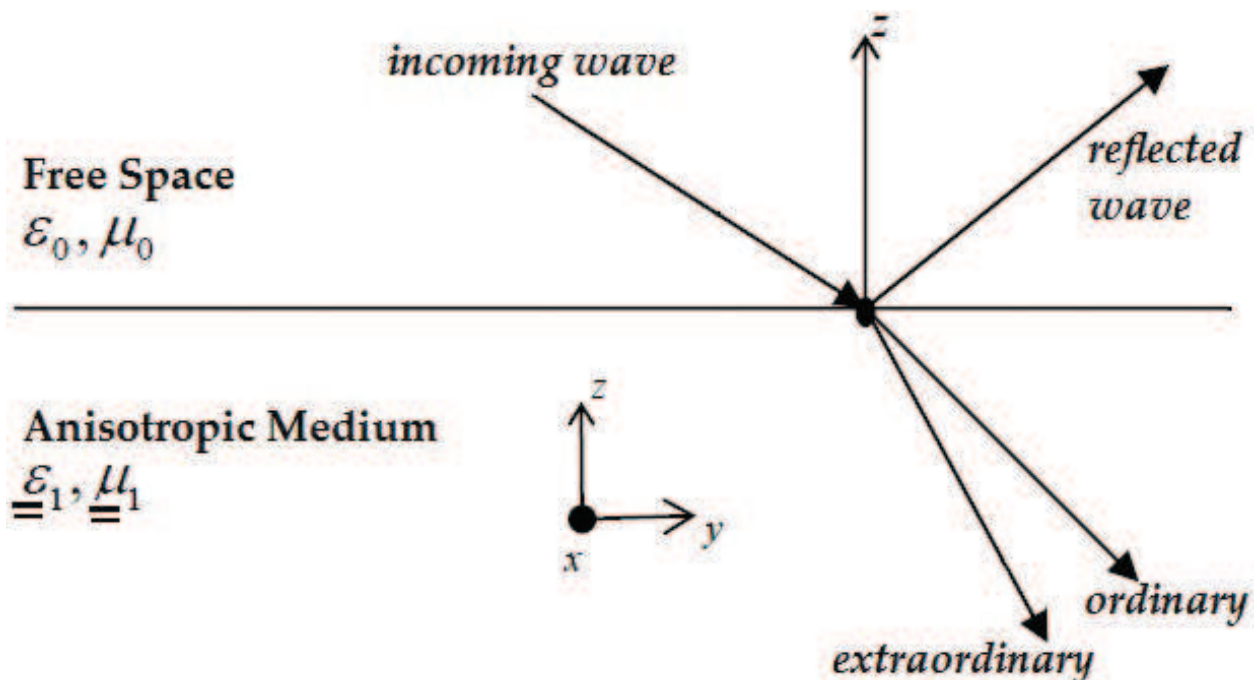

Figure 1. A plane wave incident from free space on an anisotropic boundary.
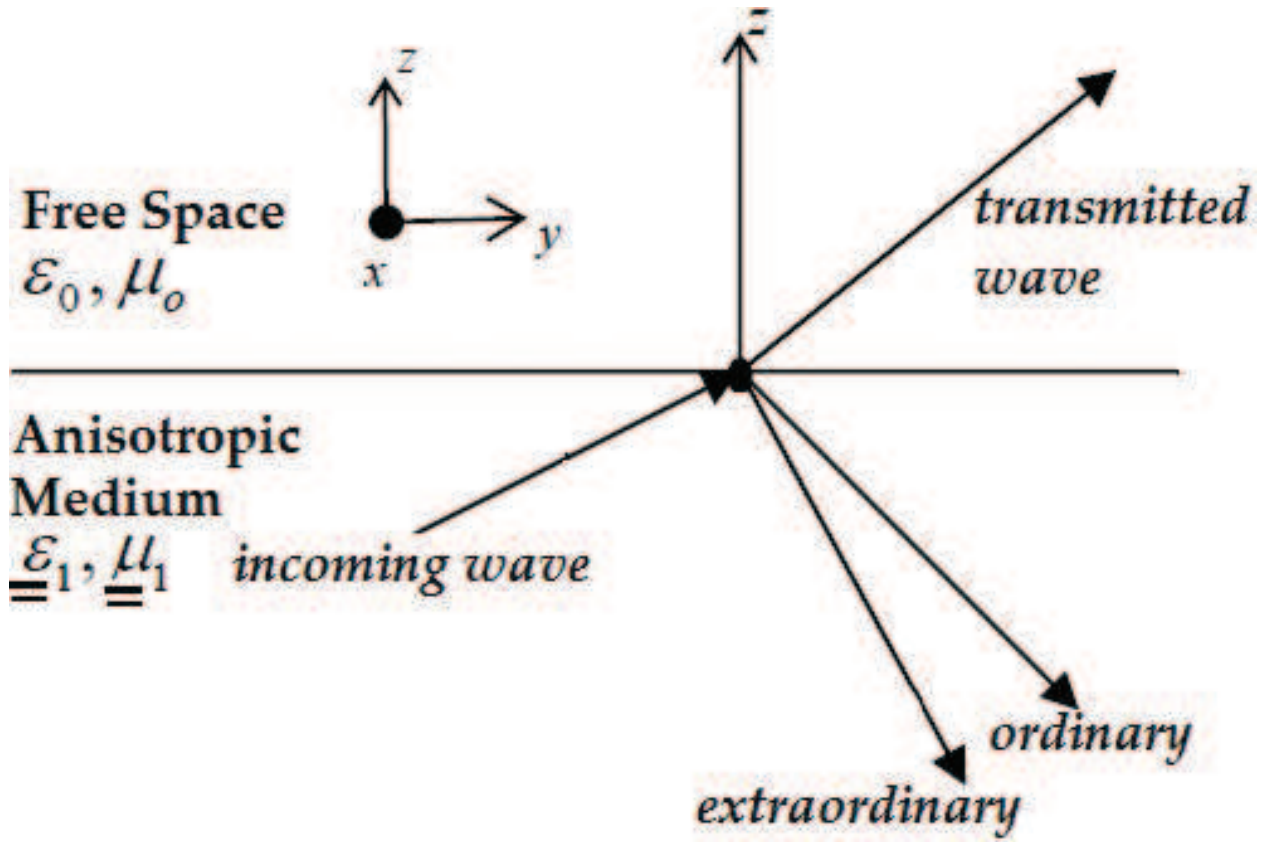
Figure 2. A plane wave incident from an anisotropic medium on a free space boundary.

Plugging (37) into (36) (equivocally we could substitute (38) into (19)) allows the generation of a polynomial equation whose solutions give the values of k_z in the anisotropic medium. Noting that $d^2/dx^2 = -k_x^2$ and $d^2/dy^2 = -k_y^2$, (36) simplifies as

$$k_o^2 k_x^2 \varepsilon_x E_z / \left(k_o^2 \mu_y \varepsilon_x - k_z^2 \right) + k_o^2 \varepsilon_y k_y^2 E_z / \left(k_o^2 \mu_x \varepsilon_y - k_z^2 \right) - k_o^2 \varepsilon_z E_z = 0. \tag{39}$$

Dividing out the $k_o^2 E_z$ term and multiplying through by both denominators gives us the following factored polynomial

