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Linear Lyapunov Cone-Systems

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1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs (Farina L. & Rinaldi S., 2000; Kaczorek T., 2001). The realization problem for positive linear systems without and with time delays has been considered in (Benvenuti L. & Farina L., 2004; Farina L. & Rinaldi S., 2000; Kaczorek T., 2004a; Kaczorek T., 2006a; Kaczorek T., 2006b; Kaczorek T. & Busłowicz M., 2004a).

The reachability, controllability to zero and observability of dynamical systems have been considered in (Klamka J., 1991). The reachability and minimum energy control of positive linear discrete-time systems have been investigated in (Busłowicz M. & Kaczorek T., 2004). The positive discrete-time systems with delays have been considered in (Kaczorek T., 2004b; Kaczorek T. & Busłowicz M., 2004b; Kaczorek T. & Busłowicz M., 2004c). The controllability and observability of Lyapunov systems have been investigated by Murty Apparao in the paper (Murty M.S.N. & Apparao B.V., 2005). The positive discrete-time and continuous-time Lyapunov systems have been considered in (Kaczorek T., 2007b; Kaczorek T. & Przyborowski P., 2007a; Kaczorek T. & Przyborowski P., 2008; Kaczorek T. & Przyborowski P., 2007e). The positive linear time-varying Lyapunov systems have been investigated in (Kaczorek T. & Przyborowski P., 2007b). The continuous-time Lyapunov cone systems have been considered in (Kaczorek T. & Przyborowski P., 2007c). The positive discrete-time Lyapunov systems with delays have been investigated in (Kaczorek T. & Przyborowski P., 2007d).

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century (Nishimoto K., 1984; Miller K. S. & Ross B., 1993; Podlubny I., 1999). This idea by engineers has been used for modelling different process in the late 1960s (Bologna M. & Grigolini P., 2003; Vinagre B. M. et al., 2002; Vinagre B. M. & Feliu V., 2002; Zaborowsky V. & Meylanov R., 2001). Mathematical fundamentals of fractional calculus are given in the monographs (Miller K. S. & Ross B., 1993; Nishimoto K., 1984; Oldham K. B. &

Spanier J, 1974; Podlubny I., 1999; Oustaloup A., 1995). The fractional order controllers have been developed in (Oldham K. B. & Spanier J., 1974; Oustaloup A., 1993; Podlubny I., 2002). A generalization of the Kalman filter for fractional order systems has been proposed in (Sierociuk D. & Dzieliński D., 2006). Some others applications of fractional order systems can be found in (Ostalczyk P., 2000; Ostalczyk P., 2004a; Ostalczyk P., 2004b; Ferreira N.M.F. & Machado I.A.T., 2003; Gałkowski K., 2005; Moshrefi-Torbati M. & Hammond K., 1998; Reyes-Melo M.E. et al., 2004; Riu D. et al., 2001; Samko S. G. et al., 1993; Dzieliński A. & Sierociuk D., 2006). In (Ortigueira M. D., 1997) a method for computation of the impulse responses from the frequency responses for the fractional standard (non-positive) discrete-time linear systems is proposed. The reachability and controllability to zero of positive fractional systems has been considered in (Kaczorek T., 2007c; Kaczorek T., 2007d). The reachability and controllability to zero of fractional cone-systems has been considered in (Kaczorek T., 2007e). The fractional discrete-time Lyapunov systems has been investigated in (Przyborowski P., 2008a) and the fractional discrete-time cone-systems in (Przyborowski P., 2008b).

The chapter is organized as follows, In the Section 2, some basic notations, definitions and lemmas will be recalled. In the Section 3, the continuous-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability and sufficient conditions for the reachability and observability will be established. In the Section 4, the discrete-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability, reachability, observability and controllability to zero will be established. In the Section 5, the fractional discrete-time linear Lyapunov cone-systems will be considered. For the systems, the necessary and sufficient conditions for being the cone-system, the reachability, observability and controllability to zero and sufficient conditions for the stability will be established. In the Section 6, the considerations will be illustrated by numerical examples.

2. Preliminaries

Let $R^{n \times m}$ be the set of real $n \times m$ matrices, $R^n = R^{n \times 1}$ and let $R_+^{n \times m}$ be the set of real $n \times m$ matrices with nonnegative entries. The set of nonnegative integers will be denoted by Z_+ .

Definition 1.

The Kronecker product $A \otimes B$ of the matrices $A = [a_{ij}] \in R^{m \times n}$ and $B \in R^{p \times q}$ is the block matrix (Kaczorek T., 1998):

$$A \otimes B = [a_{ij} B]_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \in R^{mp \times nq} \quad (1)$$

Lemma 1.

Let us consider the equation:

$$AXB = C \quad (2)$$

where: $A \in R^{m \times n}$, $B \in R^{q \times p}$, $C \in R^{m \times p}$, $X \in R^{n \times q}$

Equation (2) is equivalent to the following one:

$$(A \otimes B^T)x = c \quad (3)$$

where $x := [x_1 \ x_2 \ \dots \ x_n]^T$, $c := [c_1 \ c_2 \ \dots \ c_m]^T$, and x_i and c_i are the i th rows of the matrices X and C respectively.

Proof: See (Kaczorek T., 1998)

Lemma 2.

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A and $\mu_1, \mu_2, \dots, \mu_n$ the eigenvalues of the matrix B , then $\lambda_i + \mu_j$ for $i, j = 1, 2, \dots, n$ are the eigenvalues of the matrix:

$$\bar{A} = A \otimes I_n + I_n \otimes B^T$$

Proof: See (Kaczorek T., 1998)

Definition 2.

