We are IntechOpen,
the world's leading publisher of
Open Access books
Built by scientists, for scientists

6,900
Open access books available

185,000
International authors and editors

200M
Downloads

154
Countries delivered to

Our authors are among the

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

CLARIVATE ANALYTICS
BOOK CITATION INDEX
INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

A Perturbation Theory for Nonintegrable Equations with Small Dispersion

Georgy Omel'yanov

Additional information is available at the end of the chapter

Abstract

We describe an approach called the "weak asymptotics method" to construct multisoliton asymptotic solutions for essentially nonintegrable equations with small dispersion. This paper contains a detailed review of the method and a perturbation theory to describe the interaction of distorted solitons for equations with small perturbations. All constructions have been realized for the gKdV equation with the nonlinearity u^μ, $\mu \in (1, 5)$.

Keywords: generalized Korteweg-de Vries equation, soliton, interaction, perturbation, weak asymptotics method

2010 Mathematics Subject Classification: 35D30, 35Q53, 46F10

1. Introduction

We consider the problem of propagation and interaction of soliton-type solutions of nonlinear equations. Our basic example is the nonhomogeneous version of the generalized KdV equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^\mu}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = f\left(u, \varepsilon \frac{\partial u}{\partial x}\right), \quad x \in R^1, \quad t > 0, \tag{1}$$

where $\mu \in (1, 5)$, $\varepsilon \ll 1$, $f(u, z)$ is a known smooth function such that $f(0, 0) = 0$. Note that the restriction on μ implies both the soliton-type solution and the stability of the equation with respect to initial data (see, for example [1, 2]).

In the special case $f \equiv 0$ and $\mu = 2$ ($\mu = 3$), Eq. (1) is the famous KdV (modified KdV) equation. It is well known that KdV (mKdV) solitons are stable and interact in the elastic manner: after

the collision, they preserve the original amplitudes and velocities shifting the trajectories only (see [3] and other bibliographies devoted to the inverse scattering transform (IST) method). In the case of $\mu = 2$ ($\mu = 3$) but with $f \neq 0$, Eq. (1) is a nonintegrable one. However, using the smallness of ε (or of f for other scaling), it is possible to create a perturbation theory that describes the evolution of distorted solitons (see the approaches by Karpman and E. Maslov [4] and Kaup and Newell [5] on the basis of the IST method, and the "direct" method by V. Maslov and Omel'yanov [6]). Moreover, the approach by V. Maslov and Omel'yanov [6] can be easily extended to essentially nonintegrable equations ($\mu \neq 2, 3$), but for a single soliton only. In fact, it is impossible to use any direct method in the classical sense for the general problem of the wave interaction. To explain this proposition, let us consider the homogeneous gKdV equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^\mu}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in R^1, \quad t > 0. \tag{2}$$

It is easy to find the explicit soliton solution of (2),

$$u(x, t, \varepsilon) = A\omega(\beta(x - Vt)/\varepsilon), \quad \omega(\eta) = \cosh^{-\gamma}(\eta/\gamma), \tag{3}$$

$$\gamma = 2/(\mu - 1), \quad V = \beta^2, \quad A^{\mu-1} = V(\mu + 1)/2. \tag{4}$$

Next let us consider two-soliton initial data

$$u\mid_{t=0} = \sum_{i=1}^{2} A_i \, \omega(\beta_i(x - x_{(i,0)})/\varepsilon), \tag{5}$$

where $x_{(1,0)} > x_{(2,0)}$ and $A_2 > A_1$. Obviously, since $(x_{(2,0)} - x_{(1,0)})/\varepsilon \to \infty$ as $\varepsilon \to 0$, the sum of the waves (3)

$$u = \sum_{i=1}^{2} A_i \, \omega(\beta_i(x - V_i t - x_{(i,0)})/\varepsilon) \tag{6}$$

approximates the problem (2), (5) solution with the precision $O(\varepsilon^\infty)$ but for $t \ll 1$ only. Conversely, the sum (6) does not satisfy the gKdV equation for $t \sim O(1)$ in view of the trajectories $x = V_i t + x_{(i,0)}$ intersection at a point (x^*, t^*).

Let us consider shortly how it is possible to analyze the problem (2), (5). There are some different cases:

1. Let $A_1 \ll A_2$. Then, one can construct an asymptotic solution

$$u = W((x - \varphi_2(t))/\varepsilon, t, x, \varepsilon, \nu), \tag{7}$$

where $\nu = A_1/A_2 \ll 1$ and $W((x - \varphi_2(t))/\varepsilon, t, x, \varepsilon, \nu) = A_2 \omega(\beta_2(x - V_2 t - x_{(2,0)})/\varepsilon) + O(\nu + \varepsilon)$. Thus, to find the leading term of the asymptotics, we obtain an equation with nonlinear ordinary differential operator; whereas to construct the corrections, it is enough to analyze the linearization of this operator. This construction (with a little bit of other viewpoints) has been realized by Ostrovsky et al. [7].

2. Let $A_2 - A_1 \ll 1$. We write again the ansatz in the form (7), where $v = A_2 - A_1 \ll 1$ now, and we assume $v/\varepsilon \ll 1$. In fact, this case coincides with the problem considered in [7].

3. The amplitudes $A_2 > A_1$ are arbitrary numbers. Then, we should write a two-phase ansatz

$$u(x, t, \varepsilon) = W\big((x - \varphi_1(t))/\varepsilon, (x - \varphi_2(t))/\varepsilon, t, x, \varepsilon\big) \tag{8}$$

without any additional parameter. Substituting (8) into equation (2), we obtain for the leading term $W_0(\tau_1, \tau_2, t)$:

$$\hat{A}W_0 + \hat{B}W_0^\mu + \hat{B}^3 W_0 = 0, \quad \hat{A} = -\sum_{i=1}^2 \dot\varphi_i \frac{\partial}{\partial \tau_i}, \quad \hat{B} = \sum_{i=1}^2 \frac{\partial}{\partial \tau_i}. \tag{9}$$

Since $\dot\varphi_1 \neq \dot\varphi_2$, we can pass to new variables, $\eta = (\tau_1 - \tau_2)/(\dot\varphi_2 - \dot\varphi_1)$, $\zeta = (\dot\varphi_1 \tau_2 - \dot\varphi_2 \tau_1)/(\dot\varphi_1 - \dot\varphi_2)$, and transform equation (9) to the gKdV form (2) again

$$\frac{\partial W_0}{\partial \eta} + \frac{\partial W_0^\mu}{\partial \zeta} + \frac{\partial^3 W_0}{\partial \zeta^3} = 0. \tag{10}$$

Therefore, to construct two-phase asymptotics, we should solve (10) explicitly what is impossible for any essentially nonintegrable case.

This difficulty can be overcome by using the weak asymptotics method. The main point here is that solitons tend to distributions as $\varepsilon \to 0$. Thus, it is possible to pass to the weak description of the problem, ignore the actual shape of the multiwave solutions, and find only the main solution characteristics, that is, the time dynamics of wave amplitudes and velocities. The weak asymptotics method has been proposed at first for shock wave type solutions [8] and for soliton-type solutions [9] many years ago. Further generalizations, modifications, and adaptations to other problems can be found in publications by M. Colombeau, Danilov, Mitrovic, Omel'yanov, Shelkovich, and others, see, for example, [10–20] and references therein.