$$\left(k_o^2 \mu_y \varepsilon_x - k_z^2 \right) \left(k_o^2 \mu_x \varepsilon_y - k_z^2 \right) \varepsilon_z - k_x^2 \varepsilon_x \left(k_o^2 \mu_x \varepsilon_y - k_z^2 \right) - \varepsilon_y k_y^2 \left(k_o^2 \mu_y \varepsilon_x - k_z^2 \right) = 0. \tag{40}$$

Finally, multiplying out (40) yields a fourth order polynomial whose roots yield the four values of k_z describing the ordinary wave and extraordinary wave in the positive and negative propagation directions

$$k_z^4 \mu_z + \left[k_x^2 \mu_x + k_y^2 \mu_y - \left(\varepsilon_x \mu_y + \varepsilon_y \mu_x \right) k_o^2 \mu_z \right] k_z^2 + \left[k_o^4 \varepsilon_x \varepsilon_y \mu_x \mu_y \mu_z - k_o^2 k_x^2 \varepsilon_x \mu_y \mu_x - k_o^2 \varepsilon_y k_y^2 \mu_x \mu_y \right] = 0. \tag{41}$$

Equation (41) is directly responsible for the existence of the extraordinary wave that is characteristic of the birefringence phenomenon. In an isotropic medium, the resulting polynomial for

k_z is a second order polynomial, which yields only the values for the positive and negative propagation of the single ordinary wave.

