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Interaction of Solitons with the Electromagnetic Field in Classical Nonlinear Field Models

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Abstract

Solitary waves and solitons are briefly discussed in their historical context, especially regarding the motivation to use solitons for particle models in the nonlinear Klein-Gordon (NKG) equation. Conservation equations for charge, energy, and momentum follow from Noether's theorem in the usual way, and the NKG equation can be coupled to Maxwell's equations in the standard, gauge invariant manner. A recently proposed model is summarized in which two NKG equations are coupled to the electromagnetic field. In that model, solitons mimic the dynamical behavior of electrons and protons. A new result is then presented that follows from that model. Although the model is purely classical, it turns out that the arc spectrum of hydrogen is emitted into the electromagnetic field when small oscillations of one Klein-Gordon equation occur in the vicinity of a proton-like soliton. It is perhaps unexpected that a purely classical model can exhibit behavior suggestive of a phenomenon that is generally presumed to occur only in a quantum-mechanical context. Because of the way in which that result occurs, however, it is not clear whether there is any possible relevance to the actual physical phenomenon.

Keywords: nonlinear systems and models, solitons, nonlinear Klein-Gordon-Maxwell equations, particle models, hydrogen spectrum, nonlinear field theories

1. Introduction

When a term that acts as a restoring force is added to the usual linear wave equation, the equation

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \psi = 0 \tag{1}$$

is obtained. Schweber ([1], p. 54) points out "When Schrödinger wrote down the nonrelativistic equation now bearing his name, he also formulated the corresponding relativistic

equation." Within the year of 1926, Klein, Gordon, and at least three others independently proposed the use of Eq. (1) as the relativistic generalization of Schrödinger's equation. Eq. (1) has since become known as the Klein-Gordon equation. Instead of the original time and space variables $\hat{t}, \hat{x}, \hat{y}, \hat{z}$ that might be used in a quantum-mechanical context, we have chosen to write Eq. (1) using dimensionless variables t, x, y, z defined as

$$t = \frac{m c^2}{\hbar}\hat{t}, \quad x = \frac{mc}{\hbar}\hat{x}, \quad y = \frac{mc}{\hbar}\hat{y}, \quad z = \frac{mc}{\hbar}\hat{z}, \tag{2}$$

where $\hat{t}, \hat{x}, \hat{y}, \hat{z}$ are time and space coordinates in customary units (e.g., seconds and meters). Similarly, m is the mass of the electron, c is the speed of light, and $\hbar$ is Planck's constant divided by 2π, also in customary units. Unit distance in our dimensionless coordinates corresponds to $\hbar/(mc)$, which is about 3.86×10^{-13} m. That distance is known as the Compton wavelength of the electron (divided here by 2π), which is a fundamental unit of length for quantum-mechanical phenomena that involve electrons.

When one attempts to use the Klein-Gordon equation to describe the quantum-mechanical problem of an electron in the Coulomb field of a nucleus of atomic number Z, Eq. (1) becomes

$$\left(\frac{\partial}{\partial t} + i\,\frac{Z\,\alpha_{fs}}{r}\right)\left(\frac{\partial}{\partial t} + i\,\frac{Z\,\alpha_{fs}}{r}\right)\psi - \nabla^2\,\psi + \psi = 0, \tag{3}$$

where α_{fs} is the fine structure constant (approximately 1/137), where $i = \sqrt{-1}$, and where $r^2 = x^2 + y^2 + z^2$. Unit distance for the dimensionless quantity r is thus also $\hbar/(mc)$. Although Eq. (3) reduces properly to Schrödinger's equation in the nonrelativistic approximation, it was realized almost immediately that Eq. (3) was in fact not appropriate for the quantum-mechanical description of the electron, and the Klein-Gordon equation fell even into a degree of disrepute, especially after the remarkable insight that led to Dirac's equation in 1928 [1, 2].

The Klein-Gordon equation is, nevertheless, a simple example of a relativistic partial differential equation and, as such, it is second in importance only to the wave equation. The relativistic invariance is preserved even when the ψ term is replaced with a term that depends nonlinearly on ψ, and the resulting equation can be used, for example, in the study of nonlinear waves [3, 4]. This chapter will deal primarily with the nonlinear Klein-Gordon (NKG) equation in the form

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2\,\psi + W'\left(\psi\bar{\psi}\right)\psi = 0, \tag{4}$$

where ψ depends on time t and spatial coordinates x, y, z. Overbar denotes the complex conjugate since ψ will usually be taken as complex-valued. The real-valued function W is used to introduce an appropriate nonlinearity, and $W'(\psi\bar{\psi})$ denotes the derivative of $W(\psi\bar{\psi})$ with respect to its argument $\psi\bar{\psi}$. Let us take $W(0) = 0$, and with suitable scaling we can also take $W'(0) = 1$, so that Eq. (4) will agree with Eq. (1) when ψ is small.

Here, we are interested in the soliton solutions that occur in the NKG equation, so Section 2 is a brief historical overview of solitary waves and solitons, followed by the historical context in Section 3 that motivates the use of solitons in classical field theories. Before coupling the NKG equation to Maxwell's equations in Section 8, we need to examine the NKG equation

itself in more detail in Sections 5 and 6. An explicit example of a soliton solution is given in Section 7. In Sections 9–11, we summarize a recently proposed model in which a second NKG equation is introduced so as to model solitons that can be thought of as protons as well as electrons. In that model, the proton and electron-like solitons attract and repel each other in the desired way. Finally, in Section 12, we present new results from the same model where it is shown that small oscillations in the neighborhood of a proton-like soliton emit the well-known frequencies of the hydrogen spectrum into the electromagnetic field. Since emission of the hydrogen spectrum is a phenomenon that would normally be expected to occur only in a quantum-mechanical setting and not in the present classical field model, we interpret this result in Section 13 and suggest the future direction of the present research.