The contents of the paper are the following: in Section 2, we present a detailed survey of the weak asymptotics method application to the problem of multisoliton asymptotics and Section 3 contains new results, namely a perturbation theory to describe the evolution and collision of distorted solitons for equation (1).

2. Weak asymptotics method

2.1. Main definitions

Let us associate equation (2) with first two conservation laws written in the differential form:

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} = \varepsilon^2 \frac{\partial^3 R_j}{\partial x^3}, \quad j = 1, 2, \tag{11}$$

$$Q_1 = u, \quad P_1 = u^\mu, \quad Q_2 = u^2, \quad P_2 = 2\mu\, u^{\mu+1}/(\mu + 1) - 3\,(\varepsilon\, u_x)^2, \tag{12}$$

and $R_1 = u$, $R_2 = u^2$. Next, we define smallness in the weak sense:

Definition 1. *A function $v(t, x, \varepsilon)$ is said to be of the value $O_{\mathcal{D}'}(\varepsilon^x)$ if the relation $\int_{-\infty}^{\infty} v(t, x, \varepsilon)\Psi(x)dx = O(\varepsilon^x)$ holds uniformly in t for any test function $\Psi \in \mathcal{D}(\mathbb{R}_x^1)$. The right-hand side here is a C^∞-function for $\varepsilon = const > 0$ and a piecewise continuous function uniformly in $\varepsilon \geq 0$.*

Following [9, 17, 18], we define two-soliton weak asymptotics:

Definition 2. *A sequence $u(t, x, \varepsilon)$, belonging to $C^\infty\big(0, T; C^\infty(\mathbb{R}_x^1)\big)$ for $\varepsilon = const > 0$ and belonging to $C\big(0, T; \mathcal{D}'(\mathbb{R}_x^1)\big)$ uniformly in ε, is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (2) if the relations (11) hold uniformly in t with the accuracy $O_{\mathcal{D}'}(\varepsilon^2)$.*

Let us consider the interaction of two solitary waves for the model (2) with the initial data (5).

Following [9, 17, 18] again, we write the asymptotic ansatz in the form:

$$u = \sum_{i=1}^{2} G_i(\tau)\omega\big(\beta_i(x - \varphi_i(t, \tau, \varepsilon))/\varepsilon\big), \quad G_i(\tau) = A_i + S_i(\tau). \tag{13}$$

Here $\varphi_i = \varphi_{i0}(t) + \varepsilon\varphi_{i1}(\tau)$, where $\varphi_{i0} = V_i t + x_{(i, 0)}$ are the trajectories of noninteracting solitary waves, $\tau = \psi_0(t)/\varepsilon$ denotes the "fast time", $\psi_0(t) = \beta_1(\varphi_{20}(t) - \varphi_{10}(t))$, and the phase and amplitude corrections φ_{i1}, S_i are smooth functions such that with exponential rates

$$\varphi_{i1}(\tau) \to 0 \quad as \quad \tau \to -\infty, \quad \varphi_{i1}(\tau) \to \varphi_{i1}^\infty = const_i \quad as \quad \tau \to +\infty, \tag{14}$$

$$S_i(\tau) \to 0 \quad as \quad \tau \to \pm\infty. \tag{15}$$

2.2. Two-wave asymptotic construction

To construct the asymptotics, we should calculate the weak expansions of the terms from the left-hand sides of the relations (11). It is easy to check that

$$u = \varepsilon \sum_{i=1}^{2} a_1 \frac{G_i}{\beta_i} \delta(x - \varphi_i) + O_{\mathcal{D}'}(\varepsilon^3), \tag{16}$$

where $\delta(x)$ is the Dirac delta-function. Here and in what follows, we use the notation

$$a_k \stackrel{def}{=} \int_{-\infty}^{\infty} (\omega(\eta))^k d\eta, \quad k > 0, \quad a_2' \stackrel{def}{=} \int_{-\infty}^{\infty}(\omega'(\eta))^2 d\eta. \tag{17}$$

At the same time for any $F(u, \varepsilon\partial u/\partial x) \in C^1$, we have

$$\int_{-\infty}^{\infty} F\bigg(\sum_{i=1}^{2} G_i \omega\Big(\beta_i \frac{x - \varphi_i}{\varepsilon}\Big), \sum_{i=1}^{2} \beta_i G_i \omega'\Big(\beta_i \frac{x - \varphi_i}{\varepsilon}\Big)\bigg)\psi(x)dx$$

$$= \varepsilon \sum_{i=1}^{2} \frac{1}{\beta_i} \int_{-\infty}^{\infty} F(A_i \omega(\eta), \beta_i A_i \omega'(\eta))\psi\Big(\varphi_i + \varepsilon\frac{\eta}{\beta_i}\Big)d\eta$$

$$+ \frac{\varepsilon}{\beta_2} \int_{-\infty}^{\infty} \bigg\{ F\Big(\sum_{i=1}^{2} G_i \omega(\eta_{i2}), \sum_{i=1}^{2} \beta_i G_i \omega'(\eta_{i2})\Big)$$

$$- \sum_{i=1}^{2} F(A_i \omega(\eta_{i2}), \beta_i A_i \omega'(\eta_{i2})) \bigg\}\psi\Big(\varphi_2 + \varepsilon\frac{\eta}{\beta_2}\Big)d\eta, \tag{18}$$

where

$$\eta_{12} = \theta\eta - \sigma, \quad \eta_{22} = \eta, \quad \sigma = \beta_1(\varphi_1 - \varphi_2)/\varepsilon, \quad \theta = \beta_1/\beta_2. \tag{19}$$

We take into account that the second integrand in the right-hand side of (18) vanishes exponentially fast as $|\varphi_1 - \varphi_2|$ grows; thus, its main contribution is at the point x^*. We write

$$\varphi_{i0} = x^* + V_i(t - t^*) = x^* + \varepsilon V_i \tau/\dot{\psi}_0 \quad \text{and} \quad \varphi_i = x^* + \varepsilon\chi_i, \tag{20}$$

where $\dot{\psi}_0 = \beta_1(V_2 - V_1)$, $\chi_i = V_i\tau/\dot{\psi}_0 + \varphi_{i1}$. It remains to apply the formula

$$f(\tau)\delta(x - \varphi_i) = f(\tau)\delta(x - x^*) - \varepsilon\chi_i f(\tau)\,\delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \tag{21}$$

which holds for each φ_i of the form (20) with slowly increasing χ_i and for $f(\tau)$ from the Schwartz space. Moreover, the second term in the right-hand side of (21) is $O_{\mathcal{D}'}(\varepsilon)$. Thus, under the assumptions (14) and (15), we obtain the weak asymptotic expansion of $F(u, \varepsilon\partial u/\partial x)$ in the final form:

$$F(u, \varepsilon u_x) = \varepsilon \sum_{i=1}^{2}\left\{\frac{a_{F,i}^{(0)}}{\beta_i}\delta(x - \varphi_i) - \varepsilon\frac{a_{F,i}^{(1)}}{\beta_i}\delta'(x - x^*)\right\} + \frac{\varepsilon}{\beta_2}\left\{\mathfrak{R}_F^{(0)}\delta(x - x^*) - \varepsilon\bar{\mathfrak{R}}_F\delta'(x - x^*)\right\}$$
$$+ O_{\mathcal{D}'}(\varepsilon^3), \quad \bar{\mathfrak{R}}_F = \chi_2\mathfrak{R}_F^{(0)} + \mathfrak{R}_F^{(1)}/\beta_2, \tag{22}$$

where

$$a_{F,i}^{(n)} = \int_{-\infty}^{\infty} \eta^n F(A_i\,\omega(\eta), \beta_i A_i\,\omega'(\eta))d\eta, \tag{23}$$

$$\mathfrak{R}_F^{(n)} = \int_{-\infty}^{\infty} \eta^n\left\{F\left(\sum_{i=1}^{2} G_i\,\omega(\eta_{i2}), \sum_{i=1}^{2}\beta_i G_i\,\omega'(\eta_{i2})\right) - \sum_{i=1}^{2} F(A_i\,\omega(\eta_{i2}), \beta_i A_i\,\omega'(\eta_{i2}))\right\}d\eta. \tag{24}$$

