Monte Carlo Set-Membership Filtering for Nonlinear Dynamic Systems

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Additional information is available at the end of the chapter

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Abstract

This chapter considers the nonlinear filtering problem involving noises that are unknown and bounded. We propose a new filtering method via set-membership theory and boundary sampling technique to determine a state estimation ellipsoid. In order to guarantee the online usage, the nonlinear dynamics are linearized about the current estimate, and the remainder term is then bounded by an optimization ellipsoid, which can be described as the solution of a semi-infinite optimization problem. It is an analytically intractable problem for general nonlinear dynamic systems. Nevertheless, for a typical nonlinear dynamic system in target tracking, some certain regular properties for the remainder are analytically derived; then, we use a randomized method to approximate the semi-infinite optimization problem efficiently. Moreover, for some quadratic nonlinear dynamic systems, the semi-infinite optimization problem is equivalent to solving a semi-definite program problem. Finally, the set-membership prediction and measurement update are derived based on the recent optimization method and the online bounding ellipsoid of the remainder other than a priori bound. Numerical example shows that the proposed method performs better than the extended set-membership filter, especially in the situation of the larger noise.

Keywords: nonlinear dynamic systems, set-membership filter, randomization, semi-definite optimization, target tracking

1. Introduction

Filtering techniques for dynamic systems are widely used in practiced fields such as target tracking, signal processing, automatic control, and computer vision. The Kalman filter is a fundamental tool for solving a broad class of filtering problems with linear dynamic systems. When dynamic systems are nonlinear, some well-known generalizations include the extended

Kalman filter (EKF) and unscented Kalman filtering (UKF) [1]. These methods are based on local linear approximations of the nonlinear system where the higher order terms are ignored. Most recently, [2] proposes box particle filter to handle interval data based on interval analysis and constraint satisfaction techniques. The advantage of the box particle filter against the standard particle filter is its reduced computational complexity [3–5]. However, most of Monte Carlo filtering techniques are based on the assumptions that probability density functions of the state noise and measurement noise are known.

Actually, when the underlying probabilistic assumptions are not realistic (e.g., the main perturbation may be deterministic), it seems more natural to assume that the state noise and measurement noise are unknown but bounded [6]; then, [7] proposed set-membership estimation technique. The idea of propagating bounding ellipsoids (or boxes, polytopes, simplexes, parallelotopes, and polytopes) for systems with bounded noises has also been extensively investigated (e.g., see recent papers [6, 8–12] and references therein). Most of these methods concentrate on the linear dynamic systems.

The set-membership filtering for nonlinear dynamic systems is known to be a challenging problem. Based on ellipsoid-bounded, fuzzy-approximated, or Lipschitz-like nonlinearities, several results have been made [13–15]. These results assume that the ellipsoid bounds, the coefficients of fuzzy-approximation, or Lipschitz constants are known before filtering, which limits them in real time implementation. For example, for a typical nonlinear dynamic system in a radar, the bounds of the remainder depend on the past estimates so that they cannot be obtained before filtering. Recently, the paper [16] gives an overview of recent developments in set-theoretic methods for nonlinear systems, with a particular focus on a two-reaction model of anaerobic digestion, and the key idea of [17, 18] consists in a combination of Bayesian and set-valued estimation concepts. To our knowledge, [19, 20] develop nonlinear set-membership filters which can estimate the bounding ellipsoid of nonlinearities in real time, and the filters are called the extended set-membership filter (ESMF) and set-valued nonlinear filter (SVNF), respectively. Both [19, 20] derive the bounds of the remainder by an outer bounding box. Actually, if the remainder can be bounded by a tighter ellipsoid and using some recent advanced optimization techniques for filtering, it should be able to derive a tighter set-membership filtering for the nonlinear dynamic system.

In this chapter, when the underlying state noises and measurement noises are unknown but bounded, we propose a tighter set-membership filtering methods via set-membership estimation theory and boundary sampling technique. In order to guarantee the online usage, the nonlinear dynamics are linearized about the current estimate, and the remainder terms are then bounded by an ellipsoid, which can be formulated as the solution of a semi-infinite optimization problem. In general, it is an analytically intractable problem when dynamic systems are nonlinear. The main contributions of the paper are summarized as follows:

- For a typical nonlinear dynamic system in target tracking, we can analytically derive some regular properties for the remainder. Then, the semi-infinite optimization problem can be efficiently solved by using boundary sampling technique.

- For some quadratic nonlinear dynamic systems, using the samples on all vertices of a polyhedron, we obtain a tight bounding ellipsoid, which can cover the remainder by solving a semi-definite program (SDP) problem.
- The set-membership prediction and measurement update are derived based on the recent optimization method and the online bounding ellipsoid of the remainder other than a priori bound.

The rest of the paper is organized as follows. Preliminaries are given in Section 2. In Section 3, the bounding ellipsoid of the remainder set is calculated. In Section 4, the prediction step and the measurement update step of the set-membership filtering for nonlinear dynamic systems are derived, respectively. Examples and conclusions are given in Section 5 and Section 6, respectively.

2. Preliminaries

2.1. Problem formulation

We consider a nonlinear dynamic system:

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k) + \mathbf{w}_k, \tag{1}$$

$$\mathbf{y}_k = h_k(\mathbf{x}_k) + \mathbf{v}_k, \tag{2}$$

where $\mathbf{x}_k \in \mathcal{R}^n$ is the state of system at time k and $\mathbf{y}_k \in \mathcal{R}^{n_1}$ is the measurement. $f_k(\mathbf{x}_k)$ and $h_k(\mathbf{x}_k)$ are nonlinear functions of $\mathbf{x}_k$, $\mathbf{w}_k \in \mathcal{R}^n$ is the uncertainty of process noises or system biases, and $\mathbf{v}_k \in \mathcal{R}^{n_1}$ is the uncertainty of measurement noises or system biases. They are assumed to be confined to the specified ellipsoidal sets:

$$\mathbf{W}_k = \left\{\mathbf{w}_k : \mathbf{w}_k^T \mathbf{Q}_k^{-1} \mathbf{w}_k \le 1\right\}$$
$$\mathbf{V}_k = \left\{\mathbf{v}_k : \mathbf{v}_k^T \mathbf{R}_k^{-1} \mathbf{v}_k \le 1\right\},$$

where $\mathbf{Q}_k$ and $\mathbf{R}_k$ are the *shape matrices* of the ellipsoids $\mathbf{W}_k$ and $\mathbf{V}_k$, respectively, which are known as symmetric positive-definite matrices. At time k given that $\mathbf{x}_k$ belongs to a current bounding ellipsoid:

$$\begin{aligned}\mathcal{E}_k &= \left\{\mathbf{x} \in R^n : (\mathbf{x} - \widehat{\mathbf{x}}_k)^T (\mathbf{P}_k)^{-1} (\mathbf{x} - \widehat{\mathbf{x}}_k) \le 1\right\} \\ &= \left\{\mathbf{x} \in R^n : \mathbf{x} = \widehat{\mathbf{x}}_k + \mathbf{E}_k \mathbf{u}_k, \mathbf{P}_k = \mathbf{E}_k \mathbf{E}_k^T, \|\mathbf{u}_k\| \le 1\right\}\end{aligned} \tag{3}$$

where $\widehat{\mathbf{x}}_k$ is the center of ellipsoid $\mathcal{E}_k$ and $\mathbf{P}_k$ is a known symmetric positive-definite matrix. Moreover, we assume that when the nonlinear functions are linearized, the remainder terms can be bounded by an ellipsoid. Specifically, by Taylor's theorem, f_k and h_k can be linearized to

$$f_k(\widehat{\mathbf{x}}_k + \mathbf{E}_k\mathbf{u}_k) = f_k(\widehat{\mathbf{x}}_k) + \mathbf{J}_{f_k}\mathbf{E}_k\mathbf{u}_k + \Delta f_k(\mathbf{u}_k), \tag{4}$$

$$h_k(\widehat{\mathbf{x}}_k + \mathbf{E}_k\mathbf{u}_k) = h_k(\widehat{\mathbf{x}}_k) + \mathbf{J}_{h_k}\mathbf{E}_k\mathbf{u}_k + \Delta h_k(\mathbf{u}_k), \tag{5}$$

where $\mathbf{E}_k$ and $\mathbf{u}_k$ are defined in (3), $\mathbf{J}_{f_k} = \frac{\partial f_k(\mathbf{x}_k)}{\partial \mathbf{x}}\Big|_{\widehat{\mathbf{x}}_k}$ and $\mathbf{J}_{h_k} = \frac{\partial h_k(\mathbf{x}_k)}{\partial \mathbf{x}}\Big|_{\widehat{\mathbf{x}}_k}$ are Jacobian matrices, and $\Delta f_k(\mathbf{u}_k)$ and $\Delta h_k(\mathbf{u}_k)$ are high-order remainders, which can be bounded in an ellipsoid for all $\|\mathbf{u}_k\| \leq 1$, respectively, i.e.,