Let $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in R^{n \times n}$ be nonsingular and p_k be the k th ($k = 1, \dots, n$) its row. The set:

$$P := \left\{ X(t) \in R^{n \times n} : \bigcap_{k=1}^n p_k X_i(t) \geq 0 \right\} \quad (4)$$

where $X_i(t)$, $i = 1, \dots, n$ is the i th column of the matrix $X(t)$, is called a linear cone of the state variables generated by the matrix P . In the similar way we may define the linear cone of the inputs:

$$Q := \left\{ U(t) \in R^{m \times n} : \bigcap_{k=1}^m q_k U_i(t) \geq 0 \right\} \quad (5)$$

generated by the nonsingular matrix $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in R^{m \times m}$ and the linear cone of the outputs

$$V := \left\{ Y(t) \in R^{p \times n} : \bigcap_{k=1}^p v_k Y_i(t) \geq 0 \right\} \quad (6)$$

generated by the nonsingular matrix $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in R^{p \times p}$.

3. Continuous-time linear Lyapunov cone-systems

Consider the continuous-time linear Lyapunov system (Kaczorek T. & Przyborowski P., 2007a) described by the equations:

$$\dot{X}(t) = A_0 X(t) + X(t) A_1 + BU(t) \quad (7a)$$

$$Y(t) = CX(t) + DU(t) \quad (7b)$$

where, $X(t) \in R^{n \times n}$ is the state-space matrix, $U(t) \in R^{m \times n}$ is the input matrix, $Y(t) \in R^{p \times n}$ is the output matrix, $A_0, A_1 \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$.

The solution of the equation (1a) satisfying the initial condition $X(t_0) = X_0$ is given by (Kaczorek T. & Przyborowski P., 2007a):

$$X(t) = e^{A_0(t-t_0)} X_0 e^{A_1(t-t_0)} + \int_{t_0}^t e^{A_0(t-\tau)} BU(\tau) e^{A_1(t-\tau)} d\tau \quad (8)$$

Lemma 3.

The Lyapunov system (7) can be transformed to the equivalent standard continuous-time, nm -inputs and pn -outputs, linear system in the form:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \quad (9a)$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u}(t) \quad (9b)$$

where, $\tilde{x}(t) \in R^{n^2}$ is the state-space vector, $\tilde{u}(t) \in R^{(nm)}$ is the input vector, $\tilde{y}(t) \in R^{(pn)}$ is the output vector, $\tilde{A} \in R^{n^2 \times n^2}$, $\tilde{B} \in R^{n^2 \times (nm)}$, $\tilde{C} \in R^{(pn) \times n^2}$, $\tilde{D} \in R^{(pn) \times (nm)}$.

Proof:

The transformation is based on Lemma 1. The matrices X, U, Y are transformed to the vectors:

$$\tilde{x} = [X_1 \quad X_2 \quad \dots \quad X_n]^T, \tilde{u} = [U_1 \quad U_2 \quad \dots \quad U_m]^T, \tilde{y} = [Y_1 \quad Y_2 \quad \dots \quad Y_p]^T$$

where X_i, U_i, Y_i denotes the i th rows of the matrices X, U, Y , respectively.

The matrices of (9) are:

$$\tilde{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \tilde{B} = B \otimes I_n, \tilde{C} = C \otimes I_n, \tilde{D} = D \otimes I_n \quad (10)$$

3.1 Cone-systems

Definition 3.

The Lyapunov system (7) is called (P,Q,V)-cone-system if $X(t) \in P$ and $Y(t) \in V$ for every $X_0 \in P$ and for every input $U(t) \in Q$, $t \geq t_0$.

Note that for $P = R_+^{n \times n}$, $Q = R_+^{m \times n}$, $V = R_+^{p \times n}$ we obtain $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$ -cone system which is equivalent to the positive Lyapunov system (Kaczorek T. & Przyborowski P., 2007c).

Theorem 1.

The Lyapunov system (7) is (P, Q, V) -cone-system if and only if :

$$\hat{A}_0 = PA_0P^{-1}, \hat{A}_1 = A_1 \quad (11)$$

are the Metzler matrices and

$$\hat{B} = PBQ^{-1} \in R_+^{n \times m}, \hat{C} = VCP^{-1} \in R_+^{p \times n}, \hat{D} = VDQ^{-1} \in R_+^{p \times m}. \quad (12)$$

Proof:

Let:

$$\hat{X}(t) = PX(t), \hat{U}(t) = QU(t), \hat{Y}(t) = VY(t) \quad (13)$$

From definition 2 it follows that if $X(t) \in P$ then $\hat{X}(t) \in R_+^{n \times n}$, if $U(t) \in Q$ then $\hat{U}(t) \in R_+^{m \times n}$, and if $Y(t) \in V$ then $\hat{Y}(t) \in R_+^{p \times n}$.