4. Anisotropic rectangular waveguide

Electromagnetic wave behavior of waveguides is well understood in the literature. The mode within a waveguide that are based on the voltage and current distributions within the waveguide make up the basis for the electric and magnetic field calculations. This section derives similar formulations for a rectangular waveguide uniformly filled with an anisotropic medium as shown in **Figure 3**. **Figure 3** shows propagation in the $\underline{z}_o$-direction along the length of the waveguide. Rectangular waveguides are most commonly used for material measurement and characterization, and therefore understanding how electromagnetic waves propagate in an anisotropic waveguide is important for material characterization purposes. Furthermore, this section shows how the anisotropic derivation of waveguide behavior parallels that of a typical waveguide, and therefore how anisotropy may be applied to other waveguide geometries.

4.1. Anisotropic mode functions

Assume source free Maxwell's equations in the same form as (1) and (2). Then the transverse electromagnetic fields are defined

$$\underline{E}_{Tv}(x,y,z) = \sum_m \sum_n \left[V_v'(z)\underline{e}_v'(x,y) + V_v''(z)\underline{e}_v''(x,y) \right], \tag{42}$$

$$\underline{H}_{Tv}(x,y,z) = \sum_m \sum_n \left[I_v'(z)\underline{h}_v'(x,y) + I_v''(z)\underline{h}_v''(x,y) \right], \tag{43}$$

$$V_v(z) = V_o e^{-jk_z z}, \tag{44}$$
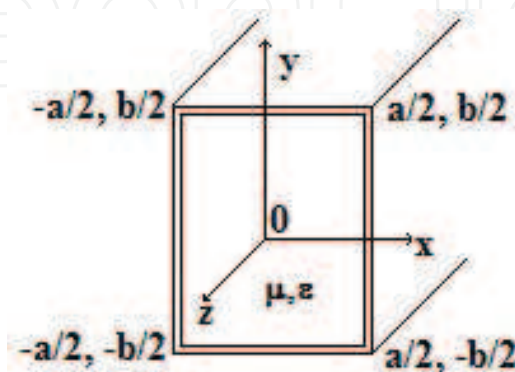
$$I_v(z) = I_o e^{-jk_z z}, \tag{45}$$

Figure 3. Cross section of a closed rectangular waveguide filled with anisotropic metamaterial and surrounded by PEC walls.

where $\underline{E}_T$ and $\underline{H}_T$ are the transverse electric and magnetic fields, $V(z)$ and $I(z)$ are the voltage and current at point z, e and h are the waveguide mode equations, and $\upsilon \in [m, n]$ is the mode number defined by the two indices m and n.

4.1.1. Incident TE mode

Assuming only a TE type mode in the waveguide sets $E_z = 0$. Then (13)–(16) become

$$E''_{x\upsilon} = -j\omega\mu_o\mu_y(d/dy)H''_{z\upsilon}/\left(k_o^2\mu_y\varepsilon_x - k_{z\upsilon}^2\right), \tag{46}$$

$$E''_{y\upsilon} = j\omega\mu_o\mu_x(d/dx)H''_{z\upsilon}/\left(k_o^2\mu_x\varepsilon_y - k_{z\upsilon}^2\right), \tag{47}$$

$$H''_{y\upsilon} = -jk_{z\upsilon}^2(d/dx)H''_{z\upsilon}/\left(k_o^2\mu_x\varepsilon_y - k_{z\upsilon}^2\right), \tag{48}$$

$$H''_{y\upsilon} = -jk_{z\upsilon}^2(d/dy)H''_{z\upsilon}/\left(k_o^2\mu_y\varepsilon_x - k_{z\upsilon}^2\right). \tag{49}$$

To solve for $H''_{z\upsilon}$ we formulate the anisotropic wave equation from (1) and (2) where (52) resembles (19)

$$\nabla \times \nabla \times \underline{H} = j\omega\varepsilon_o\underline{\underline{\varepsilon}}_r \cdot \left(-j\omega\mu_o\underline{\underline{\mu}}_r \cdot \underline{H}\right), \tag{50}$$

$$\nabla \times \nabla \times \underline{H} = j\omega\varepsilon_o\underline{\underline{\varepsilon}}_r \cdot (\nabla \times \underline{E}), \tag{51}$$

$$\nabla \times \underline{\underline{\varepsilon}}_r^{-1} \cdot (\nabla \times \underline{H}) = k_o^2\underline{\underline{\mu}}_r \cdot \underline{H}. \tag{52}$$