$$\Delta f_k(\mathbf{u}_k) \in \mathcal{E}_{f_k} = \left\{\mathbf{x} \in R^n : \left(\mathbf{x} - \mathbf{e}_{f_k}\right)^T \left(\mathbf{P}_{f_k}\right)^{-1} \left(\mathbf{x} - \mathbf{e}_{f_k}\right) \leq 1\right\}, \tag{6}$$

$$= \left\{\mathbf{x} \in R^n : \mathbf{x} = \mathbf{e}_{f_k} + \mathbf{B}_{f_k}\Delta_{f_k}, \mathbf{P}_{f_k} = \mathbf{B}_{f_k}\mathbf{B}_{f_k}^T, \|\Delta_{f_k}\| \leq 1\right\}, \tag{7}$$

$$\Delta h_k(\mathbf{u}_k) \in \mathcal{E}_{h_k} = \left\{\mathbf{x} \in R^{n_1} : (\mathbf{x} - \mathbf{e}_{h_k})^T (\mathbf{P}_{h_k})^{-1} (\mathbf{x} - \mathbf{e}_{h_k}) \leq 1\right\}, \tag{8}$$

$$= \left\{\mathbf{x} \in R^{n_1} : \mathbf{x} = \mathbf{e}_{h_k} + \mathbf{B}_{h_k}\Delta_{h_k}, \mathbf{P}_{h_k} = \mathbf{B}_{h_k}\mathbf{B}_{h_k}^T, \|\Delta_{h_k}\| \leq 1\right\}, \tag{9}$$

where $\mathbf{e}_{f_k}$ and $\mathbf{e}_{h_k}$ are the centers of the ellipsoids $\mathcal{E}_{f_k}$ and $\mathcal{E}_{h_k}$, respectively, and $\mathbf{P}_{f_k}$ and $\mathbf{P}_{h_k}$ are the shape matrices of the ellipsoids $\mathcal{E}_{f_k}$ and $\mathcal{E}_{h_k}$, respectively. Note that we do not assume that the ellipsoids $\mathcal{E}_{f_k}$ and $\mathcal{E}_{h_k}$ are given before filtering, and we will compute these ellipsoids online.

Suppose that the initial state $\mathbf{x}_0$ belongs to a given bounding ellipsoid:

$$\mathcal{E}_0 = \left\{\mathbf{x} \in R^n : (\mathbf{x} - \widehat{\mathbf{x}}_0)^T (\mathbf{P}_0)^{-1} (\mathbf{x} - \widehat{\mathbf{x}}_0) \leq 1\right\}, \tag{10}$$

where $\widehat{\mathbf{x}}_0$ is the center of ellipsoid $\mathcal{E}_0$ and $\mathbf{P}_0$ is the shape matrix of the ellipsoid $\mathcal{E}_0$ which is a known symmetric positive-definite matrix.

The proposed set-membership filter mainly contains two steps: prediction step and measurement update step. The goal of prediction step is to determine a bounding ellipsoid $\mathcal{E}_{k+1|k}$ based on the measurement $\mathbf{y}_k$ at time k, i.e., look for $\widehat{\mathbf{x}}_{k+1|k}$, $\mathbf{P}_{k+1|k}$ such that the state $\mathbf{x}_{k+1}$ belongs to

$$\mathcal{E}_{k+1|k} = \left\{\mathbf{x} \in R^n : \left(\mathbf{x} - \widehat{\mathbf{x}}_{k+1|k}\right)^T \left(\mathbf{P}_{k+1|k}\right)^{-1} \left(\mathbf{x} - \widehat{\mathbf{x}}_{k+1|k}\right) \leq 1\right\},$$

whenever (i) $\mathbf{x}_k$ is in $\mathcal{E}_k$; (ii) the processes $\mathbf{w}_k$, $\mathbf{v}_k$ are bounded in ellipsoids, i.e., $\mathbf{w}_k \in \mathbf{W}_k$, $\mathbf{v}_k \in \mathbf{V}_k$; and (iii) the remainders $\Delta f_k(\mathbf{u}_k) \in \mathcal{E}_{f_k}$ and $\Delta h_k(\mathbf{u}_k) \in \mathcal{E}_{h_k}$. The robust measurement update step is aimed to determine a bounding ellipsoid $\mathcal{E}_{k+1}$ based on the measurement $\mathbf{y}_{k+1}$ at time $k+1$, i.e., look for $\widehat{\mathbf{x}}_{k+1}$, $\mathbf{P}_{k+1}$ such that the state $\mathbf{x}_{k+1}$ belongs to

$$\mathcal{E}_{k+1} = \left\{\mathbf{x} \in R^n : (\mathbf{x} - \widehat{\mathbf{x}}_{k+1})^T (\mathbf{P}_{k+1})^{-1} (\mathbf{x} - \widehat{\mathbf{x}}_{k+1}) \leq 1\right\},$$

whenever (i) $\mathbf{x}_{k+1}$ is in $\mathcal{E}_{k+1|k}$; (ii) the measurement noises $\mathbf{v}_{k+1}$ is bounded in ellipsoid, i.e., $\mathbf{v}_{k+1} \in \mathbf{V}_{k+1}$; and (iii) the remainders $\Delta h_{k+1}(\mathbf{u}_{k+1}) \in \mathcal{E}_{h_{k+1}}$. The **key issue** is to determine two tight bounding ellipsoids $\mathcal{E}_{f_k}$ and $\mathcal{E}_{h_k}$ in real time so that the filtering algorithm can be implemented online.

3. Ellipsoidal remainder bounding

In this section, we discuss the key problem on how to adaptively determine a tighter bounding ellipsoid to cover the high-order remainders from the optimization point of view.

3.1. Ellipsoidal remainder bounding by sampling

By (4)–(5), the high-order remainders are

$$\Delta f_k(\mathbf{u}_k) = f_k(\widehat{\mathbf{x}}_k + \mathbf{E}_k\mathbf{u}_k) - f_k(\widehat{\mathbf{x}}_k) - \mathbf{J}_{f_k}\mathbf{E}_k\mathbf{u}_k,$$
$$\Delta h_k(\mathbf{u}_k) = h_k(\widehat{\mathbf{x}}_k + \mathbf{E}_k\mathbf{u}_k) - h_k(\widehat{\mathbf{x}}_k) - \mathbf{J}_{h_k}\mathbf{E}_k\mathbf{u}_k,$$

whenever $\|\mathbf{u}_k\| \le 1$. Obviously, it is a hard problem to cover a remainder by an ellipsoid since f_k and h_k are generally nonlinear functions. The outer bounding ellipsoid for $\Delta f_k(\mathbf{u}_k)$ is not uniquely defined, which can be optimized by minimizing the size $f(P)$ of the bounding ellipsoid. Thus, the optimization problem for the bounding ellipsoid of $\Delta f_k(\mathbf{u}_k)$ defined in (6) can be written as

$$\min \ f\left(\mathbf{P}_{f_k}\right) \tag{11}$$

$$\text{subject to} \ \left(\Delta f_k(\mathbf{u}_k) - \mathbf{e}_{f_k}\right)^T \left(\mathbf{P}_{f_k}\right)^{-1} \left(\Delta f_k(\mathbf{u}_k) - \mathbf{e}_{f_k}\right) \le 1, \ \text{for all} \ \|\mathbf{u}_k\| \le 1. \tag{12}$$

where $\mathbf{P}_{f_k} = \mathbf{B}_{f_k}\mathbf{B}_{f_k}^T$ and $\mathbf{e}_{f_k}$ and $\mathbf{P}_{f_k}$ are decision variables. Since the optimization problem (11)–(12) has an infinite number of constraints, it is called a semi-infinite optimization problem in [21]. In general, it is a NP-hard problem.

Remark 1. *In practice, we want to achieve a state estimation ellipsoid by minimizing its "size" at each time; it is a function of the shape matrix* $\mathbf{P}$ *denoted by* $f(\mathbf{P})$. *If we choose trace function, i.e.,* $f(\mathbf{P}) = tr(\mathbf{P})$, *it means the sum of squares of semiaxes lengths of the ellipsoid* $\mathcal{E}$. *The other common "size" of the ellipsoid is logdet*$(\mathbf{P})$, *which corresponds to the volume of the ellipsoid* $\mathcal{E}$.