2. Historical background of solitary waves and solitons

The study of solitary waves has a long history [4], which originated with observations by Scott Russell ([5], [6, p. 118]), of an isolated wave moving in a canal. Following along on horseback, he was able to observe that the wave traveled a great distance with little change in form and only gradually died out. The linear theories of water waves available at that time predicted, however, that such waveshapes would necessarily disperse, so for some time Scott Russell's observations were not taken seriously. The full equations for water waves, even in the irrotational, inviscid case, present quite a formidable boundary value problem, so various simple model equations have been examined in much greater detail. Strangely in contrast to the linear theory, a certain nonlinear model of water waves for shallow water predicted that waves of all shapes would steepen and ultimately break. Eventually, various simple, nonlinear model equations such as the equation of Korteweg and deVries (often known as the KdV equation)

$$\frac{\partial \eta}{\partial t} + (c_0 + c_1 \eta) \frac{\partial \eta}{\partial x} + \nu \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{5}$$

were formulated, where both the dispersion and steepening effects are included. In these equations, the two effects balance each other and do, in fact, allow an isolated waveshape to propagate without change of form.

The term soliton was later introduced to emphasize that solitary waves, like particles, tend to maintain their identity. When a nonlinear term is introduced into the usual linear Schrödinger equation, the nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} - \nabla^2 \psi + W'(\psi \bar{\psi}) \psi = 0, \tag{6}$$

is obtained. It has soliton solutions and is of particular interest in relation to the ideas of de Broglie and Bohm [7, 8]. Soliton solutions occur in the NKG Eq. (4) and also in the related equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \sin \psi = 0 \tag{7}$$

known (a bit flippantly) as the sine-Gordon equation.

Remarkable methods, such as the inverse scattering method, have been devised for use with certain model equations. With these methods, it is possible to find exact solutions that show solitons colliding and passing through each other while still maintaining their identity. The subject of solitary waves and solitons has been extensively described in books and review articles [9–13]. For more recent work, see [14–24] and references therein.

3. Use of solitons as particle models

The motivation for the use of solitons as particle models can best be understood by comparison with the historical development of physical theories, especially electromagnetism and quantum mechanics. Despite the successes of Maxwell's theory of electromagnetism in the late 1800s and the clarified understanding of its transformation properties according to Einstein's special theory of relativity, classical physics seemed to reach an impasse at the beginning of the twentieth century. The notion of the electron as a point particle was immediately inconsistent with electromagnetism since the electromagnetic energy around a charged point particle is easily calculated to be infinite. Moreover, if an electron were to orbit a proton like a planet around the sun, it would radiate into the electromagnetic field at a frequency based on the orbital speed and quickly spiral inward toward the nucleus.

Early in the twentieth century, however, Planck, Einstein, and Bohr advanced daring new hypotheses to describe blackbody radiation, the photoelectric effect, and the hydrogen atom, respectively. Dirac, Heisenberg, and others later developed various versions of a new quantum mechanics more sophisticated but certainly no more intuitively comprehensible. Quantum mechanics has been perpetually troubled by divergences, as when Bethe [25] somewhat jokingly referred to a quantity that "…comes out infinite in all existing theories, and has therefore always been ignored." Remarkable progress was later achieved in quantum electrodynamics by Tomonaga, Schwinger, Feynman, and others, but still only with the aid of a rather arbitrary (some would say procrustean) renormalization procedure. Dirac [2] concludes his book with the thought that "It would seem that we have followed as far as possible the path of logical development of the ideas of quantum mechanics as they are at present understood. The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics."

There have been various attempts, notably by de Broglie [7, 26], to gain insight into a possible "drastic change in the foundations of the theory" as envisioned by Dirac, but so far such attempts have had little success. One possible approach, which apparently occurred to many people, is to ask whether solitons could be used to model actual particles as localized regions where the field is large. If point particles were replaced by solitons one could hope for a theory along more classical lines where problems with infinite, divergent integrals would be avoided in a natural way. The line of research described here is meant to be a step, if only a tiny step, in that direction. A recently proposed model is summarized in Sections 9–11. A new result is presented in Section 12, where a phenomenon suggestive of quantum-mechanics occurs, but in an unexpected context.

4. Topological and nontopological solitons

It was found early on that if one tries to use the NKG Eq. (4) to form a localized, particle-like solution, where ψ is a static, real-valued function of spatial coordinates x, y, z, that such a solution turns out to be wildly unstable. Even the linear stability analysis shows that there is a mode that grows in time, so the solution either tends to collapse or to grow arbitrarily large. Hobart [27] and Derrick [28] concluded from energy considerations that a wide class of field equations would turn out to be unstable in this way. One possible way around this difficulty is for the solution to have a suitable topological property. Scott [29] and Rubinstein [30] have examined a simple example of such a topological soliton in the sine-Gordon Eq. (7). Scott's interpretation by means of a mechanical model is especially easy to visualize, where ψ represents an angle in the y, z plane, and the solution loops around the x-axis.