Here, we take into account that to define $\partial u^2/\partial t \bmod O_{\mathcal{D}'}(\varepsilon^2)$, it is necessary to calculate u^2 with the precision $O_{\mathcal{D}'}(\varepsilon^3)$. Thus, using (22) with $F(u) = u^2$ and transforming (16) with the help of (21), we obtain modulo $O_{\mathcal{D}'}(\varepsilon^3)$:

$$u = \varepsilon \sum_{i=1}^{2} a_1 K_{i0}^{(1)}\delta(x - \varphi_i) + \varepsilon \sum_{i=1}^{2} a_1 K_{i1}^{(1)}\{\delta(x - x^*) - \varepsilon\chi_i\delta'(x - x^*)\}, \tag{25}$$

$$u^2 = \varepsilon \sum_{i=1}^{2} a_2 K_{i0}^{(2)}\delta(x - \varphi_i) + \frac{\varepsilon}{\beta_2}\{\mathfrak{R}_{u^2}^{(0)}\delta(x - x^*) - \varepsilon\bar{\mathfrak{R}}_{u^2}\delta'(x - x^*)\}, \tag{26}$$

where

$$K_i^{(n)} = G_i^n/\beta_i, \quad K_{i0}^{(n)} = A_i^n/\beta_i, \quad K_{i1}^{(n)} = K_i^{(n)} - K_{i0}^{(n)}. \tag{27}$$

Calculating weak expansions for other terms from Definition 2 and substituting them into (11), we obtain linear combinations of $\varepsilon\delta'(x - \varphi_i)$, $i = 1, 2$, $\delta(x - x^*)$, and $\varepsilon\delta'(x - x^*)$. Therefore, we pass to the system:

$$a_1 V_i K_{i0}^{(1)} - a_{P_1,i}^{(0)}/\beta_i = 0, \quad a_2 V_i K_{i0}^{(2)} - a_{P_2,i}^{(0)}/\beta_i = 0, \quad i = 1, 2, \tag{28}$$

$$\sum_{i=1}^{2} K_{i1}^{(1)} = 0, \quad \mathfrak{R}_{u^2}^{(0)} = 0, \quad i = 1, 2, \tag{29}$$

$$\dot{\psi}_0 \frac{d}{d\tau} \sum_{i=1}^{2} \{K_{i0}^{(1)} \varphi_{i1} + \chi_i K_{i1}^{(1)}\} = f, \quad \dot{\psi}_0 \frac{d}{d\tau} \left\{ \sum_{i=1}^{2} a_2 K_{i0}^{(2)} \varphi_{i1} + \bar{\mathfrak{R}}_{u^2} \right\} = F, \tag{30}$$

where

$$f = \frac{1}{a_1 \beta_2} \mathfrak{R}_{P_1}^{(0)}, \quad F = \frac{1}{\beta_2} \mathfrak{R}_{P_2}^{(0)} - a_1 \dot{\psi}_0 \sum_{i=1}^{2} \varphi_{i1} \frac{dK_{i1}^{(2)}}{d\tau}. \tag{31}$$

The first four algebraic equations (28) imply again the relation (4) among A_i, β_i, and V_i. Furthermore, there exists a number $\theta^* \in (0, 1)$ such that equations (29), (30) have the required solution S_i, φ_{i1} with the properties (14) and (15) under the sufficient condition $\theta \le \theta^*$ (see [9, 17]). It is obvious that the existence of the weak asymptotics (13) with the properties (14) and (15) implies that the solitary waves interact like the KdV solitons at least in the leading term.

Theorem 1. Let $\theta \le \theta^*$. Then (13) describes mod $O_{D'}(\varepsilon^2)$ the elastic scenario of the solitary waves interaction for the μ-gKdV equation (2).

Numerical simulations ([14, 15, 17]) confirm the traced analysis, see **Figure 1**. Note that a small oscillating tail appears after the soliton collision, see [15] for detail. Obviously, this effect is similar to the "radiation" appearance for the perturbed KdV [21].

2.3. Multisoliton interaction

N-wave solutions of the form similar to waves (13) contain $2N$ free functions S_i, φ_{i1}. Thus, to describe an N-soliton collision, we should consider N conservation laws. However, nonintegrability implies the existence of a finite number of conservation laws only. For this reason, we need to involve into the consideration balance laws. For the gKdV-4 equation, the first conservation and balance laws have the form

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + \varepsilon^{-1} K_j = O_{D'}(\varepsilon^2), \tag{32}$$
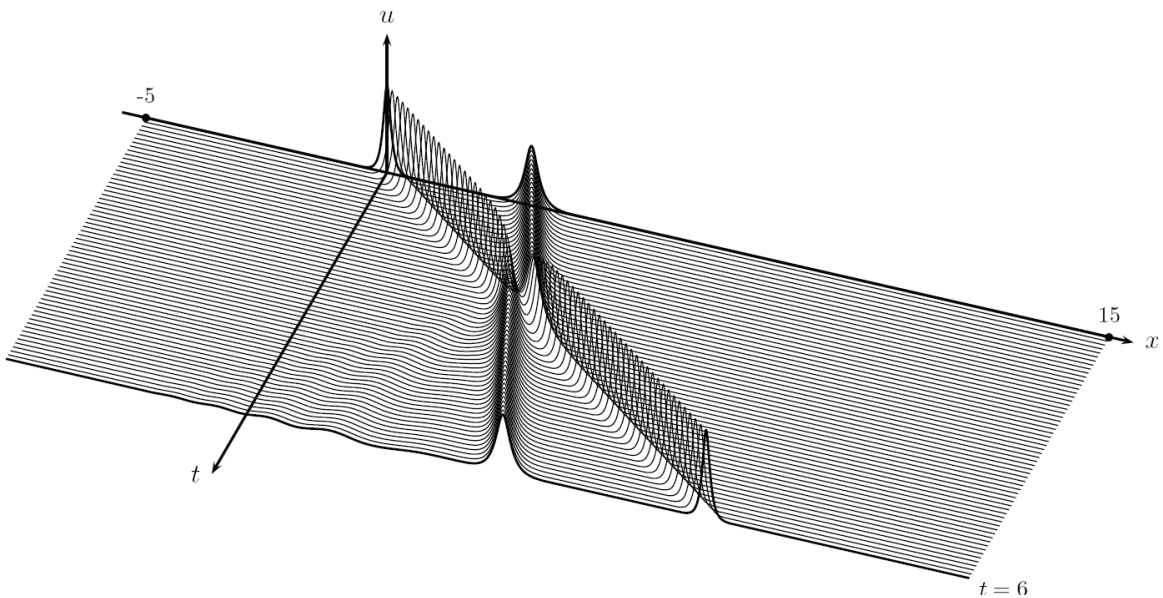