From (7) and (13) we have:

$$\begin{aligned} \dot{\hat{X}}(t) &= P\dot{X}(t) = PA_0X(t) + PX(t)A_1 + PBU(t) = PA_0P^{-1}\hat{X}(t) + PP^{-1}\hat{X}(t)A_1 + \\ &+ PBQ^{-1}\hat{U}(t) = \hat{A}_0\hat{X}(t) + \hat{X}(t)A_1 + \hat{B}\hat{U}(t) \end{aligned} \quad (14a)$$

and

$$\hat{Y}(t) = VY(t) = VCX(t) + VDU(t) = VCP^{-1}\hat{X}(t) + VDQ^{-1}\hat{U}(t) = \hat{C}\hat{X}(t) + \hat{D}\hat{U}(t) \quad (14b)$$

It is known (Kaczorek T. & Przyborowski P., 2007a) that the system (14) is positive if and only if the conditions (11) and (12) are satisfied. $\square$

3.2 Asymptotic stability

Consider the autonomous Lyapunov (P, Q, V) -cone-system:

$$\dot{X}(t) = A_0X(t) + X(t)A_1, \quad X(t_0) = X_0 \quad (15)$$

where, $X(t) \in P$ and $PA_0P^{-1}, A_1 \in R^{n \times n}$ are the Metzler matrices.

Definition 4.

The Lyapunov (P, Q, V) -cone-system (15) is called asymptotically stable if:

$$\lim_{t \rightarrow \infty} X(t) = 0 \text{ for every } X_0 \in P$$

Theorem 2.

Let us assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A_0 and $\mu_1, \mu_2, \dots, \mu_n$ the eigenvalues of the matrix A_1 . The system (15) is stable if and only if:

$$\operatorname{Re}(\lambda_i + \mu_j) < 0 \text{ for } i, j = 1, 2, \dots, n \quad (16)$$

Proof:

The theorem results directly from the theorem for asymptotic stability of standard systems (Kaczorek T., 2001), since by Lemma 2 eigenvalues of matrix $\tilde{A}$ are the sums of eigenvalues of the matrices A_0 and A_1 . $\square$

3.3 Reachability**Definition 5.**

The state $X_f \in \mathbf{P}$ of the the Lyapunov $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system (7) is called reachable at time t_f , if there exists an input $U(t) \in \mathbf{Q}$ for $t \in [t_0, t_f]$, which steers the system from the initial state $X_0 = 0$ to the state X_f .

Definition 6.

If for every state $X_f \in \mathbf{P}$ there exists $t_f > t_0$, such that the state is reachable at time t_f , then the system is called reachable.

Theorem 3.

The $(\mathbf{P}, \mathbf{Q}, \mathbf{V})$ -cone-system (7) is reachable if the matrix:

$$R_f := \int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} d\tau \quad (17)$$

is a monomial matrix (only one element in every row and in every column of the matrix is positive and the remaining are equal to zero).

The input, that steers the system from initial state $X_0 = 0$ to the state X_f is given by:

$$U(t) = Q^{-1}[(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-t)} R_f^{-1} P X_f e^{A_1(t-t_f)}] \quad (18)$$

for $t \in [t_0, t_f]$.

Proof:

If R_f is the monomial matrix, then there exists $R_f^{-1} \in R_+^{n \times n}$ and the input (18) is well-defined.

Using (8) and (18) we obtain:

$$\begin{aligned}
X(t_f) &= P^{-1} \left[\int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} R_f^{-1} P X_f e^{A_1(\tau-t_f)} e^{A_1(t_f-\tau)} d\tau \right] = \\
&= P^{-1} \left[\int_{t_0}^{t_f} e^{PA_0P^{-1}(t_f-\tau)} (PBQ^{-1})(PBQ^{-1})^T e^{(PA_0P^{-1})^T(t_f-\tau)} d\tau R_f^{-1} P X_f \right] = P^{-1} P X_f = X_f
\end{aligned} \tag{19}$$

since $e^{A_1(\tau-t_f)} e^{A_1(t_f-\tau)} = I_n$. $\square$

3.4 Dual Lyapunov cone-systems

Definition 7.

The Lyapunov system described by the equations:

$$\dot{X}(t) = A_0^T X(t) + X(t) A_0^T + C^T U(t) \tag{20a}$$

$$Y(t) = B^T X(t) + D U(t) \tag{20b}$$

is called the dual system with respect to the system (7). The matrices $A_0, A_1, B, C, D, X(t), U(t), Y(t)$ are the same as in the system (7).

3.5 Observability

Definition 8.

The state X_0 of the Lyapunov (P, Q, V) -cone- system (7) is called observable at time $t_f > 0$, if X_0 can be uniquely determined from the knowledge of the output $Y(t)$ and input $U(t)$ for $t \in [0, t_f]$.

Definition 9.

The Lyapunov (P, Q, V) -cone- system (7) is called observable, if there exists an instant $t_f > 0$, such that the system (7) is observable at time t_f .

Theorem 4.

The Lyapunov (P, Q, V) -cone-system (7) is observable if the dual system (20) is reachable i.e. if the matrix:

$$O_f := \int_{t_0}^{t_f} e^{(PA_0P^{-1})^T(t_f-\tau)} (VCP^{-1})^T (VCP^{-1}) e^{(PA_0P^{-1})(t_f-\tau)} d\tau \tag{21}$$

is a monomial matrix.

Proof:

The Lyapunov (P, Q, V) -cone-system (7) is observable if and only if the equivalent standard system (9) is observable and this implies that dual system with respect to the system (9) must be reachable thus the dual system (20) with respect to the system (7) also must be reachable. Using Theorem 3. we obtain the hypothesis of the Theorem 4. $\square$

4. Discrete-time linear Lyapunov cone-systems

Consider the discrete-time linear Lyapunov system (Kaczorek T., 2007b; Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) described by the equations:

$$X_{i+1} = A_0 X_i + X_i A_1 + B U_i \quad (22a)$$

$$Y_i = C X_i + D U_i \quad (22b)$$

where, $X_i \in R^{n \times n}$ is the state-space matrix, $U_i \in R^{m \times n}$ is the input matrix, $Y_i \in R^{p \times n}$ is the output matrix, $A_0, A_1 \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$, $i \in Z_+$.