Here, however, we will be concerned with another approach to stability that uses the NKG Eq. (4) but with a complex-valued variable ψ that has a time-dependent phase. Thus, we will be interested in possible solutions of the form

$$\psi = U(r) e^{i\omega t}, \tag{8}$$

where $r^2 = x^2 + y^2 + z^2$ and where the real-valued function $U(r)$ is exponentially small for large r. The solution is stationary in the sense that U does not depend on t, but it is not static since ψ is in effect rotating with angular frequency ω in the complex plane. Some of the early works along these lines were done by Glasko et al. [31], Zastavenko [32], and Rosen [33]. It turns out that orbital stability can be achieved in suitable circumstances. Roughly speaking, the rotation allows stability to occur just as a spinning gyroscope can be stable in a situation that would otherwise be unstable from energetic considerations. Solitons of this nature are often called nontopological solitons, in contrast to the topological solitons that occur in examples such as the sine-Gordon Eq. (7). The term soliton is sometimes reserved for particle-like solutions that have been shown to be orbitally stable; but here, we will allow the use of the term for particle-like solutions in general. One difficulty with the use of nontopological solitons as particle models is that a whole family of particle-like solutions typically occurs, as will be seen in the example of Section 7. This problem will be discussed further in Section 10. Existence and stability questions have been extensively studied for the NKG equation [15, 34]. More recently, soliton solutions with nonzero angular momentum have also been of interest [18, 19].

5. Conservation equations for the NKG equation

To understand soliton solutions for the NKG equation, it is important first to see that the NKG equation can be derived from a variational principle. The Lagrangian density is

$$L = \frac{1}{2} \frac{\partial \psi}{\partial t} \frac{\partial \overline{\psi}}{\partial t} - \frac{1}{2} \nabla \psi \cdot \nabla \overline{\psi} - \frac{1}{2} W(\psi \overline{\psi}), \tag{9}$$

and the corresponding Euler operator (with respect to ψ) is

$$O p_\psi = -\frac{\partial^2 \bar\psi}{\partial t^2} + \nabla^2 \bar\psi - W'(\psi\bar\psi)\bar\psi. \tag{10}$$

When we set the Euler operator (10) to zero (and take the negative of the complex conjugate), we obtain the NKG Eq. (4). It is well known from Noether's theorem [4, 35, 36] that each symmetry of the variational principle leads to an equation in what is referred to as conservation form. Once the equation is in conservation form, it is straightforward to integrate in order to show that a certain quantity is conserved. Symmetry with respect to time translation thus leads to conservation of energy, and symmetry with respect to translation in x, y, z space leads to conservation of momentum.

Since the variational principle that follows from Eq. (9) is unchanged under time translation, it follows that energy is conserved. Specifically, we can rewrite the expression

$-\left(O p_\psi \partial\psi/\partial t + \overline{O p_\psi}\partial\bar\psi/\partial t\right)/2$ in the following conservation form:

$$-\frac{1}{2}\left(O p_\psi \frac{\partial\psi}{\partial t} + \overline{O p_\psi}\frac{\partial\bar\psi}{\partial t}\right) = \frac{\partial}{\partial t}\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\frac{\partial\bar\psi}{\partial t} + \nabla\psi\cdot\nabla\bar\psi + W(\psi\bar\psi)\right)$$
$$-\frac{1}{2}\nabla\cdot\left(\frac{\partial\psi}{\partial t}\ \nabla\bar\psi + \frac{\partial\bar\psi}{\partial t}\nabla\psi\right). \tag{11}$$

Let

$$\mathcal{E} = \int\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\frac{\partial\bar\psi}{\partial t} + \nabla\psi\cdot\nabla\bar\psi + W(\psi\bar\psi)\right)d^3 x, \tag{12}$$

where the integral is to be taken over a certain region in x, y, z space and where d^3x denotes a volume element. Then, $\mathcal{E}$ is defined as the energy in that region. Let us suppose that Eq. (4) is satisfied, so that the right-hand side of Eq. (11) must be equal to zero. Then, if Eq. (11) is integrated over the region in question, it is clear from the divergence theorem that the energy within that region will remain constant in time, except for any energy that enters or leaves the region through the outer surface.

Similarly, since the variational principle is invariant under translation in the x direction, a conservation equation can be found for momentum in the x direction. We can thus rewrite the expression $\left(O p_\psi \partial\psi/\partial x + \overline{O p_\psi}\partial\bar\psi/\partial x\right)/2$ in the following conservation form:

$$\frac{1}{2}\left(O p_\psi \frac{\partial\psi}{\partial x} + \overline{O p_\psi}\frac{\partial\bar\psi}{\partial x}\right) = \frac{\partial}{\partial t}\left(-\frac{1}{2}\frac{\partial\psi}{\partial t}\frac{\partial\bar\psi}{\partial x} - \frac{1}{2}\frac{\partial\bar\psi}{\partial t}\frac{\partial\psi}{\partial x}\right)$$
$$+ \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial\psi}{\partial t}\frac{\partial\bar\psi}{\partial t} - \frac{1}{2}\ \nabla\psi\cdot\nabla\bar\psi - \frac{1}{2}W(\psi\bar\psi)\right) \tag{13}$$
$$+ \nabla\cdot\left(\frac{1}{2}\frac{\partial\bar\psi}{\partial x}\ \nabla\psi + \frac{1}{2}\frac{\partial\psi}{\partial x}\ \nabla\bar\psi\right).$$

Then

$$P_x = \int\left(-\frac{1}{2}\frac{\partial\psi}{\partial t}\frac{\partial\bar\psi}{\partial x} - \frac{1}{2}\frac{\partial\bar\psi}{\partial t}\frac{\partial\psi}{\partial x}\right)d^3 x \tag{14}$$

is defined as the momentum in the x direction (as contained within the region in question). Again, the use of the divergence theorem shows that momentum within the region

is conserved, except for any momentum that enters or leaves through the outer surface. Conservation of momentum in the y and z directions is similar.