Figure 1. Evolution of two solitary waves for $\mu = 4$ and $\varepsilon = 0.1$.

where Q_j, P_j, $j = 1, 2$, coincide with (12) for $\mu = 4$, $K_i = 0$, $i = 1, 2, 3$,

$$Q_3 = (\varepsilon u_x)^2 - \frac{2}{5} u^5, \quad P_3 = 16 u^3 (\varepsilon u_x)^2 - u^8 - 3 (\varepsilon^2 u_{xx})^2, \tag{33}$$

$$Q_4 = \frac{1}{2} (\varepsilon^2 u_{xx})^2 + \frac{5}{21} u^8 - \frac{10}{3} u^3 (\varepsilon u_x)^2, \quad K_4 = -(\varepsilon u_x)^5, \tag{34}$$

$$P_4 = 12 u^3 (\varepsilon^2 u_{xx})^2 - 19 u (\varepsilon u_x)^4 - \frac{3}{2} (\varepsilon^3 u_{xxx})^2 + \frac{160}{231} u^{11} - \frac{100}{3} u^6 (\varepsilon u_x)^2. \tag{35}$$

Note that the nondivergent "production" $\varepsilon^{-1} K_4$ has the same value $O(\varepsilon^{-1})$ (in the C-sense and for rapidly varying functions) as the first ones in (32).

The formal scheme of the asymptotic construction is similar to the one described above: we write the ansatz of the form (13) but with N summands, found weak representations for all terms in (32), and pass to a system similar to (28)–(30). The main obstacle here is the proof that this system admits a solution with the properties of (14), (15). This idea has been realized in [18, 19] for the problem of three soliton collisions for the gKdV-4 equation.

Theorem 2. *Let us denote A_i the amplitudes of the original solitons and $x_{(i, 0)}$ their initial positions such that $A_{i+1} > A_i$, $x_{(i, 0)} > x_{(i+1, 0)}$, and $i = 1, 2$. Let all trajectories $x = \varphi_{i0}(t)$ have an intersection point (x^*, t^*). Then, under the assumption*

$$\beta_2 / \beta_3 = v^3, \quad \beta_1 / \beta_3 = v^{3(3+\alpha)/2}, \quad \alpha \in [0, 1) \tag{36}$$

with sufficiently small $v < 1$, the three-phase asymptotic solution exists and describes $\bmod\, O_{\mathscr{D}'}(\varepsilon^2)$ the elastic scenario of the solitary waves interaction.

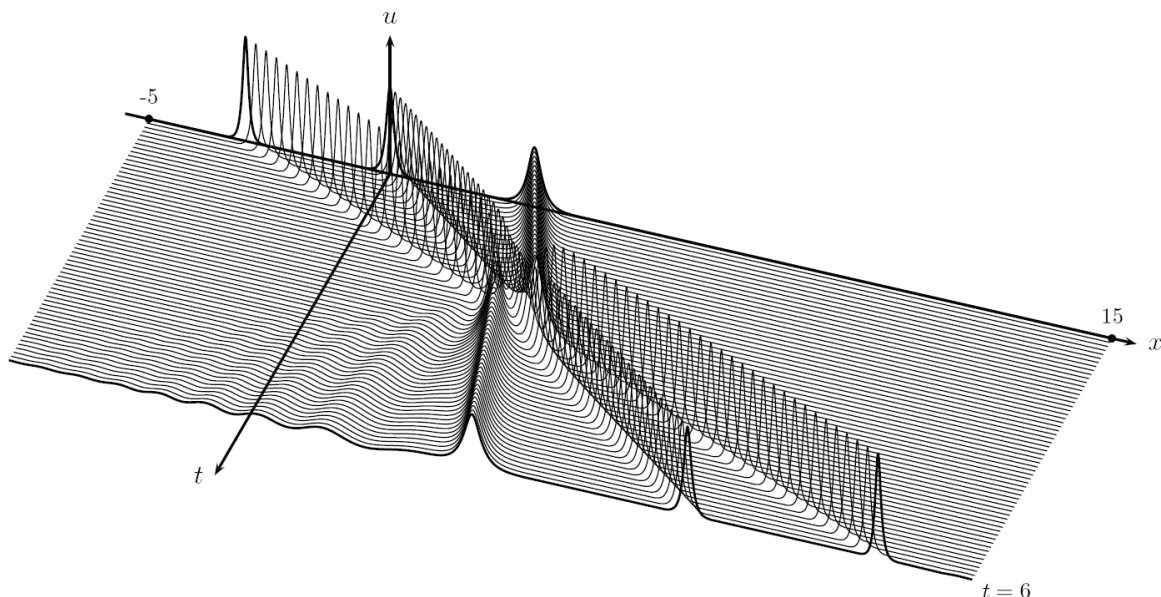**Figure 2** depicts the evolution of a three-wave solution [14].

Figure 2. Evolution of the soliton triplet with $\mu = 4$, $\varepsilon = 0.1$.

2.4. Asymptotic equivalence

Let us come back to the case of two-phase asymptotics and transform the ansatz (13) to the following form:

$$\tilde{u} = \sum_{i=1}^{2}\left\{G_i(\tau)\omega\left(\beta_i\frac{x-\varphi_i(t,\tau,\varepsilon)}{\varepsilon}\right) + \mathfrak{S}_i(\tau)W\left(\beta_i\frac{x-\varphi_i(t,\tau,\varepsilon)}{\varepsilon}\right)\right\}, \tag{37}$$

where $\mathfrak{S}_i(\tau)$, $i = 1, 2$ are arbitrary functions from the Schwartz space,

$$W(\eta) = d^{2l+1}\omega(\eta)/d\,\eta^{2l+1}, \tag{38}$$

and $l \geq 1$ is an arbitrary integer. Calculating the weak representations for $\tilde{u}$ and $\tilde{u}^2$, we obtain

$$\tilde{u} = u + O_{\mathcal{D}'}(\varepsilon^{2l+2}),$$
$$\tilde{u}^2 = \varepsilon\sum_{i=1}^{2}a_2 K_{i0}^{(2)}\delta(x-\varphi_i) + \frac{\varepsilon}{\beta_2}\mathfrak{R}_{\tilde{u}^2}^{(0)}\delta(x-x^*) + O_{\mathcal{D}'}(\varepsilon^2), \tag{39}$$

where

$$\mathfrak{R}_{\tilde{u}^2}^{(0)} = \int_{-\infty}^{\infty}\left\{\left(\sum_{i=1}^{2}(G_i\,\omega(\eta_{i2}) + \mathfrak{S}_i\,W(\eta_{i2}))\right)^2 - \sum_{i=1}^{2}(A_i\,\omega(\eta_{i2}))^2\right\}d\eta, \tag{40}$$

and u in the right-hand side in (39) is the representation (25). Thus, the difference between u of the forms (13) and (37) is arbitrarily small in the sense $\mathcal{D}'(\mathbb{R}_x)$. At the same time, instead of (29), (30), we obtain

$$\sum_{i=1}^{2}K_{i1}^{(1)} = 0, \quad \mathfrak{R}_{\tilde{u}^2}^{(0)} = 0, \quad i = 1, 2, \tag{41}$$