For a general nonlinear dynamic system, it is hard to solve the problem (11)–(12) [22]. It is this reason that the literatures [23, 24] are sought to find the particular relaxations of the original optimization problem (11)–(12). One of the typical methods is based on randomization of the parameter $\mathbf{u}_k$. Specifically, to solve the problem (11)–(12), we may take some samples from the boundary and interior points of the sphere $\|\mathbf{u}_k\| \le 1$ so that we can get a finite set of $\mathbf{u}_k^1, \ldots, \mathbf{u}_k^N$, and then the infinite constraint (12) can be approximated by N constraints based on $\mathbf{u}_k^1, \ldots, \mathbf{u}_k^N$. Moreover, by Schur complement, an approximate bounding ellipsoid for $\Delta f_k(\mathbf{u}_k)$ can be derived by solving the following SDP optimization problem:

$$\min\ f\left(\mathbf{P}_{f_k}\right) \tag{13}$$

$$\text{subject to}\ \begin{bmatrix} -1 & \left(\Delta f_k\left(\mathbf{u}_k^i\right) - \mathbf{e}_{f_k}\right)^T \\ \Delta f_k\left(\mathbf{u}_k^i\right) - \mathbf{e}_{f_k} & -\mathbf{P}_{f_k} \end{bmatrix} \preccurlyeq 0 \quad i = 1, \dots, N. \tag{14}$$

Although the randomized solution may not be feasible for all $\|\mathbf{u}_k\| \le 1$, [24] has used statistical learning techniques to provide an explicit bound on the measure of the set of original constraints that are possibly violated by the randomized solution, and they prove this measure rapidly decreases to zero as N is increasing. Therefore, the obtained randomized solution of the optimization problem (13)–(14) can be made approximately feasible for the semi-infinite optimization (11)–(12) by sampling a sufficient number of constraints.

Similarly, the outer bounding ellipsoid for $\Delta h_k(\mathbf{u}_k)$ can be derived by solving

$$\min\ f\left(\mathbf{P}_{h_k}\right) \tag{15}$$

$$\text{subject to}\ \begin{bmatrix} -1 & \left(\Delta h_k\left(\mathbf{u}_k^i\right) - \mathbf{e}_{h_k}\right)^T \\ \Delta h_k\left(\mathbf{u}_k^i\right) - \mathbf{e}_{h_k} & -\mathbf{P}_{h_k} \end{bmatrix} \preccurlyeq 0, \quad i = 1, \dots, N. \tag{16}$$

Remark 2. *Note that the bounding ellipsoid of* [19] *is derived by interval mathematics. We derive the bounding ellipsoid by solving a semi-infinite optimization problem.* **Figure 1** *illustrates the difference of two methods. It is obvious to see that the bounding ellipsoid derived by solving the SDP (13) is tighter than that obtained by interval mathematics. The cumulative effect of the conservative bounding ellipsoid at each time step may yield divergence of a filtering.*

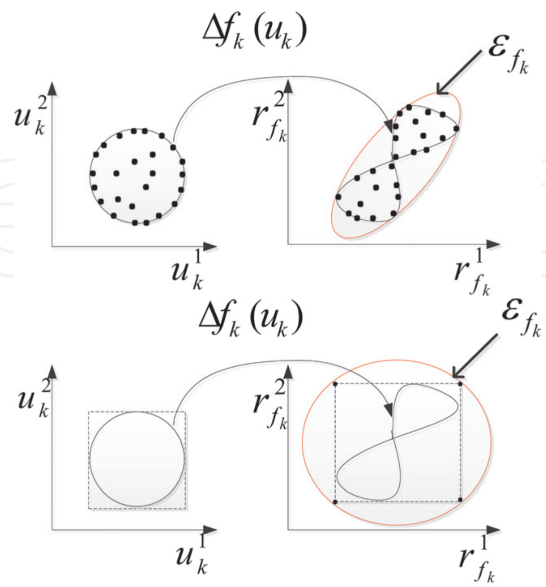

Figure 1. (Top) The bounding ellipsoid is derived by covering the solid points of the remainder which are obtained by Monte Carlo sampling. (Bottom) The bounding ellipsoid is derived by covering the vertices of the rectangle obtained by interval mathematics [19].

3.2. Ellipsoidal remainder bounding by boundary sampling

In this subsection, for a typical nonlinear dynamic system in target tracking, we discuss that the remainder can be bounded by an ellipsoid via *boundary* sampling for target tracking. Thus, the new method can reduce the computation complexity efficiently in the bounding step.

Let us consider the following nonlinear measurement Eq. [1]:

$$h(\mathbf{x}) = \begin{bmatrix} \sqrt{(\mathbf{x}(1)-a)^2 + (\mathbf{x}(2)-b)^2} \\ arctan\left(\dfrac{\mathbf{x}(2)-b}{\mathbf{x}(1)-a}\right) \end{bmatrix}, a, b \in \mathcal{R} \tag{17}$$

where $\mathbf{x}$ is a four-dimensional state variable that includes position and velocity $[x, y, \dot{x}, \dot{y}]^T$. Note that $h(\mathbf{x})$ only depends on the first two dimensions $\mathbf{x}(1)$ and $\mathbf{x}(2)$.

We discuss the relationship between the set $\{\|\mathbf{u}_k\| \ \leq 1, \mathbf{u}_k = [\mathbf{u}_k(1)\ \mathbf{u}_k(2)]\}$ and the remainder set $\{\Delta h_{k+1}(\mathbf{u}_k) : \|\mathbf{u}_k\| \ \leq 1\}$.

Proposition 1. *If we let the remainder* $g(\mathbf{u}) = h(\widehat{\mathbf{x}} + \mathbf{E}\mathbf{u}) - h(\widehat{\mathbf{x}}) - \mathbf{J}_h\mathbf{E}\mathbf{u}$ *where* $h(\widehat{\mathbf{x}})$ *is defined in (17),* $\mathbf{E}$ *is a Cholesky factorization of a positive-definite* $\mathbf{P}$ *such that ellipsoid* $\{\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{E}\mathbf{u} : \|u\| \ \leq 1\}$ *does not intersect with the radial* $\{\mathbf{x} : \mathbf{x}(1) <= a, \mathbf{x}(2) = b\}$*, and then the boundary of the remainder set* $\mathbf{S} = \{g(\mathbf{u}) : \|\mathbf{u}\| \leq 1\}$ *belongs to the set* $\{g(\mathbf{u}) : \|\mathbf{u}\| = 1\}$.

Remark 3. *Note that the ellipsoid* $\{\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{E}\mathbf{u} : \|u\| \ \leq 1\}$ *does not intersect with the radial.* $\{\mathbf{x} : \mathbf{x}(1) <= a, \mathbf{x}(2) = b\}$ *is a weak condition, which is in order to satisfy the continuity of* $g(\mathbf{u})$*, and we can verify the condition by using the distance from the ellipsoid center* $\widehat{\mathbf{x}}$ *to the radial. Moreover, if the condition is violated, i.e., the true target is near the radial, we can transform the data to a new coordinate system where the target is far way the radial, and then the assumption can be satisfied.*

The proof of Proposition 1 relies on the following three lemmas:

Lemma 1. (Remainder lemma). *The determinant of the derivative of the remainder* $g(\mathbf{u})$ *is not less than* 0, *and the equality holds if and only if* $c\mathbf{u}(1) + d\mathbf{u}(2) = 0$*, where* $c = \mathbf{E}_{11}(\widehat{\mathbf{x}}(2) - b) - \mathbf{E}_{21}(\widehat{\mathbf{x}}(1) - a)$, $d = \mathbf{E}_{12}(\widehat{\mathbf{x}}(2) - b) - \mathbf{E}_{22}(\widehat{\mathbf{x}}(1) - a)$*, and* $\mathbf{E}_{ij}$ *are the entries of the ith row and the jth column of the matrix* $\mathbf{E}$. *Meanwhile, if* $c\mathbf{u}(1) + d\mathbf{u}(2) = 0$*, then* $g(\mathbf{u}) = 0$.

Lemma 2. *If the sets* $\mathbf{S}^1 \cup \mathbf{S}^2 = \mathbf{S}^3 \cup \mathbf{S}^4$, $\mathbf{S}^3 \cap \mathbf{S}^4 = \varnothing$*, and* $\mathbf{S}^1 \subset \mathbf{S}^3$*, then* $\mathbf{S}^4 \subset \mathbf{S}^2$.

Lemma 3. (Inverse function theorem [25]) *Suppose that* $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ *is continuously differentiable in an open set containing* $\mathbf{u}$ *and* $\det(\varphi'(\mathbf{u})) \neq 0$*, then there is an open set* $\mathbf{V}$ *containing* $\mathbf{u}$ *and open set* $\mathbf{W}$ *containing* $\varphi(\mathbf{u})$ *such that* $\varphi : \mathbf{V} \to \mathbf{W}$ *has a continuous inverse* $\varphi^{-1} : \mathbf{W} \to \mathbf{V}$ *which is differentiable and for all* $\mathbf{y} \in \mathbf{W}$ *satisfies* $\left(\varphi^{-1}\right)'(\mathbf{y}) = \left[\varphi'\left(\varphi^{-1}(\mathbf{y})\right)\right]^{-1}$.