Next, because the variational principle is invariant when ψ is rotated in the complex plane, the quantity $i\left(\psi \, O \, p_\psi - \overline{\psi \, O \, p_\psi}\right)/2$ can be put in the conservation form

$$\frac{i}{2}\left(\psi \, O \, p_\psi - \overline{\psi \, O \, p_\psi}\right) = \frac{\partial}{\partial t}\left(\frac{i\overline{\psi}}{2}\frac{\partial \psi}{\partial t} - \frac{i\psi}{2}\frac{\partial \overline{\psi}}{\partial t}\right) + \nabla\cdot\left(\frac{i\psi}{2}\,\nabla\overline{\psi} - \frac{i\overline{\psi}}{2}\,\nabla\psi\right). \tag{15}$$

The quantity

$$Q = \int\left(\frac{i\overline{\psi}}{2}\frac{\partial \psi}{\partial t} - \frac{i\psi}{2}\frac{\partial \overline{\psi}}{\partial t}\right) d^3x \tag{16}$$

will be defined as the charge (contained within the region in question), for reasons that will become clear when the NKG equation is coupled to Maxwell's equations. Again, by the divergence theorem, the charge within the region must remain constant in time except for any charge that enters or leaves through the outer surface.

6. Rotating nonlinear solutions for the NKG equation

Now let us look for solutions of the NKG equation that are rotating in the complex plane with angular frequency ω, that is

$$\psi = U\,e^{i\omega t}, \tag{17}$$

where U is a real function that depends on x, y, z, but is independent of t. Then, U needs to satisfy

$$\nabla^2 U + \omega^2 U - W'(U^2)U = 0. \tag{18}$$

Generally, we will be interested in localized solutions, and U will typically be exponentially small when r is large. Substitution of expression (17) in Eqs. (16) and (12) shows that the charge of such a solution is

$$Q = -\omega\int U^2\,d^3x, \tag{19}$$

and the energy is

$$\mathscr{E} = \int\frac{1}{2}(\omega^2\,U^2 + \nabla U\cdot\nabla U + W(U^2))\,d^3x, \tag{20}$$

where the integrals are now taken over all of x, y, z space. From Eq. (19), it is clear that the charge will be positive or negative depending on the sign of ω. We are not interested here in the case where Q is zero since we want U to be nontrivial, and ω needs to be nonzero when we try for an orbitally stable soliton solution.

If we look for a radially symmetric solution $U(r)$, where $r^2 = x^2 + y^2 + z^2$, we note that

$$\nabla^2 U = \frac{1}{r} \frac{d^2(rU)}{dr^2} \tag{21}$$

so that Eq. (18) can be rewritten as

$$\frac{d^2 \mathcal{U}}{dr^2} + \omega^2 \mathcal{U} - W'(U^2)\mathcal{U} = 0, \tag{22}$$

with $\mathcal{U} = rU$.

7. An explicit soliton solution for the NKG equation

To fix ideas, let us derive the following standard result in somewhat more than the usual detail. An expedient that has been frequently used [37–40] is to let W' be a step function, in which case a simple example of a soliton solution can be worked out explicitly. Let us suppose that W' has a fixed value a_W when r is greater than a certain r_0. Then, we want $a_W - \omega^2 > 0$, so we can find a solution of Eq. (22) of the form $\mathcal{U} = B e^{-b(r-r_0)}$ for $r > r_0$, where B is a constant and $b^2 = a_W - \omega^2$. To satisfy an appropriate boundary condition at infinity, we have chosen the solution where $\mathcal{U}$ approaches 0 for large r. Next, let us suppose that W' has a somewhat smaller fixed value $a_W - b_W$ when $r < r_0$. We want the constant b_W to be a suitable value so that $a_W - b_W - \omega^2$ will be negative. Then, we can find a solution of Eq. (22) in the form $\mathcal{U} = A \sin(ar)$ for $r < r_0$, where A is a constant and $a^2 = \omega^2 - a_W + b_W$. We have chosen the sine rather than the cosine solution so as to satisfy an appropriate boundary condition at $r = 0$. Then, we note that

$$a^2 + b^2 = b_W. \tag{23}$$

We need $A \sin(ar_0) = B$ to make $\mathcal{U}$ continuous at r_0, and we need $Aa \cos(ar_0) = -Bb$ to make $d\mathcal{U}/dr$ continuous at r_0, so, altogether, we want

$$a \cos(a r_0) = -b \sin(a r_0). \tag{24}$$

There will be a whole family of rotating solutions for Eq. (18), as was mentioned in Section 4. It is convenient to use $\theta_0 = ar_0$ as a parameter for this family. Using Eqs. (23) and (24), we find that $a^2 = b_W \sin^2 \theta_0$ and $b^2 = b_W \cos^2 \theta_0$. We will be looking for solutions with θ_0 between $\pi/2$ and π (where $\sin \theta_0$ is positive but $\cos \theta_0$ is negative) so we find

$$a = \sqrt{b_W} \sin \theta_0, \quad b = -\sqrt{b_W} \cos \theta_0. \tag{25}$$

If we wish to have U take the value U_W at r_0, then

$$B = \frac{U_W \theta_0}{a}, \quad A = \frac{U_W \theta_0}{a \sin \theta_0}. \tag{26}$$

Collecting the above results, we have

$$U = \frac{\mathcal{U}}{r} = \frac{\theta_0 \, U_W \sin(\hat{r} \sin \theta_0)}{\hat{r} \sin^2 \theta_0} \tag{27}$$

for $\hat{r} \leq \hat{r}_0$ and

$$U = \frac{\mathcal{U}}{r} = \frac{\theta_0 \, U_W \, e^{(\hat{r} - \hat{r}_0) \cos \theta_0}}{\hat{r} \sin \theta_0} \tag{28}$$

for $\hat{r} > \hat{r}_0$. The result has been written in terms of scaled quantities

$$\hat{r} = \sqrt{b_W} \, r, \qquad \hat{r}_0 = \sqrt{b_W} \, r_0 = \theta_0 / \sin \theta_0. \tag{29}$$