$$\psi_0\frac{d}{d\tau}\sum_{i=1}^{2}\left\{K_{i0}^{(1)}\varphi_{i1} + \chi_i K_{i1}^{(1)}\right\} = \tilde{f}, \quad \psi_0\frac{d}{d\tau}\left\{\sum_{i=1}^{2}a_2 K_{i0}^{(2)}\varphi_{i1} + \overline{\mathfrak{R}}_{\tilde{u}^2}\right\} = \tilde{F}, \tag{42}$$

where $\tilde{f}$, $\tilde{F}$ differ from f, F in the same manner as $\mathfrak{R}_{\tilde{u}^2}^{(0)}$ differs from $\mathfrak{R}_{u^2}^{(0)}$. The system (41) and (42) have again a solution with the properties (14) and (15) [9, 12]; however, it differs from the solution of Eqs. (29) and (30) with the value $O(1)$ in the C-sense. Moreover, the asymptotic solutions (13) and (37) differ with the precision $O_{\mathcal{D}'}(\varepsilon)$ in the sense of Definition 1. This implies the principal impossibility to describe explicitly neither the real shape of the waves at the time instant of the collision nor the real ε-size displacements of the trajectories after the interaction. However, the nonuniqueness of the value $O(\varepsilon)$ is concentrated within $O(\varepsilon^{1-\nu})$-neighborhood of the time instant t^* of the interaction, $\nu > 0$. Thus, it is small in the $\mathcal{D}'(R_{x,t}^2)$ sense. We set

Definition 3. *Functions $u_1(x, t, \varepsilon)$ and $u_2(x, t, \varepsilon)$ are said to be asymptotically equivalent if for any test function $\psi \in \mathcal{D}(R^2)$*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{u_1(x,t,\varepsilon) - u_2(x,t,\varepsilon)\}\psi(x,t)dx\,dt = O(\varepsilon^2). \tag{43}$$

In this sense, the solutions (13) and (37) are asymptotically equivalent.

We now focus attention on another question: how to choose, from the set of all possible conservation and balance laws, those that allow to construct a multiphase asymptotic solution? It seems

that there is not any rule and it is possible to use arbitrary combination of the laws. Thus, there appears the next question: what is the difference between such solutions? This problem has been discussed in [20] for two-phase asymptotic solutions of the gKdV-4 equation. Let us define two-phase asymptotics in the following manner:

Definition 4. *Let $1 \le k_0 < k_1 \le 4$ and let a sequence $u_{k0, k1} = u_{k0, k1}(t, x, \varepsilon)$ belong to the same functional space as $u(t, x, \varepsilon)$ in Definition 2. Then, $u_{k0, k1}$ is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (2) if the relations (32) hold for $j = k_0$ and $j = k_1$ uniformly in t.*

A detailed analysis implies the assertion [20].

Theorem 3. *Let θ be sufficiently small. Then, the weak asymptotic solutions $u_{1, k1}$ and $u_{1, k_1'}$ of the problem (2), (5) exist and they are asymptotically equivalent for all $k_1, k_1' \in \{2, 3, 4\}$.*

3. Collision of distorted solitons

We consider now the nonhomogeneous version of the gKdV equation (1). It is easy to verify that, in the case of rapidly varying solutions, the right-hand side *f* can be treated as a "small perturbation."

An approach to construct one-phase self-similar asymptotic solutions for (1) had been created in [6] (see also [17]). Let us generalize this approach to the multiphase case. From the beginning, we state that equation (1) is associated with balance laws, the first two of which are

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + K_j = O_{\mathcal{D}'}(\varepsilon^2), \quad j = 1, 2, \tag{44}$$

where Q_j and P_j coincide with ones described in (12),

$$K_1 = -f(u, \varepsilon u_x), \quad K_2 = -uf(u, \varepsilon u_x). \tag{45}$$

Note that, in contrast to K_j in (32), productions here are regularly degenerating functions with the value $O(1)$ in the *C*-sense.

Let us first construct a two-phase version of self-similar asymptotics, which assumes a special initial data for (1) and discuss afterward how to treat it for more realistic initial data. By analogy with Definition 2, we write:

Definition 5. *Let a sequence $u = u(t, x, \varepsilon)$ belong to the same functional space as in Definition 2. Then u is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^{\bar{q}})$ solution of (1) if the relation (44) hold uniformly in $t \in (0, T)$, $\bar{q} = \min\{\mu, 2\}$.*

Generalizing one-phase asymptotics, we write the ansatz as

$$u = \sum_{i=1}^{2} \{ G_i(\tau, t)\omega(\eta_i) + \varepsilon(z_i(x, t)\mathfrak{H}(\eta_i) + \mathfrak{G}_i(\tau)\omega(\eta_i)) \}, \tag{46}$$

$$G_i(\tau, t) = A_i(t) + S_i(\tau), \quad \eta_i = \beta_i(t)(x - \varphi_i)/\varepsilon, \quad \varphi_i(t, \tau, \varepsilon) = \varphi_{i0} + \varepsilon\varphi_{i1}. \tag{47}$$

Here $A_i(t)$, $\varphi_{i0} = \varphi_{i0}(t)$, $\beta_i^2(t) = \gamma A_i^{\mu-1}(t)$, $\omega(\eta)$, $S_i(\tau)$, $\varphi_{i1}(\tau)$ are the same as in (13); $\tau = \psi_0(t)/\varepsilon$ with $\psi_0(t) = \varphi_{20}(t) - \varphi_{10}(t)$ denotes the "fast time" again; $z_i(x, t) \in \mathcal{C}^\infty$; and $\mathfrak{G}$, $\mathfrak{H}$ are smooth functions such that

$$\mathfrak{G}_i(\tau) \to 0 \quad as \quad \tau \to -\infty, \quad \mathfrak{G}_i(\tau) \to \mathfrak{G}_i^\infty = const_i \quad as \quad \tau \to +\infty, \tag{48}$$

$$\mathfrak{H}(\eta) \to 1 \quad as \quad \eta \to -\infty, \quad \mathfrak{H}(\eta) \to 0 \quad as \quad \eta \to +\infty \tag{49}$$

with exponential rates. We assume also the intersection of the trajectories $x = \varphi_{i0}(t)$, $i = 1, 2$ at a point $x^* = \varphi_{i0}(t^*)$ namely,

$$\exists t^* > 0 \text{ such that } \varphi_{10}(t^*) = \varphi_{20}(t^*), \dot{\psi}_0 \stackrel{\text{def}}{=} \frac{d}{dt}(\varphi_{20}(t) - \varphi_{10}(t))\Big|_{t=t^*} \neq 0. \tag{50}$$