Expanding the curl of (52)

$$\begin{bmatrix} \dfrac{1}{\varepsilon_z}\dfrac{d\chi''_{z\upsilon}}{dy} - \dfrac{1}{\varepsilon_y}\dfrac{d\chi''_{y\upsilon}}{dz} & 0 & 0 \\[2ex] 0 & \dfrac{1}{\varepsilon_x}\dfrac{d\chi''_{x\upsilon}}{dz} - \dfrac{1}{\varepsilon_z}\dfrac{d\chi''_{z\upsilon}}{dx} & 0 \\[2ex] 0 & 0 & \dfrac{1}{\varepsilon_y}\dfrac{d\chi''_{y\upsilon}}{dx} - \dfrac{1}{\varepsilon_x}\dfrac{d\chi''_{x\upsilon}}{dy} \end{bmatrix} \begin{bmatrix} 1 \\[2ex] 1 \\[2ex] 1 \end{bmatrix} = k_o^2 \begin{bmatrix} \mu_x H''_{x\upsilon} \\[2ex] \mu_y H''_{y\upsilon} \\[2ex] \mu_z H''_{z\upsilon} \end{bmatrix}, \tag{53}$$

$$\chi''_{x\upsilon} = (d/dy)H''_{z\upsilon} - (d/dz)H''_{y\upsilon}, \tag{54}$$

$$\chi''_{y\upsilon} = (d/dz)H''_{x\upsilon} - (d/dx)H''_{z\upsilon}, \tag{55}$$

$$\chi''_{z\upsilon} = (d/dx)H''_{y\upsilon} - (d/dy)H''_{x\upsilon}. \tag{56}$$

Isolating the $\underline{z}_o$-component of (53) gives the following relationship for $H''_{z\upsilon}$

$$\left[(d^2/dxdz)H''_{x\upsilon} - (d^2/dx^2)H''_{z\upsilon}\right]/\varepsilon_y + \left((d^2/dydz)H''_{y\upsilon} - (d^2/dy^2)H''_{z\upsilon}\right)/\varepsilon_x - k_o^2\mu_z H''_{z\upsilon} = 0, \tag{57}$$

and substituting (48) and (49) for $H_{xv}^{''}$ and $H_{yv}^{''}$ yields the following differential equation that can be solved for $H_{zv}^{''}$

$$k_o^2 \mu_x (d^2/dx^2) H_{zv}^{''} / (k_o^2 \mu_x \varepsilon_y - k_{zv}^2) + k_o^2 \mu_y (d^2/dy^2) H_{zv}^{''} / (k_o^2 \mu_y \varepsilon_x - k_{zv}^2) + k_o^2 \mu_z H_{zv}^{''} = 0. \qquad (58)$$

Assuming a solution of the form

$$H_{zv}^{''} = H_o \cos(k_{xv} x) \cos(k_{yv} y) e^{-jk_{zv} z}, \qquad (59)$$

$$k_{xv} = m\pi/a, \qquad (60)$$

$$k_{yv} = n\pi/b, \qquad (61)$$

which meets the boundary conditions at the PEC walls of the waveguide, then plugging (58) into (53) imposes the following restriction on the values of the tensors in (3) and (4)

$$\mu_x k_{xv}^2 / (k_o^2 \mu_x \varepsilon_y - k_{zv}^2) + \mu_y k_{yv}^2 / (k_o^2 \mu_y \varepsilon_x - k_{zv}^2) = \mu_z. \qquad (62)$$

Solving (62) for k_{zv} gives the following equation which yields four solutions to the propagation constant for the ordinary and extraordinary waves described in Section 3.4

$$k_{zv}^2 = \left\{ k_o^2 \left(\mu_x \varepsilon_y + \mu_y \varepsilon_x \right) - k_{xv}^2 \mu_x / \mu_z - k_{yv}^2 \mu_y / \mu_z \right.$$
$$\left. \pm \sqrt{ 4 k_{xv}^2 k_{yv}^2 \mu_x \mu_y / \mu_z^2 + \left[k_o^2 \left(\mu_x \varepsilon_y - \mu_y \varepsilon_x \right) - k_{xv}^2 \mu_x / \mu_z + k_{yv}^2 \mu_y / \mu_z \right]^2 } \right\} \Big/ 2. \qquad (63)$$