Example 1. *To illustrate Proposition* 1*, we give an example as follow: if* $a = 50$, $b = 100$, $\widehat{x} = [80\ 130]^T$*, and* $\mathbf{P} = \mathrm{diag}(500, 1000)$*, it is easy to check that* $g(\mathbf{u})$ *is continuously differentiable in set* $\mathbf{S}_1 = \{\mathbf{u} : \|\mathbf{u}\| \leq 1\}$. *We divide* $\mathbf{S}_1$ *into three parts, i.e.,* $\mathbf{S}_1 = \mathbf{A}^1 \cup \mathbf{B}^1 \cup \mathbf{C}^1$*, where* $\mathbf{A}^1 = \{\mathbf{u} : c\mathbf{u}(1) + d\mathbf{u}(2) < 0, \|\mathbf{u}\| \leq 1\}$, $\mathbf{B}^1 = \{\mathbf{u} : c\mathbf{u}(1) + d\mathbf{u}(2) > 0, \|\mathbf{u}\| \leq 1\}$, *and* $\mathbf{C}^1 = \{\mathbf{u} : c\mathbf{u}(1) + d\mathbf{u}(2) = 0,$

$\|\mathbf{u}\| \leq 1\}$. *Meanwhile, we can also divide* $\mathbf{S}$ *into the corresponding parts, such that* $\mathbf{A} = \{g(\mathbf{u}) : \mathbf{u} \in \mathbf{A}^1\}$, $\mathbf{B} = \{g(\mathbf{u}) : \mathbf{u} \in \mathbf{B}^1\}$, $\mathbf{C} = \{g(\mathbf{u}) : \mathbf{u} \in \mathbf{C}^1\}$, *and then* $\mathbf{S} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$.

Figure 2 *shows the separation area of the circle and their corresponding area of* $g(\mathbf{u})$. *Three observations can be made as follows:*

- *The remainder set is the union of two sets.*
- *The (red) line* $\mathbf{C}^1$ *is mapped to the point 0.*
- *The boundary of* $\mathbf{S}$ *belongs to the set* $\{g(\mathbf{u}) : \|\mathbf{u}\| = 1\}$.

In summary, when we take samples from the boundary, they are sufficient to derive the outer bounding ellipsoids of the remainder set. Therefore, based on Proposition 1, *the computation complexity in the bounding step of the new method can be reduced much more.*

Remark 4. *In order to further reduce the samples and cover the remainder set at the same time, we can heuristically enlarge the sampling area, such as* $\{\|\mathbf{u}_k\| = 1.1\}$; *then, the remainder set becomes a little larger. If we derive an ellipsoid to cover the little larger remainder, then this ellipsoid can cover the original remainder set* $\{\Delta h_{k+1}(\mathbf{u}_k) : \|\mathbf{u}_k\| \leq 1\}$.

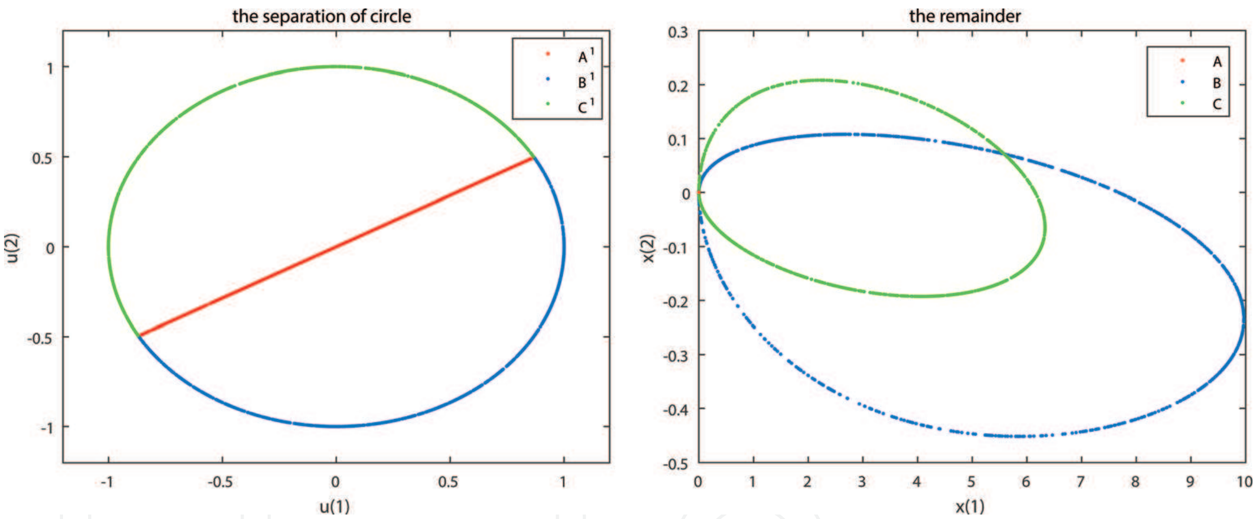

Figure 2. (Left) The separation of circle. (Right) The corresponding area of $g(u)$.

3.3. A tight solution

In this subsection, for some quadratic nonlinear dynamic systems, the semi-infinite optimization problem (11)–(12) may be equivalent to solving an SDP problem via sampling on all vertices of a polyhedron. Thus, we can obtain a tight bounding ellipsoid to cover the remainder.

We consider a quadratic nonlinear state equation:

$$f_k(\mathbf{x}_k) = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_n \end{bmatrix} \begin{bmatrix} (\mathbf{x}_k^1)^2 \\ (\mathbf{x}_k^2)^2 \\ \vdots \\ (\mathbf{x}_k^n)^2 \end{bmatrix} \tag{18}$$

where $\mathbf{x}_k \in \mathcal{R}^n$ is the state of system at time k, and $\mathbf{x}_k^i$ denoted the *ith* component of $\mathbf{x}_k$; α_i are known parameters, $i = 1, \dots, n$.

Proposition 2. *If we let the high-order remainder* $\Delta f_k(\mathbf{u}_k) = f(\widehat{\mathbf{x}}_k + \mathbf{E}_k \mathbf{u}_k) - f(\widehat{\mathbf{x}}_k) - \mathbf{J}_{f_k} \mathbf{E}_k \mathbf{u}_k$ *where* $\|\mathbf{u}_k\| \le 1$, *assume that* $f_k(\widehat{\mathbf{x}}_k)$ *is a quadratic function defined in* (18) *and* $\mathbf{E}_k$ *is a diagonal matrix; then, a tight bounding ellipsoid can be derived to cover the high-order remainder* $\Delta f_k(\mathbf{u}_k)$ *by solving the following optimization problem:*

$$\min \ f\left(\mathbf{P}_{f_k}\right) \tag{19}$$

$$\textit{subject to} \quad \left[\begin{array}{c|ccc} -1 & -\mathbf{e}^1_{f_k} & \cdots & -\mathbf{e}^n_{f_k} \\ \hline -\mathbf{e}^1_{f_k} & & & \\ \vdots & & & \\ -\mathbf{e}^2_{f_k} & & -\mathbf{P}_{f_k} & \end{array}\right] \preceq 0,$$

$$\left[\begin{array}{c|c} -1 & \begin{array}{c} -\mathbf{e}^1_{f_k} \dots \alpha_i (\mathbf{E}_k^{ii})^2 \dots - \mathbf{e}^n_{f_k} \\ - \mathbf{e}^i_{f_k} \end{array} \\ \hline -\mathbf{e}^1_{f_k} & \\ \vdots & \\ \alpha_i (\mathbf{E}_k^{ii})^2 - \mathbf{e}^i_{f_k} & \\ \vdots & \\ -\mathbf{e}^n_{f_k} & -\mathbf{P}_{f_k} \end{array}\right] \preceq 0, \tag{20}$$

$$\textit{for } i = 1, 2, \dots, n.$$

where $\mathbf{E}_k^{ii}$ is the *ith* row and *jth* column of $\mathbf{E}_k$.

In summary, we can determine the remainder bounding ellipsoid by sampling as follows.

1 For general nonlinear functions, samples may be taken from the sphere $\|\mathbf{u}_k\| \le 1$.

2 For a typical nonlinear dynamic system in target tracking or nonlinear functions, samples may be taken on the boundary of the sphere $\|\mathbf{u}_k\| \le 1$.

3 For some quadratic nonlinear functions, samples only need vertices of a polyhedron.

4. Ellipsoidal state bounding via SDP

In this section, we present the prediction step and the measurement step of the set-membership filtering by extending El Ghaoui and Calafiore's optimization method [26]. The point is that when the nonlinear dynamics are linearized on the current estimate, the uncertainties of the new linearized dynamic system include the uncertain bounding ellipsoids of the remainder terms and the noises.