It is easy to verify the weak representations with the precision $O_{\mathscr{D}'}(\varepsilon^2)$:

$$u = \varepsilon \sum_{i=1}^{2}\left\{a_1 K_{i0}^{(1)} \delta(x - \varphi_i) + a_1 K_{i1}^{(1)*} \delta(x - x^*) + z_i(x, t)H(\varphi_i - x)\right\}, \tag{51}$$

$$
\begin{aligned}
\frac{\partial u}{\partial t} =\ & a_1 \dot{\psi}_0 \frac{\partial}{\partial \tau} \sum_{i=1}^{2} K_{i1}^{(1)*} \delta(x - x^*) + \varepsilon \sum_{i=1}^{2} \frac{\partial z_i}{\partial t} H(\varphi_i - x) \\
& + \varepsilon a_1 \sum_{i=1}^{2}\left\{\frac{dK_{i1}^{(1)}}{dt} + z_i(\varphi_i, t)\frac{d\varphi_{i0}}{dt}\right\}\delta(x - \varphi_i) - \varepsilon a_1 \sum_{i=1}^{2} K_{i0}^{(1)}\frac{d\varphi_{i0}}{dt}\delta'(x - \varphi_i) \\
& + \varepsilon \sum_{i=1}^{2}\left\{\psi_0 \frac{\partial}{\partial t}\left(a_1 \frac{\mathfrak{G}_i}{\beta_i} + z_i(x^*, t)\varphi_{i1}\right) - \frac{a_1}{\beta_i^2} S_i \frac{d\beta_i}{dt}\right\}\Bigg|_{t=t^*} \delta(x - x^*) \\
& - \varepsilon a_1 \sum_{i=1}^{2}\left\{K_{i1}^{(1)}\frac{d\varphi_{i0}}{dt} + \dot{\psi}_0\left(K_i^{(1)}\frac{\partial \varphi_{i1}}{\partial \tau} + \frac{\chi_i}{\beta_i}\frac{\partial K_i^{(1)}}{\partial \tau}\right)\right\}\Bigg|_{t=t^*} \delta'(x - x^*),
\end{aligned}
\tag{52}
$$

where $H(x)$ is the Heaviside function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$; $g^* \stackrel{\text{def}}{=} g(\tau, t)\big|_{t=t^*}$, $\chi_i \stackrel{\text{def}}{=} \varphi_i(\Phi(\varepsilon\tau, t^*), \tau, \varepsilon) - x^*$, and $\Phi(\varepsilon\tau, t^*)$ is the solution of the equation $\varphi_{20}(t^* + \Phi) - \varphi_{10}(t^* + \Phi) = \varepsilon\tau$, which exists in accordance with (50).

Next, the existence of nonsoliton summands in (51) implying a correction of formula (22), namely

$$
\begin{aligned}
F(u, \varepsilon u_x) =\ & \varepsilon \sum_{i=1}^{2} a_{F,i}^{(0)} \beta_i^{-1} \delta(x - \varphi_i) + \varepsilon \beta_2^{-1} \mathfrak{R}_F^{(0)} \delta(x - x^*) \\
& + \varepsilon F_u'(0, 0) \sum_{i=1}^{2} z_i(x, t)H(\varphi_i - x) + O_{\mathscr{D}'}(\varepsilon^\eta),
\end{aligned}
\tag{53}
$$

where $a_{F,i}^{(0)}$ and $\mathfrak{R}_F^{(0)}$ are defined in (23), (24), $F_u'(0, 0) = \partial F(u, 0)/\partial u\big|_{u=0}$.

Repeating the same calculations as above, we obtain linear combinations of $\varepsilon\delta(x - \varphi_i)$, $\varepsilon\delta'(x - \varphi_i)$, $\varepsilon H(\varphi_i - x)$, $i = 1, 2$; $\delta(x - x^*)$, $\varepsilon\delta(x - x^*)$, and $\varepsilon\delta'(x - x^*)$. Equating zero, the coefficients of $\varepsilon\delta(x - \varphi_i)$ and $\varepsilon\delta'(x - \varphi_i)$ yield

$$a_1 A_i \frac{d\varphi_{i0}}{dt} = a_{P_i,i}^{(0)}, \quad a_2 \frac{d}{dt}\frac{A_i^2}{\beta_i} = \frac{a_{K_i,i}^{(0)}}{\beta_i}, \tag{54}$$

$$a_2 A_i^2 \frac{d\varphi_{i0}}{dt} = a_{P_2,i}^{(0)}, \quad a_1 \frac{d}{dt} \frac{A_i}{\beta_i} + z_i(\varphi_i, t) \frac{d\varphi_{i0}}{dt} = \frac{a_{f,i}^{(0)}}{\beta_i}. \tag{55}$$

Equation (54) forms the closed system to define $A_i(t)$ and $\varphi_{i0}(t)$. To simplify it, let us use the equalities (4) and rewrite the model equation for $\omega(\eta)$ as follows:

$$\frac{d}{d\eta} \left\{ -\omega + \frac{\mu+1}{2} \omega^\mu + \frac{d^2\omega}{d\eta^2} \right\} = 0. \tag{56}$$

Simple manipulations with (56) allow us to find relations between structural constants:

$$a_1 = (\mu+1) a_\mu / 2, \quad a_2 = (\mu+3) a_{\mu+1} / 4, \quad a_2' = (\mu-1) a_{\mu+1} / 4. \tag{57}$$

Next, we use (57), the equality $\beta_i^2 = \gamma A_i^{\mu-1}$, add the initial conditions, and obtain from (54) the Cauchy problem

$$\frac{dA_i}{dt} = -c_1 a_{K_2,i}^{(0)} A_i^{-1}, \quad \frac{d\varphi_{i0}}{dt} = \frac{a_\mu}{a_1} A_i^{\mu-1}, \quad t > 0, \tag{58}$$

$$A_i \big|_{t=0} = A_i^0, \quad \varphi_{i0} \big|_{t=0} = x_{(i,0)}, \tag{59}$$

where $c_1 = 2/(a_2(5-\mu))$; $A_i^0 > 0$ and $x_{(i,0)}$ are arbitrary numbers; and $i = 1, 2$. Note also that the first equalities in equations (54) and (55) are equivalent.

Next, equating zero the coefficients of the Heaviside functions, we obtain the equations

$$\frac{\partial z_i}{\partial t} = f_u(0,0) z_i, \quad x < \varphi_{i0}(t), \quad t > 0, \quad i = 1, 2. \tag{60}$$

In view of (58) $d\varphi_{i0}/dt > 0$, so we use the second equality in (55) to state the correct initial condition for (60)

$$z_i(x,t) \big|_{x=\varphi_{i0}(t)} = \sqrt{\gamma} a_{f,i}^{(0)} A_i^{(3-4\mu)/2}(t) + c_2 a_{K_2,i}^{(0)} A_i^{(1-3\mu)/2}(t), \quad t > 0, \tag{61}$$

$$z_i(x,t) \big|_{t=0} = z_i^0(x), \quad x \leq x_{(i,0)}, \tag{62}$$

where $c_2 = a_1(3-\mu)(1+\mu)/(2a_2(5-\mu))$, $z_i^0(x)$ is an arbitrary smooth function, which satisfies the consistency condition

$$z_i^0(x_{(i,0)}) = \left\{ \sqrt{\gamma} a_{f,i}^{(0)} A_i^{(3-4\mu)/2} + c_2 a_{K_2,i}^{(0)} A_i^{(1-3\mu)/2} \right\} \Big|_{t=0}. \tag{63}$$

We should note that the nonlinearity u^μ in (1) can require the inequality $u \geq 0$. To this end, we will assume

$$A_i(t) > 0, \quad z_i(x,t) \big|_{x=\varphi_{i0}(t)} \geq 0 \quad \text{for} \quad t \geq 0. \tag{64}$$

Furthermore, equating zero the coefficients of $\delta(x-x^*)$ and $\varepsilon\delta'(x-x^*)$ yield (29), (30) again. Consequently, the condition $\theta \leq \theta^*$ guaranties the existence of S_i, φ_{i1} with the properties of (14), (15). In particular

$$S_1 = -\theta\, S_2 \quad S_1 = \theta\beta_1^\gamma\, \lambda(\sigma)(1 + O(\theta^q)), \tag{65}$$

where

$$q = min\{1, \gamma\}, \quad \lambda(\sigma) = a_2^{-1} \int_{-\infty}^{\infty} \omega(\eta)\omega(\eta_{12})d\eta. \tag{66}$$