Equations (62) and (63) provide the criteria for determining the cutoff frequency for the propagation of modes inside the waveguide. Plugging (59) into (46)–(49) yields the following equations for the TE mode vectors in (42) and (43)

$$\underline{e}_v^{''}(x,y) = j\omega \mu_o H_o \left[\underline{x}_o \mu_y k_{yv} \cos(k_{xv} x) \sin(k_{yv} y) / (k_o^2 \mu_y \varepsilon_x - k_{zv}^2) \right.$$
$$\left. - \underline{y}_o \mu_x k_{xv} \sin(k_{xv} x) \cos(k_{yv} y) / (k_o^2 \mu_x \varepsilon_y - k_{zv}^2) \right], \qquad (64)$$

$$\underline{h}_v^{''}(x,y) = jk_{zv} H_o \left[\underline{x}_o k_{xv} \sin(k_{xv} x) \cos(k_{yv} y) / (k_o^2 \mu_x \varepsilon_y - k_{zv}^2) \right.$$
$$\left. + \underline{y}_o k_{yv} \cos(k_{xv} x) \sin(k_{yv} y) / (k_o^2 \mu_y \varepsilon_x - k_{zv}^2) \right]. \qquad (65)$$

4.1.2. Incident TM mode

Assuming only a TM type mode in the waveguide sets $H_z = 0$. Then (13)–(16) become

$$E_{xv}^{'} = -jk_{zv} (d/dx) E_{zv}^{'} / (k_o^2 \mu_y \varepsilon_x - k_{zv}^2), \qquad (66)$$

$$E_{yv}^{'} = -jk_{zv} (d/dy) E_{zv}^{'} / (k_o^2 \mu_x \varepsilon_y - k_{zv}^2), \qquad (67)$$

$$H'_{xv} = j\omega\varepsilon_o\varepsilon_y(d/dy)E'_{zv}/(k_o^2\mu_x\varepsilon_y - k_{zv}^2),$$ (68)

$$H'_{yv} = -j\omega\varepsilon_o\varepsilon_x(d/dx)E'_{zv}/\left(k_o^2\mu_y\varepsilon_x - k_{zv}^2\right).$$ (69)

Solving (66)–(69) for E_z and substituting (1) for $\nabla \times \underline{H}$ formulates the anisotropic wave equation for $\underline{E}$ where (72) resembles (20)

$$\nabla \times \nabla \times \underline{E} = -j\omega\mu_o\underline{\underline{\mu_r}} \cdot (\nabla \times \underline{H}),$$ (70)

$$\nabla \times \nabla \times \underline{E} = -j\omega\mu_o\underline{\underline{\mu_r}} \cdot \left(j\omega\varepsilon_o\underline{\underline{\varepsilon_r}} \cdot \underline{E}\right),$$ (71)

$$\nabla \times \underline{\underline{\mu_r}}^{-1} \cdot (\nabla \times \underline{E}) = k_o^2\underline{\underline{\varepsilon_r}} \cdot \underline{E}.$$ (72)

Equation (72) represents the anisotropic wave equation for the time harmonic electric field. Expanding the curl of (72) and isolating the $\underline{z_o}$ component as we did for H_z in Section 4.1.1 yields the following solution for the E_z component

$$E'_{zv} = E_o\sin(k_{xv}x)\sin(k_{yv}y)e^{-jk_{zv}z}.$$ (73)

Plugging (73) into (66)–(69) yields the following equations for the TM mode vectors in (42) and (43)

$$\underline{e}'_v(x,y) = -jk_{zv}E_o\left[\underline{x}_ok_{xv}\cos(k_{xv}x)\sin(k_{yv}y)/\left(k_o^2\mu_y\varepsilon_x - k_{zv}^2\right) + \underline{y}_o\sin(k_{xv}x)\cos(k_{yv}y)/\left(k_o^2\mu_y\varepsilon_x - k_{zv}^2\right)\right],$$ (74)

$$\underline{h}'_v(x,y) = j\omega\varepsilon_oE_o\left[\underline{x}_o\varepsilon_yk_{yv}\sin(k_{xv}x)\cos(k_{yv}y)/(k_o^2\mu_x\varepsilon_y - k_{zv}^2) - \underline{y}_o\varepsilon_xk_{xv}\cos(k_{xv}x)\sin(k_{yv}y)/k_o^2\mu_y\varepsilon_x - k_{zv}^2\right].$$ (75)

