We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900
Open access books available

185,000
International authors and editors

200M
Downloads

Our authors are among the

154
Countries delivered to

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Chapter 9

Beyond Einstein: A Polynomial Affine Model of Gravity

Oscar Castillo-Felisola

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/intechopen.70951

Abstract

We show that the effective field equations for a recently formulated polynomial affine model of gravity, in the sector of a torsion-free connection, accept general Einstein manifolds—with or without cosmological constant—as solutions. Moreover, the effective field equations are partially those obtained from a gravitational Yang-Mills theory known as the Stephenson-Kilmister-Yang (SKY) theory. Additionally, we find a generalisation of a minimally coupled massless scalar field in general relativity within a "minimally" coupled scalar field in this affine model. Finally, we present the road map to finding general solutions to the effective field equations with either isotropic or cosmologic (i.e., homogeneous and isotropic) symmetry.

Keywords: polynomial affine gravity, torsion, generalised gravity

1. Introduction

During the last century (approximately), we reached a high level of understanding of the four fundamental interactions, that is, electromagnetic, weak, strong and gravitational. However, our understanding splits into two streams: the first describes three kinds of interactions and includes them into a single model, called standard model of particle physics, while the second covers only the gravitational interactions.

The interactions within the standard model of particles are described by connection fields, modelled by gauge theories, and their quantisation procedure is successfully applied. On the other hand, the gravitational interaction, as formulated by Einstein [1] and Hilbert [2], is described by the metric field, whose model does not fit into the category of gauge theory, and its quantisation procedure is not yet well defined [3–9] (see [10] for a historical review).

INTECH
open science | open minds

© 2018 The Author(s). Licensee InTech. This chapter is distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. CC BY

The above suggests that some of the theoretical problems encountered when trying to quantise the gravitational interactions[1] are due to fact that it is formulated as a field theory for the metric and not as a theory for a connection. Therefore, there have been several attempts of describing the gravitational interaction from an affine view point, by using only the connection as fundamental field of the model [11, 12], but these descriptions were not very successful. In addition, Cartan's proposal of considering the same field equations (or action) than Einstein-Hilbert, but with more general connections, see [13–16], was left aside because the field equations, in pure gravity, impose the vanishing torsion.

After the works by Kibble [17] and Sciama [18], it was understsood that once gravity couples to matter, a nontrivial torsion compatible with the setup could exist and the search of viable affine [19–30] (and metric-affine [31]) models of gravity becomes relevant again.

Before continuing our arguments for the *necessity* of considering affine generalisations of general relativity, we shall briefly remind some basic concepts in geometry. In the antiquity, the plane geometry was built essentially with the aid of a (straight) rule and a compass. Counterintuitively, the compass was used to measure distances, while the rule was used to define parallelism. In modern differential geometry language, the object that allows us to measure distances is the metric, ($g_{\mu\nu}$), while the one associated with the concept of parallelism is the connection ($\Gamma^{\lambda}{}_{\mu\nu}$).

Although the concepts of distance and parallelism are independent, there exists a (unique) particular case in which the concepts relate with each other, and thus one needs just of the metric: while the connection is a potential for the metric. This particular case is known as Riemannian geometry, and general relativity stands on such particular construction.

The geometries with general connection are called with the adjective affine and are characterised by its curvature ($\mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho}$), torsion ($\mathcal{T}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$), and the metricity condition ($\mathcal{Q}_{\lambda\mu\nu} = \nabla^{\Gamma}_{\lambda} g_{\mu\nu}$) [31–33]. Notice that in Riemannian geometries, either $\mathcal{T}^{\lambda}{}_{\mu\nu}$ or $\mathcal{Q}_{\lambda\mu\nu}$ vanishes, and the only quantity that characterises the manifolds is the curvature.

An argument to consider affine geometries, to describe gravitational interactions, is that it introduces new degrees of freedom into the model (other than a spin-two field), which might be interpreted as matter—from the view point of general relativity—contributing to the dark sector of the universe [34, 35], inflation [36], exotic cosmologies [37], and diverse particle physics effects [38–46].

In the following sections, we will analyse a recently proposed model called *polynomial affine gravity* [29, 30], which is built up with an affine connection as sole field, and under the premise of preserving the whole group of diffeomorphisms.