4.1. Prediction step

Proposition 3. *At time $k+1$, based on measurements $\mathbf{y}_k$, the bounding ellipsoids of the state, and the remainders $\mathcal{E}_k$, $\mathcal{E}_{f_k}$, and $\mathcal{E}_{h_k}$, a predicted bounding ellipsoid $\mathcal{E}_{k+1|k} = \left\{\mathbf{x} : \left(\mathbf{x} - \widehat{\mathbf{x}}_{k+1|k}\right)^T \left(\mathbf{P}_{k+1|k}\right)^{-1} \left(\mathbf{x} - \widehat{\mathbf{x}}_{k+1|k}\right) \leq 1\right\}$ can be obtained by solving the optimization problem in the variables $\mathbf{P}_{k+1|k}$ and $\widehat{\mathbf{x}}_{k+1|k}$ and nonnegative scalars $\tau^u \geq 0, \tau^w \geq 0, \tau^v \geq 0, \tau^f \geq 0, \tau^h \geq 0$:*

$$\min\ f\left(\mathbf{P}_{k+1|k}\right) \tag{21}$$

$$\textit{subject to} \quad -\tau^u \leq 0,\ -\tau^w \leq 0,\ -\tau^v \leq 0, \tag{22}$$

$$-\tau^f \leq 0,\ -\tau^h \leq 0,\ -\mathbf{P}_{k+1|k} \prec 0, \tag{23}$$

$$\begin{bmatrix} -\mathbf{P}_{k+1|k} & \Phi_{k+1|k}\left(\Psi_{k+1|k}\right)_\perp \\ \left(\Phi_{k+1|k}\left(\Psi_{k+1|k}\right)_\perp\right)^T & -\left(\Psi_{k+1|k}\right)_\perp^T \Xi \left(\Psi_{k+1|k}\right)_\perp \end{bmatrix} \preccurlyeq 0, \tag{24}$$

where

$$\Phi_{k+1|k} = \left[f_k(\widehat{\mathbf{x}}_k) + \mathbf{e}_{f_k} - \widehat{\mathbf{x}}_{k+1|k}, \mathbf{J}_{f_k}\mathbf{E}_k, \mathbf{I}, 0, \mathbf{B}_{f_k}, 0\right], 0 \in \mathcal{R}^{n, n_1}, \tag{25}$$

$$\Psi_{k+1|k} = \left[h_k(\widehat{\mathbf{x}}_k) + \mathbf{e}_{h_k} - \mathbf{y}_k, \mathbf{J}_{h_k}\mathbf{E}_k, 0, \mathbf{I}, 0, \mathbf{B}_{h_k}\right]. \tag{26}$$

$\left(\Psi_{k+1|k}\right)_\perp$ is the orthogonal complement of $\Psi_{k+1|k}$. $\mathbf{E}_k$ is the Cholesky factorization of $\mathbf{P}_k$, i.e., $\mathbf{P}_k = \mathbf{E}_k(\mathbf{E}_k)^T$. $\mathbf{e}_{f_k}$, $\mathbf{e}_{h_k}$, $\mathbf{B}_{f_k}$, and $\mathbf{B}_{h_k}$ are denoted by (7) and (9), respectively. $\mathbf{J}_{f_k} = \frac{\partial f_k(\mathbf{x}_k)}{\partial \mathbf{x}}\Big|_{\widehat{\mathbf{x}}_k}$ and $\mathbf{J}_{h_k} = \frac{\partial h_k(\mathbf{x}_k)}{\partial \mathbf{x}}\Big|_{\widehat{\mathbf{x}}_k}$:

$$\Xi = \operatorname{diag}\left(1 - \tau^u - \tau^w - \tau^v - \tau^f - \tau^h, \tau^u I, \tau^w \mathbf{Q}_k^{-1}, \tau^v \mathbf{R}_k^{-1}, \tau^f I, \tau^h I\right). \tag{27}$$

Remark 5. *The objective function (21) is aimed at minimizing the shape matrix of the predicted ellipsoid, and the constraints (22)–(24) ensure that the true state is contained in predicted bounding ellipsoid $\mathcal{E}_{k+1|k}$. Notice that if $f(P) = tr(P)$, the optimization problems* (13)–(16), (21)–(24), and (28)–(31) *are SDP problems, which can be efficiently solved by modern interior-point methods* [27]. *According to the guidelines in* [28], *the computational complexity of solving an SDP problem is $O\left(\max(m,n)^4 n^{1/2} \log 1/\epsilon\right)$, where n is the number of the states. With the development of convex optimization technology technique, one can also use first-order optimizing algorithm. The computational complexity may be reduced further* (*see* [29]).

4.2. Measurement update step

Similarly, we can derive the measurement update step of the nonlinear filtering.

Proposition 4. *At time $k+1$, based on measurement $\mathbf{y}_{k+1}$, the predicted bounding ellipsoid $\mathcal{E}_{k+1|k}$, and the bounding ellipsoid of the remainder $\mathcal{E}_{h_{k+1}}$, an estimated bounding ellipsoid*

$\mathcal{E}_{k+1} = \left\{ \mathbf{x} : (\mathbf{x} - \widehat{\mathbf{x}}_{k+1})^T (\mathbf{P}_{k+1})^{-1} (\mathbf{x} - \widehat{\mathbf{x}}_{k+1}) \le 1 \right\}$ *can be obtained by solving the optimization problem in the variables* $\mathbf{P}_{k+1}$ *and* $\widehat{\mathbf{x}}_{k+1}$ *and nonnegative scalars* $\tau^u \ge 0$, $\tau^v \ge 0$, $\tau^h \ge 0$:

$$\min\ f(\mathbf{P}_{k+1}) \tag{28}$$

$$\textit{subject to} \quad -\tau^u \le 0,\ -\tau^v \le 0,\ -\tau^h \le 0, \tag{29}$$

$$-\mathbf{P}_{k+1} \prec 0, \tag{30}$$

$$\begin{bmatrix} -\mathbf{P}_{k+1} & \Phi_{k+1}(\Psi_{k+1})_\perp \\ \left(\Phi_{k+1}(\Psi_{k+1})_\perp\right)^T & -(\Psi_{k+1})_\perp^T \Xi (\Psi_{k+1})_\perp \end{bmatrix} \preccurlyeq 0, \tag{31}$$

where

$$\Phi_{k+1} = \left[\widehat{\mathbf{x}}_{k+1|k} - \widehat{\mathbf{x}}_{k+1}, \mathbf{E}_{k+1|k}, 0, 0\right],\quad 0 \in \mathcal{R}^{n, n_1}, \tag{32}$$

$$\Psi_{k+1} = \left[h_{k+1}\left(\widehat{\mathbf{x}}_{k+1|k}\right) + \mathbf{e}_{h_{k+1}} - \mathbf{y}_{k+1}, \mathbf{J}_{h_{k+1|k}} \mathbf{E}_{k+1|k}, \mathbf{I}, \mathbf{B}_{h_{k+1}}\right]. \tag{33}$$

$(\Psi_{k+1})_\perp$ *is the orthogonal complement of* Ψ_{k+1}. $\mathbf{E}_{k+1|k}$ *is the Cholesky factorization of* $\mathbf{P}_{k+1|k}$*, i.e.,* $\mathbf{P}_{k+1|k} = \mathbf{E}_{k+1|k}\left(\mathbf{E}_{k+1|k}\right)^T$. $\widehat{\mathbf{x}}_{k+1|k}$ *is the center of the predicted bounding ellipsoid* $\mathcal{E}_{k+1|k}$. $\mathbf{e}_{h_{k+1}}$ *and* $\mathbf{B}_{h_{k+1}}$ *are denoted by* (9) *at the time step* $k+1$. $\mathbf{J}_{h_{k+1|k}} = \frac{\partial h_{k+1}(\mathbf{x}_k)}{\partial \mathbf{x}}\Big|_{\widehat{\mathbf{x}}_{k+1|k}}$:

$$\Xi = \mathrm{diag}\left(1 - \tau^u - \tau^v - \tau^h, \tau^u \mathbf{I}, \tau^v \mathbf{R}_{k+1}^{-1}, \tau^h I\right). \tag{34}$$

4.3. Sampling-based ellipsoidal bounding filter algorithm

- Step 1: (Initialization step) Set $k = 0$ and initial values $(\widehat{\mathbf{x}}_0, \mathbf{P}_0)$ such that $\mathbf{x}_0 \in \mathcal{E}_0$.
- Step 2: (Bounding step) Take samples $\mathbf{u}_k^1, \dots, \mathbf{u}_k^N$ from the sphere $\|\mathbf{u}_k\| \le 1$, and then determine two bounding ellipsoids to cover the remainders Δf_k and Δh_k by (13)–(14) and (15)–(16), respectively.
- Step 3: (Prediction step) Optimize the center and shape matrix of the state prediction ellipsoid $\left(\widehat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k}\right)$ such that $\mathbf{x}_{k+1|k} \in \mathcal{E}_{k+1|k}$ by solving the optimization problem (21)–(24).
- Step 4: (Bounding step) Take samples $\mathbf{u}_{k+1|k}^1, \dots, \mathbf{u}_{k+1|k}^N$ from the sphere $\|\mathbf{u}_{k+1|k}\| \le 1$, and then determine one bounding ellipsoid to cover the remainder Δh_{k+1} by (15)–(16).
- Step 5: (Measurement update step) Optimize the center and shape matrix of the state estimation ellipsoid $(\widehat{\mathbf{x}}_{k+1}, \mathbf{P}_{k+1})$ such that $\mathbf{x}_{k+1} \in \mathcal{E}_{k+1}$ by solving the optimization problem (28)–(31).
- Step 6: Set $k = k + 1$ and go to Step 2.