4.2. Anisotropic transverse resonance

This section describes the derivation of an anisotropic transverse resonance condition established between resonant walls of a rectangular waveguide. Assume an infinite rectangular waveguide partially loaded with an anisotropic medium, then $w(z)$ represents the width of the anisotropic medium at any point z along the direction of propagation as shown in **Figure 4**.
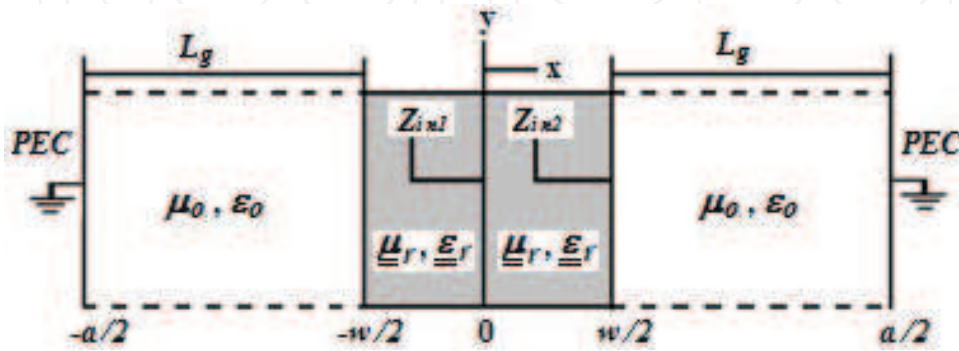

Figure 4. Symmetrically loaded transmission line model with a short at either end.

At any length z along the waveguide, the assumption that the horizontal distance between two resonant walls represented as a partially filled parallel plate waveguide is a valid presumption. We can calculate $L_g(z)$ as the unknown distance between the edge of the anisotropic medium and the cavity wall based on a transverse resonance condition in the $\underline{x}_o$-direction. However, we first need to derive the characteristic impedance of the anisotropic region in the transmission line model.

4.2.1. Electromagnetic fields in free space regions

Calculating the fields in the free space region of the waveguide begins with Maxwell's source free Eqs. (1) and (2) and the equations for the individual vector components of the electromagnetic fields (13)–(16). Using the standard derivation of the wave equation for H_z in free space from (1) and (2) shows

$$\nabla_T \times \nabla_T \times \underline{H}_v = j\omega\varepsilon_o(\nabla_T \times \underline{E}_v) = \nabla_T(\nabla_T \cdot \underline{H}_v) - \nabla_T^2 H_{zv}, \tag{76}$$

$$j\omega\varepsilon_o(-j\omega\mu_o H_{zv}) + \nabla_T^2 H_{zv} = 0, \tag{77}$$

$$[(d^2/dx^2) + (d^2/dy^2) + k_o^2]H_{zv} = 0. \tag{78}$$

Utilizing (52) and setting $k_{zo} = 0$ due to the assumption of the transverse resonance condition in the free space region of the waveguide will lead to the solution to H_z. Assuming the dominate mode to be TE_{10} because $a \geq 2b$, then $k_{yo} = 0$ for the first resonance at cutoff [1]. With $k_{yo} = 0$ no variation of the fields in the $\underline{y}_o$ direction and $(d^2/dy^2)H_z = 0$ then (78) becomes

$$(d^2/dx^2 + k_o^2)H_z = 0. \tag{79}$$

Equation (80) is a standard differential equation with a known solution [17]

$$H_{zv} = Ae^{-jk_ox} + Be^{+jk_ox}, \tag{80}$$

where A and B are yet to be determined coefficients. Substituting (80) into (13)–(16) yields the expression for E_y

$$E_y = k_o\omega\mu_o\left(Ae^{-jk_ox} - Be^{+jk_ox}\right)/k_o^2 = Z_o\left(Ae^{-jk_ox} - Be^{+jk_ox}\right). \tag{81}$$

Accounting for the restrictions imposed by the transverse resonance conditions on E_z, k_{zo} and k_{yo}, then $E_x = 0$, $H_x = 0$ and $H_y = 0$ as well.

4.2.2. Electromagnetic fields in anisotropic region

Starting with (1) and (2) for the source free Maxwell's equations in an anisotropic medium, the vector components (13)–(16) led to the derivation of the dispersion Eqs. (26) and (38) for H_z and E_z, respectively.