2. Polynomial affine gravity: The model

The connection is the field that allows us to define the notion of parallelism as follows. Given a connection $\widehat{\Gamma}^{\mu}_{\rho\sigma}$, one defines a *covariant* derivative, ($\nabla^{\widehat{\Gamma}}$), such that, if the directional derivative

[1]The issue is not only related to quantization, but also to generality. Mathematically, the only ingredient needed to define the curvature is the connection. Thus, considering connections defined by a more fundamental field is not the most general set up.

of a geometrical object, $\mathcal{V}$, along a vector (X) vanishes, one says that the object is parallel transported along (the integral curve defined by) the vector

$$\nabla_X^{\widehat{\Gamma}}\mathcal{V} = 0.$$

The affine connection accepts a decomposition on irreducible components as

$$\widehat{\Gamma}^{\mu}_{\rho\sigma} = \widehat{\Gamma}^{\mu}_{(\rho\sigma)} + \widehat{\Gamma}^{\mu}_{[\rho\sigma]} = \Gamma^{\mu}{}_{\rho\sigma} + \varepsilon_{\rho\sigma\lambda\kappa}T^{\mu,\lambda\kappa} + A_{[\rho}\delta^{\mu}_{\nu]}, \tag{1}$$

where $\Gamma^{\mu}{}_{\rho\sigma} = \widehat{\Gamma}^{\mu}_{(\rho\sigma)}$ is symmetric in the lower indices, A_μ is a vector field corresponding to the trace of torsion and $T^{\mu,\lambda\kappa}$ is a Curtright-like field [47],[2] satisfying $T^{\kappa,\mu\nu} = -T^{\kappa,\nu\mu}$ and $\varepsilon_{\lambda\kappa\mu\nu}T^{\kappa,\mu\nu} = 0$.[3]

Using the above decomposition, we need to build the most general action preserving diffeomorphisms. In order to guarantee the correct transformation of the Lagrangian density, the geometrical objects used to write down the action are a Curtright ($T^{\mu,\nu\lambda}$), a vector (A_μ), the covariant derivative defined with the Levi-Civita connection (∇_μ), both Levi-Civita tensors ($\varepsilon_{\mu\nu\lambda\rho}$ and $\varepsilon^{\mu\nu\lambda\rho}$) and the Riemannian curvature ($R_{\mu\nu}{}^{\lambda}{}_{\rho}$). Since the Riemannian curvature is defined as the commutator of the covariant derivative, it is not an independent field, so it will be left out of the analysis, and only five ingredients remain. In [30], a method of *dimensional analysis* was introduced to ensure that all possible terms were taken into account, and the general action—up to boundary and topological terms—is