The solution of the equation (22a) satisfying the initial condition X_0 is given by (Kaczorek T., 2007b):

$$X_i = \sum_{k=0}^i \frac{i!}{k!(i-k)!} A_0^k X_0 A_1^{i-k} + \sum_{j=0}^{i-1} \sum_{k=0}^j \frac{j!}{k!(j-k)!} A_0^k B U_{i-j-1} A_1^{i-k}, i \in Z_+ \quad (23)$$

Lemma 4.

The Lyapunov system (22) can be transformed to the equivalent standard discrete-time, nm -inputs and pn -outputs, linear system in the form:

$$\bar{x}_{i+1} = \bar{A} \bar{x}_i + \bar{B} u_i \quad (24a)$$

$$\bar{y}_i = \bar{C} \bar{x}_i + \bar{D} u_i \quad (24b)$$

where, $\bar{x}_i \in R^{n^2}$ is the state-space vector, $\bar{u}_i \in R^{(nm)}$ is the input vector, $\bar{y}_i \in R^{(pn)}$ is the output vector, $\bar{A} \in R^{n^2 \times n^2}$, $\bar{B} \in R^{n^2 \times (nm)}$, $\bar{C} \in R^{(pn) \times n^2}$, $\bar{D} \in R^{(pn) \times (nm)}$, $i \in Z_+$.

Proof:

The proof is similar to the one of Lemma 3.

The matrices of (24) have the form:

$$\bar{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \bar{B} = B \otimes I_n, \bar{C} = C \otimes I_n, \bar{D} = D \otimes I_n \quad (25)$$

4.1 Cone-systems

Definition 10.

The Lyapunov system (22) is called (P, Q, V) -cone-system if $X_i \in P$ and $Y_i \in V$ for every $X_0 \in P$ and for every input $U_i \in Q$, $i \in Z_+$.

Note that for $P = R_+^{n \times n}$, $Q = R_+^{m \times n}$, $V = R_+^{p \times n}$ we obtain $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$ -cone system which is equivalent to the positive Lyapunov system (Kaczorek T., 2007b).

Theorem 5.

The Lyapunov system (22) is (P, Q, V) -cone-system if and only if :

$$\hat{A}_0 = PA_0P^{-1} = \left[\hat{a}_{ij}^0 \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}, \hat{A}_1 = A_1 = \left[\hat{a}_{ij}^1 \right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad (26)$$

are the Metzler matrices satisfying

$$\hat{a}_{kk}^0 + \hat{a}_{ll}^1 \geq 0 \text{ for every } k, l = 1, \dots, n \quad (27)$$

and

$$\hat{B} = PBQ^{-1} \in R_+^{n \times m}, \hat{C} = VCP^{-1} \in R_+^{p \times n}, \hat{D} = VDQ^{-1} \in R_+^{p \times m} \quad (28)$$

Proof:

Let:

$$\hat{X}_i = PX_i, \hat{U}_i = QU_i, \hat{Y}_i = VY_i \quad (29)$$

From definition 2 it follows that if $X_i \in P$ then $\hat{X}_i \in R_+^{n \times n}$, if $U_i \in Q$ then $\hat{U}_i \in R_+^{m \times n}$, and if $Y_i \in V$ then $\hat{Y}_i \in R_+^{p \times n}$.

From (22) and (29) we have:

$$\begin{aligned} \hat{X}_{i+1} &= PX_{i+1} = PA_0X_i + PX_iA_1 + PBQ^{-1}QU_i = PA_0P^{-1}\hat{X}_i + PP^{-1}\hat{X}_iA_1 + \\ &+ PBQ^{-1}\hat{U}_i = \hat{A}_0\hat{X}_i + \hat{X}_iA_1 + \hat{B}\hat{U}_i \end{aligned} \quad (30a)$$

and

$$\hat{Y}_i = VY_i = VCP^{-1}\hat{X}_i + VDQ^{-1}\hat{U}_i = \hat{C}\hat{X}_i + \hat{D}\hat{U}_i \quad (30b)$$

The Lyapunov system (30) is positive if and only if, the equivalent standard system is positive. By the theorem for the positivity of the standard discrete-time systems, the matrices $(\hat{A}_0 \otimes I_n + I_n \otimes \hat{A}_1^T), (\hat{B} \otimes I_n), (\hat{C} \otimes I_n), (\hat{D} \otimes I_n)$ have to be the matrices with nonnegative entries, so from (30) follows the hypothesis of the Theorem 5. $\square$

4.2 Asymptotic stability

Consider the autonomous Lyapunov (P, Q, V) -cone-system:

$$X_{i+1} = A_0X_i + X_iA_1 \quad (31)$$

where, $X_i \in P, i \in Z_+$.

Definition 11.

The Lyapunov (P, Q, V) -cone-system (15) is called asymptotically stable if:

$$\lim_{i \rightarrow \infty} X_i = 0 \text{ for every } X_0 \in P$$

Theorem 6.