The last step of the construction is the determination of $\mathfrak{G}_i(\tau)$, $i = 1, 2$. By setting the coefficients of $\varepsilon\delta(x - x^*)$ zero, we obtain

$$\frac{\partial}{\partial\tau} \sum_{i=1}^{2} \frac{\mathfrak{G}_i}{\beta_i^*} = \mathfrak{F}_1, \quad \frac{\partial}{\partial\tau} \sum_{i=1}^{2} \left(\frac{G_i}{\beta_i} + \frac{G_{\bar{\imath}}}{\beta_2}\lambda(\sigma)\right)\bigg|_{t=t^*} \mathfrak{G}_i = \mathfrak{F}_2, \tag{67}$$

where $\bar{\imath} = 2$ for $i = 1$ and $\bar{\imath} = 1$ for $i = 2$,

$$\mathfrak{F}_1 = -\sum_{i=1}^{2}\left\{\frac{z_i(x^*, t)}{a_1}\frac{d\varphi_{i1}}{d\tau} + \dot{\psi}_0^{-1}\frac{\partial}{\partial t}\frac{S_i}{\beta_i}\right\}\bigg|_{t=t^*} + \frac{1}{\beta_2}\mathfrak{R}_f^{(0)}\bigg|_{t=t^*}, \tag{68}$$

$$\mathfrak{F}_2 = -\frac{\partial}{\partial\tau}\sum_{i=1}^{2}\left\{a_H z_i(x^*, t)\frac{S_i}{\beta_i} + z_{\bar{\imath}}(x^*, t)\frac{G_i}{\beta_2}\lambda_{H,i}(\sigma)\right\}\bigg|_{t=t^*} + \frac{\mathfrak{R}_{K_2}^{(0)}}{c_H}\bigg|_{t=t^*}, \tag{69}$$

$$a_H = a_1/(2\,a_2), \quad c_H = 2\,a_2\,\dot{\psi}_0\beta_2, \quad \lambda_{H,i}(\sigma) = a_2^{-1}\int_{-\infty}^{\infty}\omega(\eta_{i2})\mathfrak{H}(\eta_{\bar{\imath}2})d\eta.$$

Calculating the determinant Δ of the matrix in the left-hand part of (67) and using (65), we conclude

$$\Delta = \left(G_2 - G_1 + \lambda(G_1 - \theta\,G_2)\right)\big|_{t=t^*} = \beta_2^\gamma\left(1 - \theta^\gamma - \lambda(\theta - \theta^\gamma) - \lambda^2\,\theta^\gamma(1 + O(\theta^q))\right)\big|_{t=t^*}. \tag{70}$$

Obviously, $\Delta \neq 0$ for sufficiently small θ. Since the right-hand sides $\mathfrak{F}_i$ belong to the Schwartz space, the functions $\mathfrak{G}_i$ exist and satisfy the assumption (48).

Henceforth, we pass to the final result:

Theorem 4. *Let θ be sufficiently small and let the assumptions (50), (63), and (64), if it is necessary, be fulfilled. Then, the self-similar two-wave weak asymptotic mod $O(\varepsilon^{\bar{q}})$ solution of the equation (1) exists and has the form (46).*

Let us finally stress that the self-similarity implies a special choice of the initial data: for the classical asymptotics in the C-sense, there appears a very restrictive condition for small correction of the soliton $A(0)\omega((x - x_0)/\varepsilon)$ (see [6, 17]), and for weak asymptotics, there appears the restriction (63). If it is violated, then the perturbed soliton generates a rapidly oscillating tail of the amplitude $o(1)$ ("radiation") instead of the smooth tails $\varepsilon u^-(x, t)$ (see [21] and numerical results [14, 15, 17]). Nowadays, this radiation phenomenon can be described analytically only for integrable equations, so that we should use self-similar approximation for essentially nonintegrable equations. However, the smooth tail $\varepsilon u^-(x, t)$, which can be treated
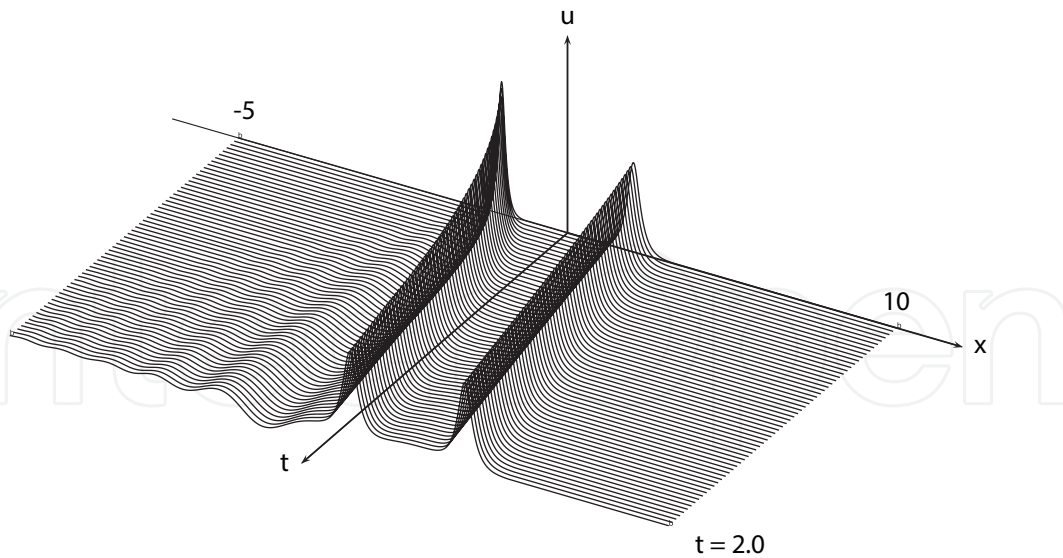
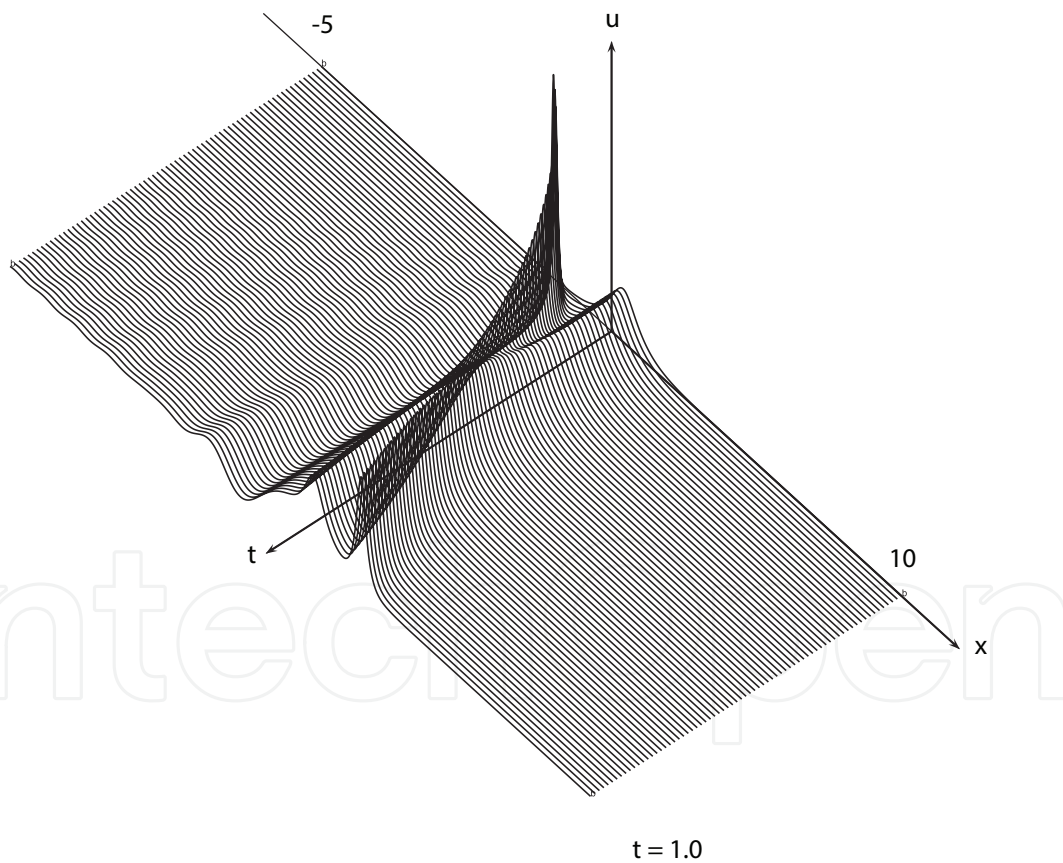**Figure 3.** Example of noninteracting solitary waves, $\mu = 4$, $f = u(1 - u)$.