5. Numerical example in target tracking

In this section, we compare the performance between the proposed set-membership filter and extended set-membership filter (ESMF) [19], which can also be implemented online for target tracking with a nonlinear dynamic system, when the state noises and measurement noises are unknown but bounded.

By considering a two-dimensional Cartesian, coordinate system as follows [1]:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k + \mathbf{w}_k, \tag{35}$$

$$\mathbf{y}_k = \begin{bmatrix} \sqrt{(\mathbf{x}_k(1))^2 + (\mathbf{x}_k(2))^2} \\ \arctan\left(\dfrac{\mathbf{x}_k(2)}{\mathbf{x}_k(1)}\right) \end{bmatrix} + \mathbf{v}_k, \tag{36}$$

$\mathbf{x}$ is a four-dimensional state variable that includes position and velocity $[x, y, \dot{x}, \dot{y}]^T$, and $T = 1$ s is the time sampling interval. The process noise and measurement noise assumed to be confined to specified ellipsoidal sets:

$$\mathbf{W}_k = \left\{ \mathbf{w}_k : \mathbf{w}_k^T \mathbf{Q}_k^{-1} \mathbf{w}_k \leq 1 \right\}$$
$$\mathbf{V}_k = \left\{ \mathbf{v}_k : \mathbf{v}_k^T \mathbf{R}_k^{-1} \mathbf{v}_k \leq 1 \right\}.$$

where

$$\mathbf{Q}_k = \sigma^2 \begin{bmatrix} \dfrac{T^3}{3} & 0 & \dfrac{T^2}{2} & 0 \\ 0 & \dfrac{T^3}{3} & 0 & \dfrac{T^2}{2} \\ \dfrac{T^2}{2} & 0 & T & 0 \\ 0 & \dfrac{T^2}{2} & 0 & T \end{bmatrix}, \mathbf{R}_k = q \cdot \begin{bmatrix} 3^2 & 0 \\ 0 & 1^2 \end{bmatrix}.$$

The target acceleration σ^2 equals to 10. The parameter q is used to control the uncertainty of the measurement noise. In the example, the target starts at the point $(50, 30)$ with a velocity of $(5, 5)$. The center and the shape matrix of the initial bounding ellipsoid are $\widehat{\mathbf{x}}_0 = [49.5 \ \ 29.5 \ \ 5 \ \ 5]^T$ and $\mathbf{P}_0 = \mathrm{diag}([5, 5, 2, 2])$, respectively.

The following simulation results include three parts: the first part is about the size of the remainder bounding ellipsoid, the second part is about the root-mean-square error (RMSE) of the state estimation, and the third part is about the computation time. They are illustrated and

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discussed by the number of samples, the time steps, and the uncertain parameter q of the measurement noise, respectively:

• In **Figure 3**, the log size of the remainder bounding ellipsoid is plotted as a function of the time steps with the uncertain parameter $q = 0.01$. It shows that the size of the new method is much smaller than that of the ESMF, i.e., the new method derives a tighter ellipsoid to cover the remainder. Moreover, we use 30 samples to calculate a remainder bounding ellipsoid on a time step based on solving the optimization problem (15)–(16). The corresponding bounding ellipsoid is presented in **Figure 4**. It shows that the bounding ellipsoid can cover all points of the remainder set with a very small size. In **Figures 5** and **6**, the average size of the remainder bounding ellipsoid through the time steps 1–20 is plotted as a function of the uncertain parameter q of the measurement noise. The larger q means that the measurement noise is more uncertain. Thus, **Figures 5** and **6** show that when the uncertainty of the measurement noise is increasing, the size of the remainder bounding ellipsoid of ESMF is quickly increasing; however, that of the new method is slowly increasing and relatively stable.

• RMSE of the state estimation along the position direction is plotted as a function of the time steps in **Figure 7**. It shows that RMSE of the new method is less than that of ESMF. The reason may be that the new method derives a tighter ellipsoid to cover the remainder, which can be seen in the **Figure 4**. **Figure 7** also shows that RMSEs of the proposed filter based on 30 and 40 samples are almost same. The reason may be that the remainder bounding ellipsoid is same when the number of samples is more than 30. In **Figure 8**, the average RMSE of the state estimation through the time steps 1 to 20 is plotted as a function of the uncertain parameter q. It

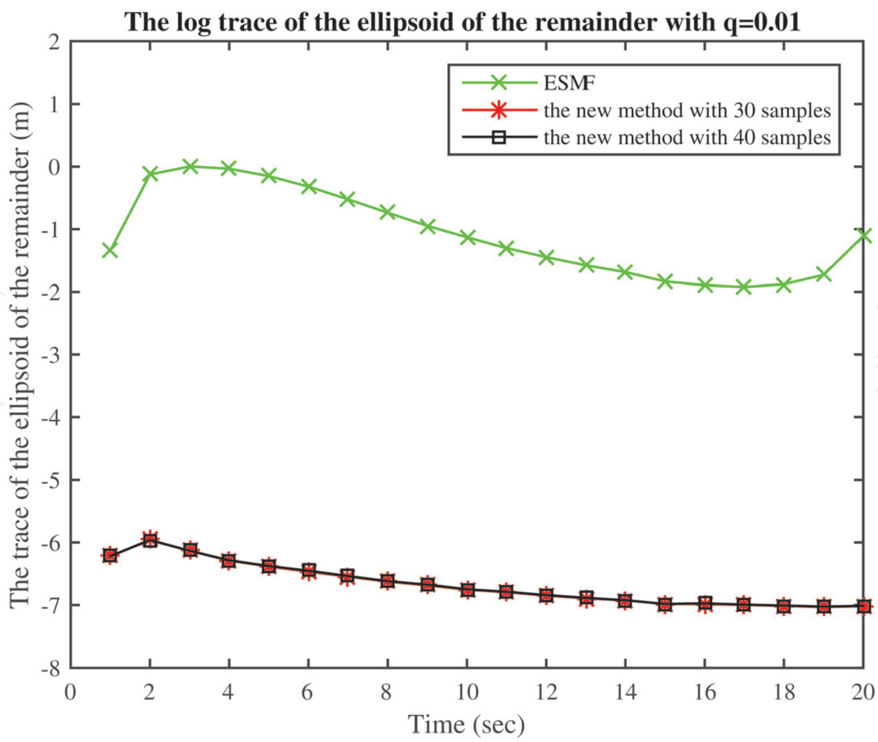

Figure 3. The log size of the remainder bounding ellipsoid is plotted as a function of time steps with $q = 0.01$.

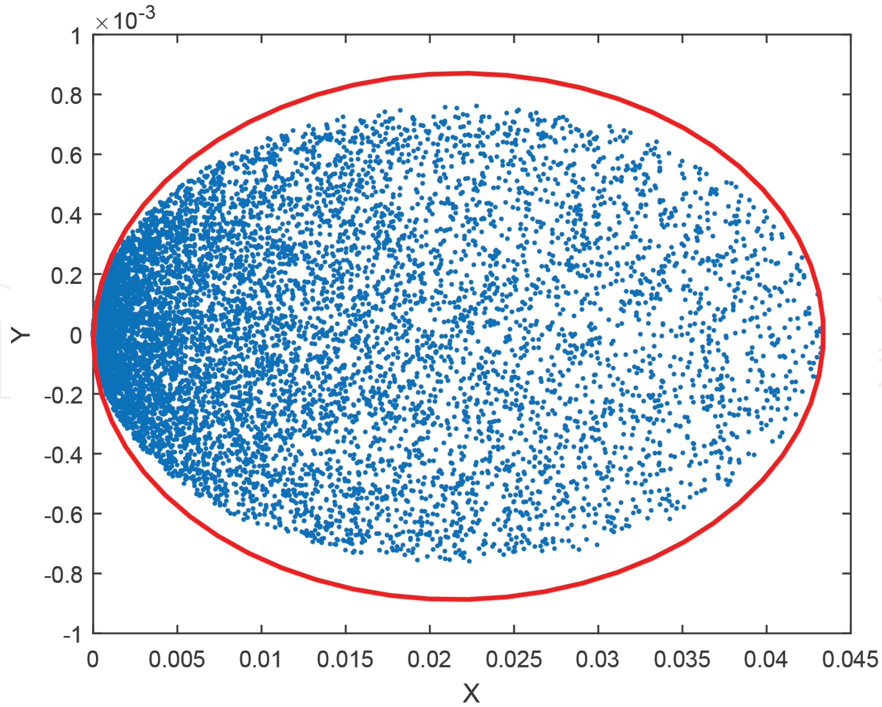

Figure 4. The remainder bounding ellipsoid on a time step based on 30 samples from the boundary.