Let us assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A_0 and $\mu_1, \mu_2, \dots, \mu_n$ the eigenvalues of the matrix A_1 . The system (31) is stable if and only if:

$$|\lambda_i + \mu_j| < 1 \text{ for } i, j = 1, 2, \dots, n \quad (32)$$

Proof:

The theorem results directly from the theorem for asymptotic stability of standard systems (Kaczorek T., 2001), since by Lemma 2 eigenvalues of matrix $\bar{A}$ are the sums of eigenvalues of the matrices A_0 and A_1 . $\square$

4.3 Reachability**Definition 12.**

The Lyapunov (P, Q, V) -cone-system (22) is called reachable if for any given $X_f \in \mathbf{P}$ there exist $q \in \mathbf{Z}_+, q > 0$ and an input sequence $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$ that steers the state of the system from $X_0 = 0$ to X_f , i.e. $X_q = X_f$.

Theorem 7.

The Lyapunov (P, Q, V) -cone-system (22) is reachable:

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n, a \in R$, if and only if the matrix:

$$R_n = [PBQ^{-1} \quad \bar{A}_0(PBQ^{-1}) \quad \dots \quad \bar{A}_0^{n-1}(PBQ^{-1})] \quad (33)$$

contains n linearly independent monomial columns, $\bar{A}_0 = PA_0P^{-1} + A_1$.

b) For $A_1 \neq aI_n, a \in R$, if and only if the matrix PBQ^{-1} contains n linearly independent monomial columns.

Proof:

From (26),(28),(29) and from the definitions 2 and 12, we have that the discrete-time Lyapunov (P, Q, V) -cone-system (22) is reachable if and only if the positive discrete-time Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is reachable – so from the theorem for the reachability of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 7. $\square$

4.4 Controllability to zero**Definition 13.**

The Lyapunov (P, Q, V) -cone-system (22) is called controllable to zero if for any given nonzero $X_0 \in \mathbf{P}$ there exist $q \in \mathbf{Z}_+, q > 0$ and an input sequence $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$ that steers the state of the system from X_0 to $X_q = X_f = 0$.

Theorem 8.

The Lyapunov (P, Q, V) -cone-system (22) is controllable to zero:

- a) in a finite number of steps (not greater than n^2) if and only if the matrix $(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T)$ is nilpotent, i.e. has all zero eigenvalues.
- b) in an infinite number of steps if and only if the system is asymptotically stable.

Proof:

From (26),(28),(29) and from the definitions 2 and 13, we have that the discrete-time Lyapunov (P, Q, V) -cone-system (22) is controllable to zero if and only if the positive discrete-time Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is controllable to zero – so from the theorem for the controllability to zero of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 8. $\square$

Lemma 5.

If the matrices PA_0P^{-1} and A_1 are nilpotent then the matrix $(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T)$ is also nilpotent with the nilpotency index $\nu \leq 2n$.

Proof: See (Kaczorek T. & Przyborowski P, 2008).

4.5 Dual Lyapunov cone-systems**Definition 14.**

The Lyapunov system described by the equations:

$$X_{i+1} = A_0^T X_i + X_i A_1^T + C^T U_i \quad (34a)$$

$$Y_i = B^T X_i + D U_i \quad (35b)$$

is called the dual system respect to the system (22). The matrices $A_0, A_1, B, C, D, X_i, U_i, Y_i$ are the same as in the system (22).

4.6 Observability**Definition 15.**

The Lyapunov (P, Q, V) -cone-system (22) is called observable in q -steps, if X_0 can be uniquely determined from the knowledge of the output Y_i and $U_i = 0, i \in Z_+$ for $i \in [0, q]$.

Definition 16.

The Lyapunov (P, Q, V) -cone-system (22) is called observable, if there exists a natural number $q \geq 1$, such that the system (22) is observable in q -steps.

Theorem 9.

The Lyapunov (P, Q, V) -cone-system (22) is observable:

- a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n, a \in R$, if and only if the matrix:

$$O_n = \begin{bmatrix} VCP^{-1} \\ (VCP^{-1})\bar{A}_0 \\ \vdots \\ (VCP^{-1})\bar{A}^{n-1} \end{bmatrix} \quad (33)$$

contains n linearly independent monomial rows, $\bar{A}_0 = PA_0P^{-1} + A_1$.

b) For $A_1 \neq aI_n, a \in R$, if and only if the matrix VCP^{-1} contains n linearly independent monomial rows.

Proof:

From (26),(28),(29) and from the definitions 2 and 15, we have that the discrete-time Lyapunov (P, Q, V) -cone-system (22) is observable if and only if the positive discrete-time

Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is observable – so from the theorem for the observability of positive discrete-time Lyapunov systems (Kaczorek T. & Przyborowski P., 2007e; Kaczorek T. & Przyborowski P., 2008) follows the hypothesis of the theorem 9. $\square$

5. Fractional discrete-time linear Lyapunov cone-systems

Consider the fractional discrete-time linear Lyapunov system (Przyborowski P., 2008a; Przyborowski P., 2008b) described by the equations:

$$\Delta^N X_{i+1} = A_0 X_i + X_i A_1 + B U_i \quad (34a)$$

$$Y_i = C X_i + D U_i \quad (34b)$$

where, $X_i \in R^{n \times n}$ is the state-space matrix, $U_i \in R^{m \times n}$ is the input matrix, $Y_i \in R^{p \times n}$ is the output matrix, $A_0, A_1 \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}, D \in R^{p \times m}, i \in Z_+$ and

$$\Delta^N X_i = \frac{1}{h^N} \sum_{j=0}^i (-1)^j \binom{N}{j} X_{i-j}, \quad \binom{N}{j} = \begin{cases} 1 & \text{for } j=0 \\ \frac{N(N-1) \cdots (N-j+1)}{j!} & \text{for } j=1, 2, \dots \end{cases}$$

is the Grünwald-Letnikov N -order ($N \in R, 0 < N \leq 1$) fractional difference, and h is the sampling interval.