Figure 4. Example of interacting solitary waves, $\mu = 4$, $f = u(1 - u)$, $\varepsilon = 0.1$.

as an average of the radiation, describes sufficiently well the tendency of the radiation amplitude behavior, see graphics depicted in **Figures 3** and **4**, and other numerical results in [15, 17]).

Acknowledgements

Author details

Georgy Omel'yanov

Address all correspondence to: omel@mat.uson.mx

University of Sonora, Hermosillo, Mexico

References

[1] Bona J, Souganidis P, Strauss W. Stability and instability of solitary waves of Korteweg-de Vries type. Proceedings of the Royal Society of London A. 1987;**411**(1841):395-412. DOI: 10.1098/rspa.1987.0073

[2] Faminskii A. Cauchy problem for the Korteweg-de Vries equation and its generalizations. Journal of Soviet Mathematics. 1990;**50**(1):1381-1420. Available from: https://link.springer.com/article/10.1007/BF01097030

[3] Ablowitz M, Segur H. Solitons and the Inverse Scattering Transform. Philadelphia: SIAM; 1981. Available from: http://dx.doi.org/10.1137/1.9781611970883.bm

[4] Karpman V, Maslov E. Perturbation theory for solitons. Journal of Experimental and Theoretical Physics. 1977;**46**(2):281-291. Available from: http://jetp.ac.ru/cgi-bin/dn

[5] Kaup D, Newell A. Solitons as particles, oscillators and in slowly changing media: A singular perturbation theory. Proceedings of the Royal Society of London Series A. 1978;**361**:413-446. DOI: 10.1098/rspa.1978.0110

[6] Maslov V, Omel'yanov G. Asymptotic soliton-form solutions of equations with small dispersion. Russian Mathematical Surveys. 1981;**36**:73-149. Available from: http://iopscience.iop.org/0036-0279/36/3/R02

[7] Gorshkov K, Ostrovsky L. Interaction of solitons in nonintegrable systems: Direct perturbation method and applications. Physica D. 1981;**3**:428-438. Available from: https://doi.org/10.1016/0167-2789(81)90146-9

[8] Danilov V, Shelkovich V. Generalized solutions of nonlinear differential equations and the Maslov algebras of distributions. Integral Transformations and Special Functions. 1997;**6**:137-146. Available from: http://dx.doi.org/10.1080/10652469808819161

[9] Danilov V, Omel'yanov G. Weak asymptotics method and the interaction of infinitely narrow delta-solitons. Nonlinear Analysis: Theory Methods & Applications. 2003;**54**:773-799 Available from: https://doi.org/10.1016/S0362-546X(03)00104-4

[10] Abreu E, Colombeau M, Panov E. Weak asymptotic methods for scalar equations and systems. Journal of Mathematical Analysis and Applications. 2016;**444**(2):1203-1232. Available from: https://doi.org/10.1016/j.jmaa.2016.06.047

[11] Danilov V, Mitrovic D. Shock wave formation process for a multidimensional scalar conservation law. Quarterly of Applied Mathematics. 2011;**69**(4):613-634. Available from: doi: 10.1090/S0033-569X-2011-01234-9

[12] Danilov V, Omel'yanov G, Shelkovich V. Weak asymptotics method and interaction of nonlinear waves. In: Karasev M, editor. Asymptotic Methods for Wave and Quantum Problems. Trans. Ser. 2. Vol. 208. Providence, RI: American Mathematical Society; 2003. p. 33-164. Available from: https://goo.gl/aPPhqG

[13] Danilov V, Shelkovich V. Dynamics of propagation and interaction of delta-shock waves in conservation law systems. Journal of Differential Equations. 2005;**211**(2):333-381. Available from: https://goo.gl/Y6g4vv

[14] Garcia M, Omel'yanov G. Interaction of solitary waves for the generalized KdV equation. Communications in Nonlinear Science and Numerical Simulation. 2012;**17**(8):3204-3218. Available from: https://doi.org/10.1016/j.cnsns.2011.12.001

[15] Garcia M, Omel'yanov G. Interaction of solitons and the effect of radiation for the generalized KdV equation. Communications in Nonlinear Science and Numerical Simulation. 2014;**19**(8):2724-2733. Available from: https://doi.org/10.1016/j.cnsns.2013.06.024

[16] Kalisch H, Mitrovic D. Singular solutions of a fully nonlinear 2×2 system of conservation laws. Proceedings of the Edinburgh Mathematical Society II. 2012;**55**:711-729. DOI: https://doi.org/10.1017/S0013091512000065

[17] Omel'yanov G, Valdez-Grijalva M. Asymptotics for a C^1-version of the KdV equation. Nonlinear Phenomena in Complex Systems. 2014;**17**(2):106-115. Available from: http://elib.bsu.by/handle/123456789/116849

[18] Omel'yanov G. Soliton-type asymptotics for non-integrable equations: A survey. Mathematical Methods in The Applied Sciences. 2015;**38**(10):2062-2071. DOI: 10.1002/mma.3198 2014

[19] Omel'yanov G. Propagation and interaction of solitons for nonintegrable equations. Russian Journal of Mathematical Physics. 2016;**23**(2):225-243. DOI: 10.1134/S1061920816020084

[20] Omel'yanov G. Multi-soliton Collision for essentially nonintegrable equations. In: Oberguggenberger M et al., editors. Generalized Functions and Fourier Analysis, Series: Operator Theory: Advances and Applications. Birkhäuser, Cham. 2017;**260**:153-170. DOI: https://doi.org/10.1007/978-3-319-51911-1_9

[21] Kalyakin L. Perturbation of the Korteweg-de Vries soliton. Theoretical and Mathematical Physics. 1992;**92**:736-747. Available from: https://goo.gl/aPPhqG