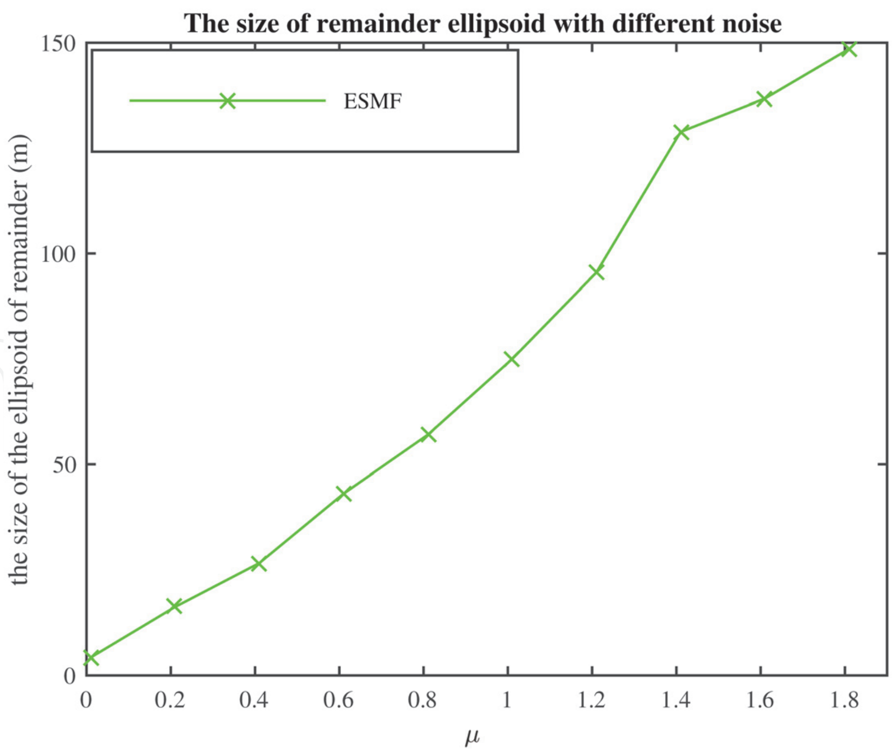

Figure 5. The average size of the remainder bounding ellipsoid is plotted as a function of the uncertain parameter q by ESMF.

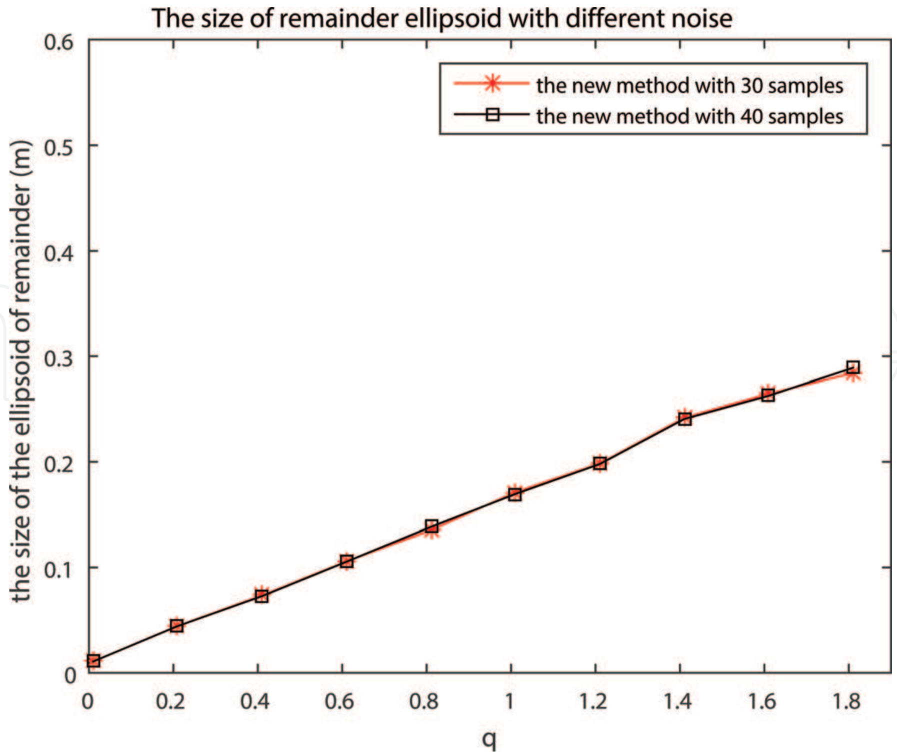

Figure 6. The average size of the remainder bounding ellipsoid is plotted as a function of the uncertain parameter q by the new method.

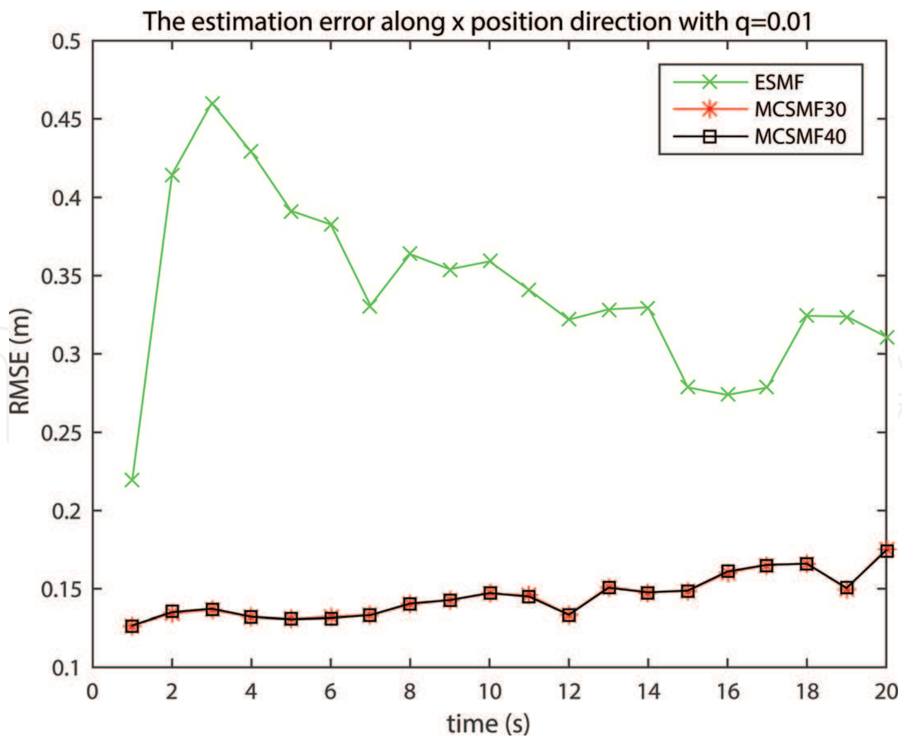

Figure 7. The RMSE of the state estimation is plotted as a function of time steps with $q = 0.01$.

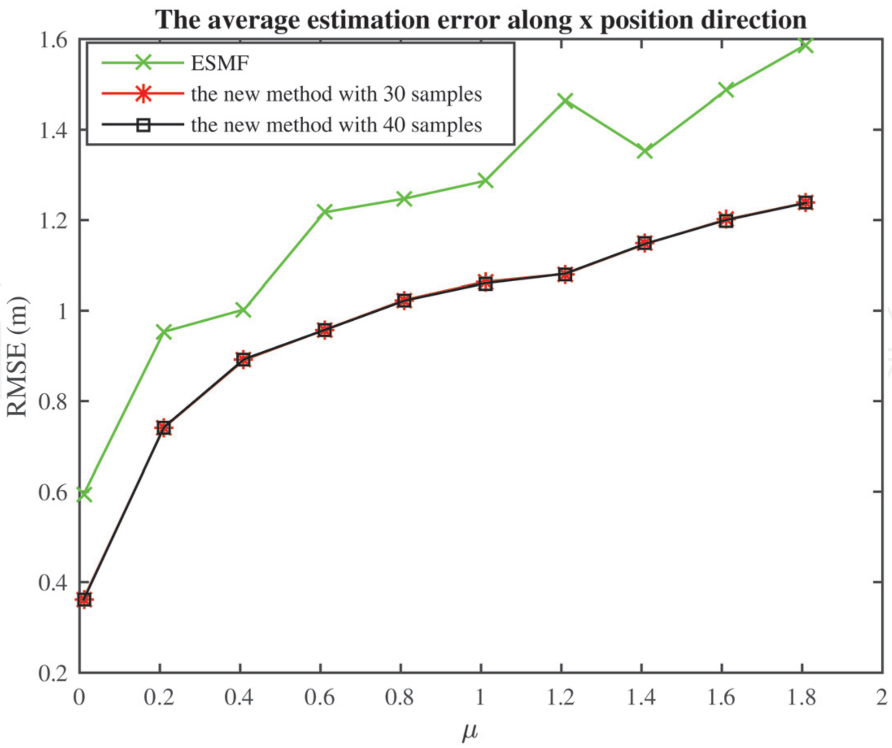

Figure 8. The average RMSE of the state estimation through the time steps 1 to 20 is plotted as a function of the uncertain parameter q.