The equations (34) can be written in the form:

$$X_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} X_{i-j+1} = A_0 X_i + X_i A_1 + B U_i \quad (35a)$$

$$Y_i = C X_i + D U_i \quad (35b)$$

Lemma 6.

The fractional Lyapunov system (34) can be transformed to the equivalent fractional discrete-time, nm -inputs and pn -outputs, linear system in the form:

$$\Delta^N \hat{x}_{i+1} = \hat{A} \hat{x}_i + \hat{B} \hat{u}_i \quad (36a)$$

$$\hat{y}_i = \hat{C} \hat{x}_i + \hat{D} \hat{u}_i \quad (36b)$$

where, $\hat{x}_i \in R^{n^2}$ is the state-space vector, $\hat{u}_i \in R^{(nm)}$ is the input vector, $\hat{y}_i \in R^{(pn)}$ is the output vector, $\hat{A} \in R^{n^2 \times n^2}$, $\hat{B} \in R^{n^2 \times (nm)}$, $\hat{C} \in R^{(pn) \times n^2}$, $\hat{D} \in R^{(pn) \times (nm)}$, $i \in Z_+$.

Proof:

The proof is similar to the one of Lemma 3.

The matrices of (36) have the form:

$$\hat{A} = (A_0 \otimes I_n + I_n \otimes A_1^T), \hat{B} = B \otimes I_n, \hat{C} = C \otimes I_n, \hat{D} = D \otimes I_n \quad (37)$$

5.1 Cone-systems**Definition 17.**

The fractional Lyapunov system (22) is called (P, Q, V) -cone-system if $X_i \in P$ and $Y_i \in V$ for every $X_0 \in P$ and for every input $U_i \in Q$, $i \in Z_+$.

Note that for $P = R_+^{n \times n}$, $Q = R_+^{m \times n}$, $V = R_+^{p \times n}$ we obtain $(R_+^{n \times n}, R_+^{m \times n}, R_+^{p \times n})$ -cone system which is equivalent to the fractional positive Lyapunov system (Przyborowski P., 2008a).

Theorem 10.

The fractional Lyapunov system (34) is (P, Q, V) -cone-system if and only if :

$$\hat{A}_0 = P A_0 P^{-1} = [\hat{a}_{ij}^0]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}, \hat{A}_1 = A_1 = [\hat{a}_{ij}^1]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \quad (38)$$

are the Metzler matrices satisfying

$$\hat{a}_{kk}^0 + \hat{a}_{ll}^1 + N \geq 0 \text{ for every } k, l = 1, \dots, n \quad (39)$$

and

$$\hat{B} = P B Q^{-1} \in R_+^{n \times m}, \hat{C} = V C P^{-1} \in R_+^{p \times n}, \hat{D} = V D Q^{-1} \in R_+^{p \times m} \quad (40)$$

Proof:

Let:

$$\hat{X}_i = P X_i, \hat{U}_i = Q U_i, \hat{Y}_i = V Y_i \quad (41)$$

From definition 2 it follows that if $X_i \in P$ then $\hat{X}_i \in R_+^{n \times n}$, if $U_i \in Q$ then $\hat{U}_i \in R_+^{m \times n}$, and if $Y_i \in V$ then $\hat{Y}_i \in R_+^{p \times n}$.

From (34) and (41) we have:

$$\begin{aligned}\hat{X}_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} \hat{X}_{i-j+1} &= P X_{i+1} + \sum_{j=1}^{i+1} (-1)^j \binom{N}{j} P X_{i-j+1} = \\ &= P A_0 X_i + P X_i A_1 + P B U_i = P A_0 P^{-1} \hat{X}_i + P P^{-1} \hat{X}_i A_1 + P B Q^{-1} \hat{U}_i = \\ &= \hat{A}_0 \hat{X}_i + \hat{X}_i A_1 + \hat{B} \hat{U}_i\end{aligned}\quad (41a)$$

and

$$\hat{Y}_i = V Y_i = V C X_i + V D U_i = V C P^{-1} \hat{X}_i + V D Q^{-1} \hat{U}_i = \hat{C} \hat{X}_i + \hat{D} \hat{U}_i \quad (42b)$$

It is known (Przyborowski P., 2008a) that the system (34) is positive if and only if the conditions (41a) and (42b) are satisfied. $\square$

5.2. Stability

Consider the autonomous fractional Lyapunov (P, Q, V) -cone-system:

$$\Delta^N X_{i+1} = A_0 X_i + X_i A_1 \quad (43)$$

where, $X_i \in P$, $i \in Z_+$.

Definition 18. (Dzieliński A. & Sierociuk D., 2006)

The fractional Lyapunov (P, Q, V) -cone- system (43) is called stable in finite relative time if for $\alpha, \beta \in R_+$, $\alpha < \beta$, $\alpha, \beta < \infty$, $k = 1, \dots, n$; $M, N \in Z_+$:

$$\|X_i^k\| < \alpha \quad \text{for } i = 0, -1, \dots, -N$$

implies

$$\|X_i^k\| < \beta \quad \text{for } i = 0, 1, \dots, M$$

where X_i^k is the k th column of the matrix X_i .

Theorem 11.