Then, we need $W'(U^2) = a_W$ for $u^2 < u_W^2$ and $W'(U^2) = a_W - b_W$ for $u^2 > u_W^2$. Given values of a_W and b_W, as well as a value of the parameter θ_0, we find that ω is to be determined from the equation

$$\omega^2 = a_W - b_W \cos^2 \theta_0. \tag{30}$$

The shape of the solution, as given by the dependence of U on $\hat{r}$ in Eqs. (27) and (28), depends on θ_0 but not on a_W or b_W. For now, let us take $a_W = 1$. It is easy to obtain U in terms of the original variable r using Eq. (29). Eq. (29) shows that the physical size of the soliton, i.e., its size in x, y, z space, will be larger when b_W is smaller.

The charge Q and energy $\mathcal{E}$ of such a soliton solution follow from the integrals (19) and (20). It turns out that the integral for the charge can be worked out explicitly as

$$\begin{aligned} Q &= -\omega \int_0^\infty U^2 \, 4\pi \, r^2 \, dr \\ &= -\sqrt{1 - b_W \cos^2 \theta_0} \, \frac{2\pi(\theta_0 \cos \theta_0 - \sin \theta_0) \, b_W^{-3/2} \, U_W^2 \, \theta_0^2}{(\sin \theta_0)^5 \cos \theta_0}. \end{aligned} \tag{31}$$

If a graph of charge Q is plotted (say for $b_W = 0.1$) with respect to the parameter θ_0, it is seen that a minimum value of $|Q|$ occurs. Expressed in another way, the family of solutions bifurcates as $|Q|$ is increased.

8. Coupling of the NKG equation to the electromagnetic field

Next, we want to couple the NKG equation to electromagnetism to obtain the nonlinear Klein-Gordon-Maxwell (NKGM) system of equations, as has been done by Rosen [37], Morris [41], and others. Again it is important to express the coupling of the NKG equation to the electromagnetic field in terms of a variational principle so that the conservation laws of interest can be obtained from Noether's theorem. Although Maxwell's equations are usually expressed in terms of the quantities $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}$, it turns out that, to get a suitable variational principle, we need to work instead with the scalar potential A_0 and the vector potential $\mathbf{A}$, which are defined in terms of the relations

$$\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}, \qquad \nabla \times \mathbf{A} = \mathbf{B}. \tag{32}$$

Here, the notation A_0 is used for the scalar potential to emphasize that it can be thought of as the time component of a Lorentz 4-vector, while $\mathbf{A} = (A_1, A_2, A_3)$ gives the three spatial components. The gauge condition can then be taken in the form

$$\frac{\partial A_0}{\partial t} - \nabla \cdot \mathbf{A} = 0. \tag{33}$$

It should be noted that $A_0 = -\Phi$ has the opposite sign from the symbol Φ that is often used to designate the scalar potential [42]. Eq. (4) can be coupled to the electromagnetic field in the usual gauge invariant way by replacing $\partial/\partial t$ by $\partial/\partial t - iA_0$ and by replacing ∇ by $\nabla - i\mathbf{A}$, where $i = \sqrt{-1}$. The resulting equation for ψ is

$$\left(\frac{\partial}{\partial t} - i A_0\right)\left(\frac{\partial}{\partial t} - i A_0\right)\psi - (\nabla - i\mathbf{A}) \cdot (\nabla - i\mathbf{A})\,\psi + W'(\psi\bar{\psi})\psi = 0. \tag{34}$$

When oscillations of ψ are small so that $W'(\psi\bar{\psi})$ is effectively 1, we want Eq. (34) to reduce to the linear Klein-Gordon equation. In particular, for the example of the Coulomb field of a nucleus of atomic number Z, where $\mathbf{A}$ will be 0, we find that Eq. (34) reduces to

$$\left(\frac{\partial}{\partial t} - i A_0\right)\left(\frac{\partial}{\partial t} - i A_0\right)\psi - \nabla^2\psi + \psi = 0. \tag{35}$$

Then by comparison with Eq. (3), it is clear that a nucleus of atomic number Z should generate a potential well

$$A_0 = -\frac{Z\,\alpha_{fs}}{r}. \tag{36}$$

The Lagrangian density for the electromagnetic field can be expressed in terms of A_0 and $\mathbf{A}$, as mentioned, but it involves a tensor quantity that is not convenient to express in terms of the usual vector notation. Details are given in [24]. When the equations for ψ, A_0, and $\mathbf{A}$ are derived as Euler equations of a variational principle, as in [24], it turns out that A_0 and $\mathbf{A}$ are to satisfy

$$\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 = \frac{i\psi}{2}\frac{\partial\bar{\psi}}{\partial t} - \frac{i\bar{\psi}}{2}\frac{\partial\psi}{\partial t} - A_0\,\psi\bar{\psi} \tag{37}$$

and

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{i\psi}{2}\,\nabla\bar{\psi} - \frac{i\bar{\psi}}{2}\,\nabla\psi - \mathbf{A}\psi\bar{\psi}, \tag{38}$$

respectively. Substantial work has been done regarding existence of solitons in the NKGM system [43–45], in some cases even with nonzero angular momentum [16]. Stability of solitons in the NKGM system is a difficult subject, but some progress has been made [17, 21, 44], especially in the case of small coupling of the NKG equation to the electromagnetic field. The effect of an external field on a soliton has also been examined in the NKGM context by several authors [20, 46, 47].