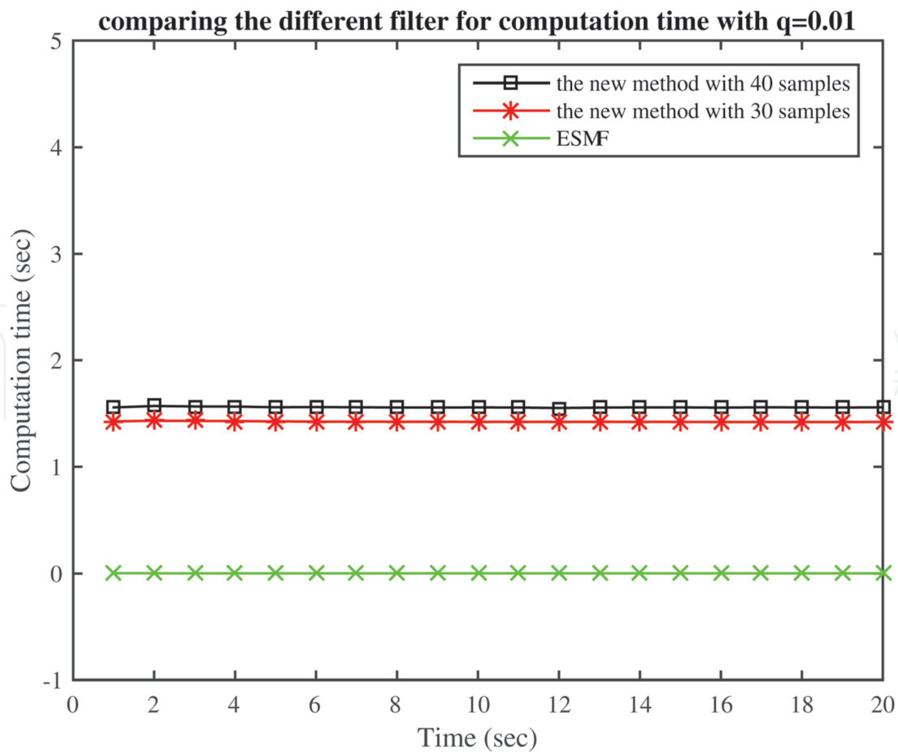

Figure 9. The computation time of the proposed state bounding filter and ESMF at each time step.

shows that the average RMSE of the state estimation based on the new method is also less than that of ESMF. The larger uncertain parameter q is a better performance of the new method than that of ESMF. In summary, **Figures 5–8** indicate that the new method performs much better than ESMF, especially in the situation of the larger noise.

- Since the predictive step and measurement update step of the new method are calculated by solving an SDP, the computation time of the new method is greater than that of ESMF, which can be seen in the right of **Figure 9**, but it may be tolerated and be done in polynomial time.

6. Conclusion

In order to deal with the nonlinear dynamic systems with unknown but bounded noise, we have proposed a new filtering method via set-membership theory and boundary sampling technique to determine a state estimation ellipsoid. To guarantee the online usage, the nonlinear dynamics are linearized about the current estimate, and the remainder terms are then bounded by an ellipsoid, which can be written as the solution of a semi-infinite optimization problem. For a typical nonlinear dynamic system in target tracking, the semi-infinite optimization problem can be efficiently approximated by a randomized method. Moreover, for some quadratic nonlinear dynamic systems, using the samples on all vertices of a polyhedron, we obtain a tight bounding ellipsoid, which covers the remainder by solving an SDP problem. Finally, the set-membership prediction and measurement update are derived based on the recent optimization method and the online bounding ellipsoid of the remainder other than a priori bound, so that a tighter set-membership filter can be achieved. Numerical example shows that the proposed method performs much better than ESMF, especially in the situation of the larger noise. Future work will include that the multisensor fusion, multiple target tracking, and various applications such as sensor management and placement for structures and different types of wireless networks.

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References

[1] Bar-Shalom Y, Li X, Kirubarajan T. Estimation with Applications to Tracking and Navigation. New York: Wiley; 2001

[2] Abdallah F, Gning A, Bonnifait P. Box particle filtering for nonlinear state estimation using interval analysis. Automatica. 2008;**44**:807-815

[3] Gning A, Ristic B, Mihaylova L, Abdallah F. An introduction to box particle filtering. IEEE Signal Processing Magazine. 2013;**30**:166-171

[4] Freitas AD, Mihaylova L, Gning A, Angelova D, Kadirkamanathan V. Autonomous crowds tracking with box particle filtering and convolution particle filtering. Automatica. 2016;**69**: 380-394

[5] Merlinge N, Dahia K, Piet-Lahanier H. A box regularized particle filter for terrain navigation with highly non-linear measurements. IFAC-Papers OnLine. 2016;**49**:361-366

[6] Polyak BT, Nazin SA, Durieu C, Walter E. Ellipsoidal parameter or state estimation under model uncertainty. Automatica. 2004;**40**:1171-1179

[7] Schweppe FC. Recursive state estimation: Unknown but bounded errors and system inputs. IEEE Transactions on Automatic Control. 1968;**13**:22-28

[8] Jaulin L. Nonlinear bounded-error state estimation of continuous-time systems. Automatica. 2002;**38**:1079-1082

[9] Rohou S, Jaulin L, Mihaylova L, Bars FL. Guaranteed computation of robot trajectories. Robotics and Autonomous Systems. 2017;**97**:76-84

[10] Calafiore G, El Ghaoui L. Ellipsoidal bounds for uncertain equations and dynamical systems. Automatica. 2004;**40**:773-787

[11] Shen X, Zhu Y, Song E, Luo Y. Minimizing Euclidian state estimation error for linear uncertain dynamic systems based on multisensor and multi-algorithm fusion. IEEE Transactions on Information Theroy. 2011;**57**:7131-7146

[12] Durieu C, Walter E, Polyak BT. Multi-input multi-output ellipsoidal state bounding. Journal of Optimization Theory and Applications. 2001;**111**:273-303

[13] Shamma JS, Tu K. Approximate set-valued observers for nonlinear systems. IEEE Transactions on Automatic Control. 1997;**42**:648-658

[14] Morrell DR, Stirlling WC. An extended set-valued Kalman filter. Proceeding of ISIPTA. 2003:396-407

[15] Wei G, Wang Z, Shen B. Error-constrained filtering for a class of nonlinear time-varying delay systems with non-Gaussian noises. IEEE Transaction on Automatic Control. 2010; **55**:2876-2882

[16] Chachuat B, Houska B, Paulen R, Perić N, Rajyaguru J, Villanueva ME. Set-theoretic approaches in analysis, estimation and control of nonlinear systems. IFAC-PapersOnLine. 2015;**48**:981-998

[17] Noack B, Klumpp V, Hanebeck UD. State estimation with sets of densities considering stochastic and systematic errors. In: Proceedings of the IEEE International Conference on Information Fusion; 2009

[18] Noack B, Klumpp V, Petkov N, Hanebeck UD. Bounding linearization errors with sets of densities in approximate Kalman filtering. Proceedings of the IEEE International Conference on Information Fusion. 2010

[19] Scholte E, Campbell ME. A nonlinear set-membership filter for on-line applications. International Journal of Robust and Nonlinear Control. 2003;**13**:1337-1358

[20] Calafiore G. Reliable localization using set-valued nonlinear filters. IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans. 2005;**35**:189-197

[21] Boyd S, Vandenberghe L. Convex Optimization. Cambridge, United Kingdom: Cambridge University Press; 2004

[22] Ben-Tal A, Nemirovski A. Robust convex optimization. Mathematics of Operations Research. 1998;**23**:769-805

[23] Still G. Discretization in semi-infinite programming: The rate of convergence. Mathematical programming. 2001;**91**:53-69

[24] Calafiore G, Campi M. Uncertain convex programs: Randomized solutions and confidence levels. Mathematical Programming. 2005;**102**:25-46

[25] Spivak M. Calculus on manifolds. New York: Benjamin; 1965

[26] El Ghaoui L, Calafiore G. Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. IEEE Transactions on Automatic Control. 2001;**36**:1084-1089

[27] Nesterov Y, Nemirovski A. Interior point polynomial methods in convex programming: Theorey and applications. Philadelphia, PA: SIAM;1994

[28] Vandenberghe L, Boyd S. Semidefinite programming. SIAM Review. 1996;**38**:49-95

[29] Lan G, Lu Z, Monteiro RD. Primal-dual first-order methods with $\mathcal{O}(1/\epsilon)$ iteration-complexity for cone programming. Mathematical Programming. 2011;**126**:1-29