$$\begin{aligned}
S[\Gamma,T,A] = \int \mathrm{d}^4x \Big[& B_1 R_{\mu\nu}{}^{\mu}{}_{\rho} T^{\nu,\alpha\beta}T^{\rho,\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} + B_2 R_{\mu\nu}{}^{\sigma}{}_{\rho} T^{\beta,\mu\nu}T^{\rho,\gamma\delta}\epsilon_{\sigma\beta\gamma\delta} + B_3 R_{\mu\nu}{}^{\mu}{}_{\rho} T^{\nu,\rho\sigma}A_\sigma + B_4 R_{\mu\nu}{}^{\sigma}{}_{\rho} T^{\rho,\mu\nu}A_\sigma \\
& + B_5 R_{\mu\nu}{}^{\rho}{}_{\rho} T^{\sigma,\mu\nu}A_\sigma + C_1 R_{\mu\rho}{}^{\mu}{}_{\nu}\nabla_\sigma T^{\nu,\rho\sigma} + C_2 R_{\mu\nu}{}^{\rho}{}_{\rho}\nabla_\sigma T^{\sigma,\mu\nu} + D_1 T^{\alpha,\mu\nu}T^{\beta,\rho\sigma}\nabla_\gamma T^{(\lambda,\kappa)\gamma}\epsilon_{\beta\mu\nu\lambda}\epsilon_{\alpha\rho\sigma\kappa} \\
& + D_2 T^{\alpha,\mu\nu}T^{\lambda,\beta\gamma}\nabla_\lambda T^{\delta,\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma} + D_3 T^{\mu,\alpha\beta}T^{\lambda,\nu\gamma}\nabla_\lambda T^{\delta,\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma} + D_4 T^{\lambda,\mu\nu}T^{\kappa,\rho\sigma}\nabla_{(\lambda}A_{\kappa)}\epsilon_{\mu\nu\rho\sigma} \\
& + D_5 T^{\lambda,\mu\nu}\nabla_{[\lambda}T^{\kappa,\rho\sigma}A_{\kappa]}\epsilon_{\mu\nu\rho\sigma} + D_6 T^{\lambda,\mu\nu}A_\nu\nabla_{(\lambda}A_{\mu)} + D_7 T^{\lambda,\mu\nu}A_\lambda\nabla_{[\mu}A_{\nu]} \\
& + E_1\nabla_{(\rho}T^{\rho,\mu\nu}\nabla_{\sigma)}T^{\sigma,\lambda\kappa}\epsilon_{\mu\nu\lambda\kappa} + E_2\nabla_{(\lambda}T^{\lambda,\mu\nu}\nabla_{\mu)}A_\nu + T^{\alpha,\beta\gamma}T^{\delta,\eta\kappa}T^{\lambda,\mu\nu}T^{\rho,\sigma\tau}(F_1\epsilon_{\beta\gamma\eta\kappa}\epsilon_{\alpha\rho\mu\nu}\epsilon_{\delta\lambda\sigma\tau} \\
& + F_2\epsilon_{\beta\lambda\eta\kappa}\epsilon_{\gamma\rho\mu\nu}\epsilon_{\alpha\delta\sigma\tau}) + F_3 T^{\rho,\alpha\beta}T^{\gamma,\mu\nu}T^{\lambda,\sigma\tau}A_\tau\epsilon_{\alpha\beta\gamma\lambda}\epsilon_{\mu\nu\rho\sigma} + F_4 T^{\eta,\alpha\beta}T^{\kappa,\gamma\delta}A_\eta A_\kappa\epsilon_{\alpha\beta\gamma\delta}\Big].
\end{aligned} \tag{2}$$

Despite the complex structure of the action, it shows very interesting features: (a) The structure is rigid (it does not accept extra terms), which forbids the appearance of counter-terms, if one would like to quantise the model; (b) all coupling constants are dimensionless, which might be a hint of conformal invariance of the model; (c) the action turns out to be power-counting renormalisable, which does not guarantee renormalisability, but is a nice feature; and (d) the structure of the model yields no three-point graviton vertices, which might allow to overcome the *no-go* theorems found in Refs. [48, 49].

[2] Notice that the Curtright-like field is defined as the quasi-Hodge dual of the traceless part of the torsion.

[3] Since no metric is present, the epsilon symbols are not related by raising (lowering) their indices, but instead we demand that $\varepsilon^{\delta\eta\lambda\kappa}\varepsilon_{\mu\nu\rho\sigma} = 4!\,\delta^{\delta}_{[\mu}\delta^{\eta}_{\nu}\delta^{\lambda}_{\rho}\delta^{\kappa}_{\sigma]}$.

3. Limit of vanishing torsion

We now want to restrict ourselves to the limit of vanishing torsion, which simplifies the comparison between our model and general relativity. The vanishing torsion limit—equivalent to take $T^{\lambda,\,\mu\nu} \to 0$ and $A_\mu \to 0$—cannot be taken at the action level, but in the field equations, and the limit is a consistent truncation of the whole field equations [30].

The only nontrivial field equation after the limit will be the one for the Curtright-like field, $T^{\nu,\,\mu\rho}$:

$$\nabla_{[\rho} R_{\mu]\nu} + \kappa \nabla_\nu R_{\mu\rho}{}^{\lambda}{}_{\lambda} = 0, \tag{3}$$

with κ a constant related with the original couplings of the model. These field equations are simpler if one restricts to connections compatible with a volume form, also known as equi-affine [32, 50, 51], which assures that the Ricci tensor of the connection is symmetric and the contraction of the last indices vanishes; thus the equation is

$$\nabla_{[\rho} R_{\mu]\nu} = 0. \tag{4}$$

Eq. (4) is a generalisation of Einstein's field equation in vacuum. This can be seem as follows: all Einstein manifolds posses Ricci tensor proportional to the metric, $R_{\mu\nu} \propto g_{\mu\nu}$; the metricity condition thus ensures that every vacuum solution to the Einstein's equations solves the (simplified) field equations of our model.