9. A model with interacting solitons

Recently, a model (call it Model One) has been proposed [24] in which a second NKG equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + M^2 W'\left(\frac{\phi\bar{\phi}}{M^2}\right)\phi = 0, \tag{39}$$

is also coupled to the electromagnetic field, but scaled with the constant M so that solitons in the ϕ field can be thought of as protons, whereas solitons in the ψ field are to be regarded as electrons. It must be stated at the outset that Model One is a very primitive model that clearly does not represent the universe in which we live; nevertheless, as an example in applied mathematics it has interesting features that seem worth examining. Although Model One is purely classical in nature, we will show in Section 12 that it exhibits behavior suggestive of a phenomenon that is generally presumed to occur only in a quantum-mechanical context.

In addition to Eq. (34) for ψ, we now have an equation

$$\left(\frac{\partial}{\partial t} - i A_0\right)\left(\frac{\partial}{\partial t} - i A_0\right)\phi - (\nabla - i\mathbf{A}) \cdot (\nabla - i\mathbf{A})\phi + M^2 W'\left(\frac{\phi\bar{\phi}}{M^2}\right)\phi = 0 \tag{40}$$

for ϕ, and Eqs. (37) and (38) are to be replaced by

$$\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 = \frac{i\psi}{2}\frac{\partial\bar{\psi}}{\partial t} - \frac{i\bar{\psi}}{2}\frac{\partial\psi}{\partial t} - A_0\,\psi\bar{\psi} + \frac{i\phi}{2}\frac{\partial\bar{\phi}}{\partial t} - \frac{i\bar{\phi}}{2}\frac{\partial\phi}{\partial t} - A_0\,\phi\bar{\phi} \tag{41}$$

and

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{i\psi}{2}\,\nabla\bar{\psi} - \frac{i\bar{\psi}}{2}\,\nabla\psi - \mathbf{A}\psi\bar{\psi} + \frac{i\phi}{2}\,\nabla\bar{\phi} - \frac{i\bar{\phi}}{2}\,\nabla\phi - \mathbf{A}\phi\bar{\phi}, \tag{42}$$

respectively.

10. Electron-like solitons

Let us first investigate solutions of the system (34), (40)–(42) where ψ rotates in the complex plane and $\phi = 0$. Also, we will take the scalar field A_0 to be constant in time and set $\mathbf{A}$ to zero, that is

$$\psi = U(x, y, z)\, e^{i\omega t}, \quad \phi = 0, \quad A_0 = V(x, y, z), \quad \mathbf{A} = 0, \tag{43}$$

where U and V are real-valued. Then, Eqs. (34) and (40)–(42) reduce to

$$\nabla^2 U + (\omega - V)^2 U - W'(U^2)U = 0, \tag{44}$$

$$\nabla^2 V = (-\omega + V)\, U^2. \tag{45}$$

We want to find a soliton solution that satisfies both Eqs. (44) and (45) and can be thought of as an electron. Suppose that we start with a soliton solution for Eq. (18) of the kind that was previously discussed. We want the coupling to the electromagnetic field in Eqs. (44) and (45) to result in only a small change, so that U will still be spherically symmetric and localized, with U exponentially small for large r. The right-hand side of Eq. (45) is then nearly zero for large r, so the solution for V will be nearly proportional to $1/r$ for large r. If we can arrange for V to approach α_{fs}/r for large r we see by comparison with Eq. (36) that the soliton will have one negative unit of elementary charge, so we can think of it as an electron-like soliton.

In the present NKGM context, it turns out Eq. (16) is to be replaced by

$$Q = \int \left(\frac{i\bar{\psi}}{2} \frac{\partial \psi}{\partial t} - \frac{i\psi}{2} \frac{\partial \bar{\psi}}{\partial t} + \psi\bar{\psi} A_0 \right) d^3 x \tag{46}$$

so for a solution of the form (43), the charge is now

$$Q = \int (-\omega + V) U^2 d^3 x. \tag{47}$$

The divergence theorem together with Eq. (45) shows that for large r, where U is small, V is approximately $-Q/(4\pi r)$, and so, by comparison with Eq. (36), $Q = 4\pi\alpha_{fs}$ corresponds to one unit of elementary charge. Positive ω in a solution of the form (43), which gives counterclockwise rotation in the complex plane, thus gives negative Q in agreement with the usual convention for electron charge. There will be a corresponding solution with negative ω and clockwise rotation which can then be thought of as a positron.

As mentioned in Section 4, there is a fundamental difficulty with the use of nontopological solitons as particle models. A whole family of soliton solutions typically exists, and the different solutions will have various values of charge, as was illustrated in the example of Section 7. Thus, it is not yet clear whether a preferred solution occurs in practice that could be regarded as defining one unit of elementary charge. Morris [41] has made a suggestion in this regard, but it seems that the resolution of the question is beyond the scope of present research on models of this nature since it may well involve complex, possibly chaotic, interactions of many solitons. A second problem is that the stability proofs currently available apply only in the limit of small coupling, so it is not clear whether orbital stability is achieved when the coupling is sufficient to correspond to a meaningful physical case. Future research should resolve this question, although perhaps in a somewhat nonrigorous manner based at least in part on numerical calculation. Despite these two difficulties let us assume for present purposes that a suitable orbitally stable solution of the form (43) is available which can then be thought of as an electron.