The cutoff frequency or resonance of a rectangular waveguide is determined when the propagation constant in the direction of resonance, in the case the $\underline{x_o}$-direction, is 0 [1]. By definition, when the waveguide's dominant mode $v = 1$ propagates, then $k_{x1} > 0$ and the guide is resonant whereas when the mode attenuates then $k_{x1} < 0$ and there is no resonance. Therefore, the resonance first manifests itself when $k_{x1} = 0$. For the dominate mode to be TE_{10} then $a \geq 2b$ and $d^2 H_z/dy^2 = 0$. Simplifying (28) with these substitutions produces a simpler form to solve for H_z

$$\left[(d^2/dx^2) + k_o^2 \mu_z \varepsilon_y \right] H_z = 0, \tag{82}$$

$$\beta = k_o \sqrt{\mu_z \varepsilon_y}. \tag{83}$$

Solving (82) for H_z and plugging the result into (13)–(16) yields

$$H_z = Ce^{-j\beta x} + De^{+j\beta x}, \tag{84}$$

$$E_y = Z_o \beta \left(Ce^{-j\beta x} - De^{+j\beta x} \right)/k_o \varepsilon_y = Z_o \sqrt{\mu_z/\varepsilon_y} \left(Ce^{-j\beta x} - De^{+j\beta x} \right), \tag{85}$$

We can see from (13)–(16) that based on our resonance conditions on E_z, k_{z1} and k_{y1} that $E_x = 0$, $H_x = 0$ and $H_y = 0$.

4.2.3. Characteristic impedances of the two regions

The first boundary condition exists at the perfect electric conductor (PEC) boundary when $x = -a/2$ and $\underline{E}(x, y, z) = 0$

$$E_y \big|_{x=-a/2} = 0 \rightarrow Ae^{jk_o a/2} = Be^{-jk_o a/2}, \tag{86}$$

$$A = Be^{-jk_o a}. \tag{87}$$

Plugging (87) into (80) and (81) yields

$$E_y = Z_o Be^{-jk_o a/2} \left[e^{-jk_o(x+a/2)} - e^{+jk_o(x+a/2)} \right], \tag{88}$$

$$E_y = -2Z_o Be^{-jk_o a/2} \sin[k_o(x + a/2)]. \tag{89}$$

Similarly,

$$H_z = 2Be^{-jk_o a/2} \cos[k_o(x + a/2)]. \tag{90}$$

Equations (89) and (90) solve for the impedance of the free space region as $Z = -E_y/H_z$

$$Z_o = -E_y/H_z = jZ_o \tan[k_o(x + a/2)], \tag{91}$$

within the region $0 \leq (x + a/2) \leq (a-w)/2$. The second boundary condition exists at $x = -w/2$ where the tangential fields at the boundary are equal. In this case, there are two tangential fields in E_y and H_z. At the boundary, we have the following three conditions

$$E_y^-\Big|_{x=-w/2} = E_y^+\Big|_{x=-w/2'} \tag{92}$$

$$H_z^-\Big|_{x=-w/2} = H_z^+\Big|_{x=-w/2'} \tag{93}$$

$$Z_o\big|_{x=-w/2}^- = Z_1\big|_{x=-w/2}^+. \tag{94}$$

Plugging Eqs. (80) and (81) into (92) and (93) yields the following set of equations

$$-2jBe^{-jk_o(a/2)}\sin[k_o(a-w)/2] = \sqrt{(\mu_z/\varepsilon_y)}\left(Ce^{-j\beta_1(w/2)} - De^{+j\beta_1(w/2)}\right), \tag{95}$$

$$2Be^{-jk_o(a/2)}\cos[k_o(a-w)/2] = Ce^{-j\beta_1(w/2)} - De^{+j\beta_1(w/2)}. \tag{96}$$

Equations (95) and (96) give two equations to solve for three unknowns. Match equation (91) to the impedance in the anisotropic region at $x = -w/2$ to solve for the third unknown. Now solve for $Z = -E_y/H_z$ from (89) and (90)

$$Z_1 = Z_o\sqrt{(\mu_z/\varepsilon_y)}\left[\left(De^{+j\beta_1 x} - Ce^{-j\beta_1 x}\right)/\left(De^{+j\beta_1 x} + Ce^{-j\beta_1 x}\right)\right]$$
$$= Z_o\sqrt{(\mu_z/\varepsilon_y)}\left[\left(1 - \rho e^{-j2\beta_1 x}\right)/\left(1 + \rho e^{-j2\beta_1 x}\right)\right], \tag{97}$$

$$|Z_1| = Z_o\sqrt{(\mu_z/\varepsilon_y)}, \tag{98}$$

where $\rho = C/D$. Now apply boundary condition (94) to (91) and (97) in order to yield an expression for ρ

$$Z_o\sqrt{(\mu_z/\varepsilon_y)}\left[\left(1 - \rho e^{j\beta_1 w}\right)/1 + \rho e^{j\beta_1 w}\right] = jZ_o\tan[k_o(a-w)/2], \tag{99}$$

$$\left(1 - \rho e^{j\beta_1 w}\right)/1 + \rho e^{j\beta_1 w} = j\sqrt{(\varepsilon_y/\mu_z)}\tan[k_o(a-w)/2] = j\Psi, \tag{100}$$

$$\rho = e^{-j\beta_1 w}(1 - j\Psi)/(1 + j\Psi). \tag{101}$$

Substituting (101) into (97) yields the last equation along with (95) and (96) to solve for B, C and D.