The fractional Lyapunov (P, Q, V) -cone- system (34) is stable in the meaning of the definition 18 if:

$$\|\hat{A} + I_{n^2} N\| + \sum_{j=2}^{i+1} \left\| (-1)^j \binom{N}{j} I_{n^2} \right\| < 1 \quad (44)$$

where $\hat{A} = A_0 \otimes I_n + I_n \otimes A_1^T$ and $\|W\|$ denotes the norm of the matrix W , defined as $\max_l |\lambda_l|$, where λ_l is the l th eigenvalue of the matrix W .

Proof:

The theorem results directly from the theorem of asymptotic stability of standard fractional systems (Dzieliński A. & Sierociuk D., 2006). $\square$

5.3 Reachability

Definition 19.

The fractional Lyapunov (P, Q, V) -cone-system (34) is called reachable if for any given $X_f \in \mathbf{P}$ there exists $q \in \mathbf{Z}_+, q > 0$ and an input sequence $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$ that steers the state of the system from $X_0 = 0$ to $X_q = X_f$, i.e. $X_q = X_f$.

Theorem 12.

The fractional Lyapunov (P, Q, V) -cone-system (34) is reachable:

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n, a \in R$, if and only if the matrix:

$$R_n = \begin{bmatrix} PBQ^{-1}, & (\bar{A}_0 + I_n N)PBQ^{-1} \end{bmatrix} \quad (45)$$

contains n linearly independent monomial columns, $\bar{A}_0 = PA_0P^{-1} + A_1$.

b) For $A_1 \neq aI_n, a \in R$, if and only if the matrix PBQ^{-1} contains n linearly independent monomial columns.

Proof:

From (38),(39),(40) and from the definitions 2 and 19, we have that the fractional discrete-time Lyapunov (P, Q, V) -cone-system (34) is reachable if and only if the fractional positive discrete-time Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is reachable – so from the theorem for the reachability of positive discrete-time Lyapunov systems (Przyborowski P., 2008a). follows the hypothesis of the theorem 12. $\square$

5.4 Controllability to zero

Definition 20.

The fractional Lyapunov (P, Q, V) -cone-system (34) is called controllable to zero if for any given nonzero $X_0 \in \mathbf{P}$ there exist $q \in \mathbf{Z}_+, q > 0$ and an input sequence $U_i \in \mathbf{Q}, i = 0, 1, \dots, q-1$ that steers the state of the system from X_0 to $X_q = X_f = 0$.

Theorem 13.

The fractional Lyapunov (P, Q, V) -cone-system (34) is controllable to zero if and only if $q = 2$ and:

$$PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T + (I_n N) \otimes I_n = 0 \quad (46)$$

Proof:

From (38),(39),(40) and from the definitions 2 and 19, we have that the fractional discrete-time Lyapunov (P, Q, V) -cone-system (34) is controllable to zero if and only if the fractional positive discrete-time Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is controllable – so from the theorem for the controllability of positive discrete-time Lyapunov systems (Przyborowski P., 2008a). follows the hypothesis of the theorem 13. $\square$

5.5 Dual fractional Lyapunov cone-systems

Definition 21.

The fractional Lyapunov system described by the equations:

$$\Delta^N X_{i+1} = A_0^T X_i + X_i A_1^T + C^T U_i \quad (47a)$$

$$Y_i = B^T X_i + D U_i \quad (47b)$$

is called the dual system respect to the system (34). The matrices $A_0, A_1, B, C, D, X_i, U_i, Y_i$ are the same as in the system (34).

5.6 Observability

Definition 22.

The fractional Lyapunov (P, Q, V) -cone-system (34) is called observable in q -steps, if X_0 can be uniquely determined from the knowledge of the output Y_i and $U_i = 0, i \in Z_+$ for $i \in [0, q]$.

Definition 23.

The fractional Lyapunov (P, Q, V) -cone-system (34) is called observable, if there exists a natural number $q \geq 1$, such that the system (34) is observable in q -steps.

Theorem 14.

The fractional Lyapunov (P, Q, V) -cone-system (34) is observable:

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n, a \in R$, if and only if the matrix:

$$O_n = \begin{bmatrix} VCP^{-1} \\ VCP^{-1}(\bar{A}_0 + I_n N) \end{bmatrix} \quad (48)$$

contains n linearly independent monomial rows, $\bar{A}_0 = PA_0P^{-1} + A_1$.

b) For $A_1 \neq aI_n, a \in R$, if and only if the matrix VCP^{-1} contains n linearly independent monomial rows.

Proof:

From (38),(39),(40) and from the definitions 2 and 20, we have that the fractional discrete-time Lyapunov (P, Q, V) -cone-system (34) is controllable to zero if and only if the fractional positive discrete-time Lyapunov system, with the matrices $\hat{A}_0, \hat{A}_1, \hat{B}, \hat{C}, \hat{D}$, is observable – so from the theorem for the observability of positive discrete-time Lyapunov systems (Przyborowski P., 2008a). follows the hypothesis of the theorem 14. $\square$

6. Examples

Consider the state, input and output cones generated by the matrices

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (49)$$

6.1 Example 1

Consider the continuous-time Lyapunov system (7) with the matrices

$$\begin{aligned} A_0 &= \frac{1}{3} \begin{bmatrix} -7 & -4 \\ -2 & -5 \end{bmatrix}, A_1 = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, B = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (50)$$

This system is (P,Q,V)-cone-system with P, Q and V defined by (49) since:

$$\begin{aligned} \hat{A}_0 &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -7 & -4 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \\ \hat{A}_1 &= \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

are the Metzler matrices and

$$\begin{aligned} \hat{B} &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \hat{C} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ \hat{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

are matrices with nonnegative entries.