11. Proton-like solitons

Now let us look for solutions of the form

$$\psi = 0, \quad \phi = \tilde{U}(x, y, z)\, e^{-i\tilde{\omega}t}, \quad A_0 = \tilde{V}(x, y, z), \quad \mathbf{A} = 0, \tag{48}$$

for real-valued functions $\tilde{u}$ and $\tilde{v}$. Then $\tilde{u}$ and $\tilde{v}$ need to satisfy

$$\nabla^2 \tilde{U} + (\tilde{\omega} + \tilde{V})^2 \tilde{U} - M^2 \, W'\left(\frac{\tilde{U}^2}{M^2}\right)\tilde{U} \;=\; 0, \quad \nabla^2 \tilde{V} \;=\; (\tilde{\omega} + \tilde{V})\, \tilde{U}^2. \tag{49}$$

If a solution $U = f(r)$ and $V = g(r)$ has been found for Eqs. (44) and (45) with a certain value of ω, then

$$\tilde{U} \;=\; M \, f(Mr), \quad \tilde{V} \;=\; -M \, g(Mr) \tag{50}$$

gives a solution for Eq. (49) with $\tilde{\omega} = M\omega$. We are now looking for a positively charged soliton, so we have purposely introduced the minus sign in the solution form (48) so as to obtain clockwise rotation in the complex plane but still allow $\tilde{\omega}$ to be taken as positive. The new solution has one positive unit of elementary charge, but its energy is larger by a factor of M than that of the electron-like soliton (43). We want to interpret a solution of the form (48) as a proton, so we will refer to ϕ as the proton field, and take M to be the appropriate value, approximately 1836, to give the desired mass ratio between the proton and electron.

Solitons in the electron field ψ and the proton field ϕ interact at a distance through the electromagnetic field. It turns out that, as desired, like charges repel and opposite charges attract, and the magnitude of the interaction agrees with Coulomb's law to a good approximation when W satisfies an appropriate condition. Details are available in [24]. By contrast, some early attempts to investigate interaction of solitons [48, 49] led to obviously undesired results such as attraction of like charges.

12. An unexpected result of the model

Although Model One was set up from purely classical considerations to describe the dynamical interaction of electron and proton-like solitons, a startling and perhaps unexpected result occurs. When small oscillations in the electron field occur in the vicinity of a proton-like soliton, it turns out that, because of the nature of the coupling to the electromagnetic field, only the difference frequencies are radiated, and it will be seen that the radiation is just the familiar arc spectrum of hydrogen.

Because of the scaling (50) needed for Eq. (39), the spatial size of a proton-like soliton is much smaller (by a factor of 1836) than that of an electron-like soliton even though the magnitude of the charge is the same. Consequently, in the vicinity of a proton-like soliton, Eq. (36) with $Z = 1$ will be a good approximation, and the electric potential V will be nearly equal to $-\alpha_{fs}/r$ except for very small values of r.

Now let us suppose that small oscillations are excited in the ψ field in the vicinity of a proton-like soliton, so that various modes of oscillation occur. Since the oscillations are small, we can set $W'(U^2) = 1$ and factor Eq. (44) to get

$$\nabla^2 U + (\omega - V - 1)(\omega - V + 1)U = 0, \tag{51}$$

where V now represents the potential well caused by the ϕ field of the proton-like soliton. Any contribution to the V field caused by the ψ field can be neglected if the oscillations in the ψ field are sufficiently small. Let us suppose that $|V|$ is typically much smaller than 1 and that ω is near to 1. Then, we can approximate $\omega - V + 1$ as 2 in the Klein-Gordon Eq. (51) to obtain the usual Schrödinger equation approximation:

$$-\frac{1}{2}\nabla^2 U + VU = (\omega - 1)U. \tag{52}$$

Here, V describes a potential well, but the quantity $\omega - 1$ is related to an actual oscillation frequency ω, which is unlike the traditional use of Schrödinger's equation, where the right-hand side of the equation is regarded as related to an energy.

The main point of interest here is that solutions of the soliton equations lead in a natural way to Eq. (52), which of course results in frequencies of the hydrogen spectrum. For the $V = -\alpha_{fs}/r$ potential well of a proton-like soliton, it is then straightforward to solve Eq. (52) according to the well-known solution of Schrödinger's equation for a hydrogen atom [50]. Eigenfunctions occur in the form

$$U = R(r) P_l^m(\cos\theta)(\cos(m\varphi) + \sin(m\varphi)), \tag{53}$$

where the spherical harmonics are written here in terms of the associated Legendre polynomials P_l^m and the usual spherical coordinates θ and φ. In Eq. (53), the usual factor $e^{im\varphi}$ has been replaced by sinusoids since a real-valued formulation for U is desired. A real-valued solution for the radial function $R(r)$ can be obtained in the standard way. The corresponding frequencies are

$$\omega = 1 - \frac{\alpha_{fs}^2}{2n^2}, \quad n = 1, 2, \dots, \tag{54}$$

with $l = 0, 1, \dots, n-1$ and $m = -l, \dots, l$.