4.2.4. Anisotropic transverse resonance condition

To simplify the calculation, consider **Figure 4** as slice of **Figure 3** in only one direction that is partially filled with an anisotropic medium. **Figure 4** represents a transmission line representation that allows for a solution to L_g in terms of w for a given wavelength. Now use standard transmission line theory to calculate the input impedance Z_{in} at $x = 0$ from both directions. Transmission line theory says that as we approach the same point in a transmission line from either direction the input impedances should be equal. Then by symmetry the transverse resonance condition simplifies to $Z_{in} = 0$ from either direction.

Starting with the short located at $x = a/2$, calculate Z_{in2} at $x = w/2$ as

$$Z_{in2} = jZ_o \tan(k_o L_g).$$ (102)

Now calculate Z_{in1} at $x = 0$ as

$$Z_{in1} = Z_1 [Z_{in2} + jZ_1 \tan(\beta_1 w/2)] / [Z_1 + jZ_{in2} \tan(\beta_1 w/2)].$$ (103)

The symmetric transverse resonance condition simplifies (103) to

$$Z_1 + jZ_{in2} \tan(\beta_1 w/2) = 0.$$ (104)

Plugging (98) and (101) into (104) yields the following equation for L_g [14]

$$L_g = \lambda \tan^{-1} \left[\sqrt{(\mu_z/\varepsilon_y)} / \tan(\pi w \sqrt{\mu_z \varepsilon_y}/\lambda) \right] / (2\pi).$$ (105)

where λ is wavelength. Importantly, the solution of (105) shows that the transverse resonance only depends on two of the six $\underline{\underline{\varepsilon_r}}$ and $\underline{\underline{\mu_r}}$ components. This means that maintaining a constant resonance in a waveguide or cavity relies on the clever engineering of ε_y and μ_z and leaves designers free to adjust the other components as they see fit to enhance performance in other ways. Furthermore, if $\varepsilon_y = \mu_z = 1$ then the resonance in the $\underline{x_o}$ direction will see the anisotropic substrate as air, while other tensor elements can be utilized to achieve performance attributed to materials with an arbitrarily high refractive index.

4.2.5. Suppression of birefringence in a rectangular waveguide

Section 3.4 discusses the phenomenon of birefringence in an unbounded anisotropic half-space by deriving the existence of a fourth order polynomial for the wavenumber in the propagation direction. However, for low order resonances, a rectangular waveguide suppresses the birefringence inherent to anisotropic media by suppressing propagation in the vertical direction of the waveguide. In other words, $k_y = 0$ and $d^2/dy^2 = 0$ assuming the horizontal dimension of the waveguide is at least twice the size of the vertical dimension or $a \geq 2b$ in **Figure 1** [3]. The bound on the waveguide geometry simplifies (39), the dispersion equation for E_z in an anisotropic waveguide, to

$$k_o^2 k_x^2 \varepsilon_x E_z / \left(k_o^2 \mu_y \varepsilon_x - k_z^2 \right) - k_o^2 \varepsilon_z E_z = 0,$$ (106)

and results in the following second order polynomial for k_z

$$k_z^2 = \varepsilon_x \left(k_o^2 \mu_y - k_x^2/\varepsilon_z \right).$$ (107)

The suppression of the k_y term in (39) yields a second order differential equation for the wave number in the propagation direction, thereby eliminating the property of birefringence for this case.

5. Conclusions

Recently engineered materials have come to play an important role in state of the art designs electromagnetic devices and especially antennas. Many of these engineered materials have inherent anisotropic properties. Anisotropic media yield characteristics such as conformal surfaces, focusing and refraction of electromagnetic waves as they propagate through a material, high impedance surfaces for artificial magnetic conductors as well as high index, low loss, and lightweight ferrite materials. This chapter analyzes the properties of electromagnetic wave propagation in anisotropic media, and presents research including plane wave solutions to propagation in anisotropic media, a mathematical derivation of birefringence in anisotropic media, modal decomposition of rectangular waveguides filled with anisotropic media, and the full derivation of anisotropic transverse resonance in a partially loaded waveguide.

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