A_0 has the eigenvalues: $\lambda_1 = -1, \lambda_2 = -3$ and A_1 has the eigenvalues: $\mu_1 = -4, \mu_2 = -1$ therefore the system is asymptotically stable, since all the sums of the eigenvalues:

$$(\lambda_1 + \mu_1) = -5, (\lambda_1 + \mu_2) = -2, (\lambda_2 + \mu_1) = -7, (\lambda_2 + \mu_2) = -4$$

have negative real parts.

For this system the reachability matrix

$$R_f = \int_0^{t_f} \begin{bmatrix} e^{(-t_f+\tau)^2} & 0 \\ 0 & 4e^{3(-t_f+\tau)^2} \end{bmatrix} d\tau$$

and the observability matrix

$$O_f = \int_0^{t_f} \begin{bmatrix} 4e^{(-t_f+\tau)^2} & 0 \\ 0 & e^{3(-t_f+\tau)^2} \end{bmatrix} d\tau$$

are the monomial matrices for every $t_f > 0$. Therefore, the system is reachable and observable.

6.2 Example 2

Consider the discrete-time Lyapunov system (22) with the matrices

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, B = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (50)$$

This system is (P, Q, V) -cone-system with P, Q and V defined by (49) since:

$$\hat{A}_0 = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

are the Metzler matrices satisfying conditions:

$$\begin{aligned} \hat{a}_{11}^0 + \hat{a}_{11}^1 &= 0.3 + 0.2 = 0.5 > 0, \quad \hat{a}_{22}^0 + \hat{a}_{11}^1 = 0.2 + 0.2 = 0.4 > 0 \\ \hat{a}_{11}^0 + \hat{a}_{22}^1 &= 0.3 + 0.5 = 0.8 > 0, \quad \hat{a}_{22}^0 + \hat{a}_{22}^1 = 0.2 + 0.5 = 0.7 > 0 \end{aligned}$$

and

$$\begin{aligned} \hat{B} &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ \hat{D} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

are matrices with nonnegative entries.

A_0 has the eigenvalues: $\lambda_1 = 0.1, \lambda_2 = 0.4$ and A_1 has the eigenvalues: $\mu_1 = 0.2, \mu_2 = 0.5$ therefore the system is asymptotically stable, since all the eigenvalues:

$$(\lambda_1 + \mu_1) = 0.3, (\lambda_1 + \mu_2) = 0.6, (\lambda_2 + \mu_1) = 0.6, (\lambda_2 + \mu_2) = 0.9$$

have moduli less than one.

The system is reachable and observable because the matrix PBQ^{-1} has $n = 2$ monomial columns, and the matrix VCP^{-1} has $n = 2$ monomial rows.

The system is not controllable to zero in finite number of steps since the matrix

$$(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T) = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix}$$

is not a the nilpotent matrix, but the system is controllable to zero in the infinite number of steps since it is asymptotically stable.

6.3 Example 3

Consider the discrete-time Lyapunov system (34) with $N = \frac{1}{4}$ and the matrices

$$A_0 = \begin{bmatrix} -0.5 & -0.8 \\ 0.6 & 0.9 \end{bmatrix}, A_1 = \begin{bmatrix} 0.17 & 0 \\ 1 & 0.4 \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad (51)$$

$$C = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (n = 2)$$

This system is (P, Q, V) -cone-system with P, Q and V defined by (49) since:

$$\hat{A}_0 = \begin{bmatrix} 0.3 & 2 \\ 0 & 0.1 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 0.17 & 0 \\ 1 & 0.4 \end{bmatrix}, \hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \hat{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The system is (P, Q, V) -cone-system because:

$$\hat{a}_{11}^0 + \hat{a}_{11}^1 + N = 0.3 + 0.17 + 0.25 = 0.72 > 0, \hat{a}_{22}^0 + \hat{a}_{11}^1 + N = 0.1 + 0.17 + 0.25 = 0.52 > 0$$

$$\hat{a}_{11}^0 + \hat{a}_{22}^1 + N = 0.3 + 0.4 + 0.25 = 0.95 > 0, \hat{a}_{22}^0 + \hat{a}_{22}^1 + N = 0.1 + 0.4 + 0.25 = 0.75 > 0$$

and the matrices $\hat{B}, \hat{C}, \hat{D}$ have nonnegative entries.

For the instant $i = 100$ we have

$$\left\| \bar{A} + I_{n^2} N \right\| + \sum_{j=2}^{i+1} \left\| (-1)^j \binom{N}{j} I_{n^2} \right\| = 0.8268 < 1$$

so the system is stable in the meaning of the the definition 18.

The system is reachable and observable because the matrix PBQ^{-1} has $n = 2$ monomial columns, and the matrix VCP^{-1} has $n = 2$ monomial rows.

The system is not controllable to zero in finite number of steps since the matrix

$$(PA_0P^{-1} \otimes I_n + I_n \otimes A_1^T + (I_n N) \otimes I_n) = \begin{bmatrix} 0.72 & 0 & 0 & 0 \\ 0 & 0.52 & 0 & 0 \\ 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 0.75 \end{bmatrix}$$

is not a zero matrix.

7. Conclusions

In this paper three types of systems have been considered. For the continuous-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability and sufficient conditions for the reachability and observability have been established. For the discrete-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the asymptotic stability, reachability, observability and controllability to zero have been established. For the fractional discrete-time linear Lyapunov cone-systems, the necessary and sufficient conditions for being the cone-system, the reachability, observability and controllability to zero and sufficient conditions for the stability have been established. The considerations have been illustrated on the numerical examples.

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