When small oscillations in the electron field ψ are allowed in the vicinity of a proton-like soliton, the overall solution (48) will be slightly changed, so we need to investigate how the electron field source terms

$$S_0 = \frac{i\psi}{2}\frac{\partial\bar{\psi}}{\partial t} - \frac{i\bar{\psi}}{2}\frac{\partial\psi}{\partial t} - A_0\psi\bar{\psi} \tag{55}$$

and

$$\mathbf{S} = \frac{i\psi}{2}\nabla\bar{\psi} - \frac{i\bar{\psi}}{2}\nabla\psi - \mathbf{A}\psi\bar{\psi} \tag{56}$$

in Eqs. (41) and (42) affect the electromagnetic field A_0, $\mathbf{A}$. First, let us try $\psi = U(x, y, z)e^{i\omega t}$, where U is a solution of the form (53) and ω is the corresponding frequency given by Eq. (54). Since the oscillation is to be small, we will assume that an appropriate small constant is absorbed in $R(r)$. Substitution in Eqs. (55) and (56) shows that $S_0 = (\omega - V)U^2$ and $\mathbf{S} = 0$. Since the time dependence has canceled out, we conclude that no radiation into the electromagnetic field occurs

when the ψ field is restricted to a single mode of oscillation. The nonzero value of S_0 causes a change in the static V potential field, but the change in the potential well is small since the ψ oscillation is taken as small (in the evaluation of Eqs. (55) and (56) for present purposes, we have used the original values of A_0 and $\mathbf{A}$ from the proton solution (48) since any change in A_0 and $\mathbf{A}$ caused by the ψ oscillation will be of higher order).

Next, let us consider the case where small oscillations occur in two different modes in the ψ field, so that

$$\psi = U(x, y, z) \, e^{i\omega t} + \hat{U}(x, y, z) \, e^{i\hat{\omega} t}. \tag{57}$$

Substitution in Eqs. (55) and (56) shows

$$S_0 = (\omega - V) \, U^2 + (\hat{\omega} - V) \, \hat{U}^2 + (\omega + \hat{\omega} - 2V) \, U\hat{U} \cos(\omega - \hat{\omega})t, \tag{58}$$

$$\mathbf{S} = (\hat{U} \, \nabla U - U \, \nabla \hat{U}) \sin(\omega - \hat{\omega})t. \tag{59}$$

Since S_0 and $\mathbf{S}$ are now time-varying it is clear that radiation into the electromagnetic field is a possibility, but from the form of the sine and cosine terms it turns out that the radiation can only occur at the difference frequency $\omega - \hat{\omega}$.

It should be noted that charge is not necessarily quantized in Model One, and that the small oscillations in the ψ field indeed represent a small amount of charge. In this example, the V field is not to be noticeably changed from that of the proton-like soliton, however, so the small oscillations in the ψ field must correspond to only a minuscule amount of charge, much less than one elementary charge. Then, Eqs. (41) and (42) are to be solved using the source terms (55), (56) in the form (58), (59). Since charge cannot be radiated into the electromagnetic field, small amounts of charge must be transferred between the trapped modes in the ψ field, and the charge must go to those modes in which the ratio of energy to charge is smaller. It is expected that there will be a selection rule since the two modes in question will not effectively radiate into the electromagnetic field in some cases because of their relative symmetry.

Thus, we find in Model One that radiation is emitted into the electromagnetic field when small oscillations in the ψ field occur in the potential well created by a proton-like soliton. That radiation turns out to be the well-known spectrum of hydrogen, namely, difference frequencies of the form $\omega - \hat{\omega}$, where ω and $\hat{\omega}$ are given by Eq. (54) with different values of n.

13. Interpretation and summary

Although Model One, Eqs. (34) and (40)–(42), was set up to let solitons mimic the dynamical behavior of electrons and protons, it turns out that small oscillations of the electron field ψ in the vicinity of a proton-like soliton have modes of oscillation that correspond to the various terms in the Grotrian diagram for hydrogen. Thus, a quantum-like phenomenon automatically occurs even though the model is of a purely classical nature. It has seemed

in the past that such a result was beyond the realm of a possible classical model since physical oscillation of a charge would presumably radiate away energy at the actual oscillation frequency, whereas only difference frequencies are in fact observed in the hydrogen spectrum.

Here, it should be noted that three circumstances intrinsic to Model One contribute to allow this quantum-like result to occur. First, the standard, i.e., gauge invariant, coupling of the ψ and ϕ fields to the electromagnetic field dictates that the electric potential created by the proton-like soliton is experienced by the ψ field as a potential well that allows trapped oscillations. Secondly, the great difference in scale (by a factor of 1836) needed in Eq. (50) assures that the ψ field automatically experiences almost a pure $V = -\alpha_{fs}/r$ potential well, which leads to frequencies that correspond to the well-known terms in the Grotrian diagram for hydrogen. Finally, standard coupling of the ψ field to the electromagnetic field again automatically implies that only the difference frequencies are radiated.

The emission of the hydrogen spectrum in Model One has some interesting aspects but it occurs in an unexpected context. When oscillations of the ψ field occur in the vicinity of a proton-like soliton, only small oscillations are allowed or the $V = -\alpha_{fs}/r$ potential well created by the proton-like soliton would be substantially changed and the spectrum typical of hydrogen would not be observed. Thus, in the situation in question, an electron-like soliton need not, indeed must not, be in the vicinity at all. When we investigate Model One, then, whether we like it or not, within that model the hydrogen spectrum is emitted from the vicinity of an isolated proton-like soliton. In other words, in Model One, the hydrogen arc spectrum is emitted from the equivalent of a hydrogen *ion* rather than a neutral hydrogen atom, which is in startling contrast to the accepted understanding of the actual physical phenomenon. It remains to be seen whether such a result might have any possible physical significance, but it seems of interest, nevertheless, to see how a purely classical model can behave in a way suggestive of a quantum mechanical phenomenon.

Relativistic effects have not been considered in the above treatment of Model One since it is well-known that the Klein-Gordon equation gives wrong answers for fine-structure corrections, especially those that involve the anomalous Zeeman effect. It is contemplated that the NKG equations might be replaced with nonlinear Dirac equations in a more advanced Model Two. Such a model will be comparatively difficult to examine, however, so it seems that further study of Model One will still be of interest.

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