Moreover, Eq. (4) is related through the second Bianchi identity to the harmonic curvature condition [52]:

$$\nabla_\lambda R_{\mu\nu}{}^{\lambda}{}_{\rho} = 0. \tag{5}$$

Eqs. (4) and (5) accept a geometrical interpretation equivalent to that of the field equations of a pure Yang-Mills theory, which in the language of differential forms are

$$\mathcal{D}\boldsymbol{F} = 0, \quad \mathcal{D}\star\boldsymbol{F} = 0, \tag{6}$$

where $\boldsymbol{F} = \mathcal{D}\boldsymbol{A}$ is the field strength two-form (the curvature two-form of the connection in the principal bundle; see, for example, [53, 54]), and the operator $\star\star$ denotes the Hodge star. Now, these Yang-Mills field equations are obtained from the variation of the action functional:

$$S_{\mathrm{YM}} = \int \mathrm{Tr}(\boldsymbol{F}\star\boldsymbol{F}), \tag{7}$$

and the Jacobi identity for the covariant derivative.

Similarly, Eq. (4) (equivalently Eq. (5)) can be obtained from an effective gravitational Yang-Mills functional action [55–57]:

http://dx.doi.org/10.5772/intechopen.70951

$$S_{\mathrm{YM}} = \int \mathrm{Tr}(\mathcal{R}\star\mathcal{R}) = \int \left(\mathcal{R}^a{}_b \star \mathcal{R}^b{}_a\right), \tag{8}$$

where $\mathcal{R} \in \Omega^2(\mathcal{M}, T^*\mathcal{M} \otimes T\mathcal{M})$ is the curvature two-form, the operator $\star$ denotes the Hodge star and the trace is taken on the bundle indices (see [53]).

The gravitational model described by the action in Eq. (8) is called Stephenson-Kilmister-Yang (or SKY for short), and its physical interpretation relies—as in general relativity—in the fact that the metric is the fundamental field for describing the gravitational interaction. In that case, the field equations are third-order partial differential equations, and there are several undesirable behaviours due to this characteristic of the equations. However, in our model the field mediating the gravitational interaction is the connection, and therefore, the Eqs. (4) and (5) are second-order field equations for the components of the connection.

It is worth noticing that, according to the arguments in [48, 49], the SKY theory is not renormalisable. However, it is possible that the polynomial affine gravity could be renormalisable, in the sense that SKY is an effective description for the torsionless limit of polynomial affine gravity.

4. Polynomial affine gravity coupled to a scalar field

In the standard formulation of physical theories (even in flat spacetimes), the metric is a required ingredient. The metric and its inverse define a homomorphism between the tangent and cotangent bundles, allowing to build the kinetic energy term in the action.

$$\partial_\mu\phi\partial^\mu\phi = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

Therefore, the inclusion of matter within models with no necessity of a metric is a nontrivial task.

Inspired in the method of *dimensional analysis* introduced in Ref. [30], we attempt to couple a scalar field to the polynomial affine gravity by defining the most general, symmetric $\binom{2}{0}$-tensor density, $\mathfrak{g}^{\mu\nu}$, built with the available fields, and use it to build Lagrangian densities for the matter content. It can be shown that such a density is given by

$$\mathfrak{g}^{\mu\nu} = \alpha\, \nabla_\lambda T^{\mu,\nu\lambda} + \beta\, A_\lambda T^{\mu,\nu\lambda} + \gamma\, \varepsilon_{\lambda\kappa\rho\sigma} T^{\mu,\lambda\kappa} T^{\nu,\rho\sigma}, \tag{9}$$

with α, β and γ arbitrary coefficients.

Therefore, the action defined by the "kinetic term" is

$$S_\phi = -\int \mathrm{d}^4x \left(\alpha\, \nabla_\lambda T^{\mu,\nu\lambda} + \beta\, A_\lambda T^{\mu,\nu\lambda} + \gamma\, \varepsilon_{\lambda\kappa\rho\sigma} T^{\mu,\lambda\kappa} T^{\nu,\rho\sigma}\right)\partial_\mu\phi\partial_\nu\phi. \tag{10}$$

Remarkably, it induces a nontrivial contribution to the field equations once we restrict to the torsionless sector. The field nontrivial equations, when the scalar field is turned on, is[4]

$$\nabla_{[\sigma}R_{\rho]\mu}{}^{\mu}{}_{\nu} - C_2\nabla_{\nu}R_{\rho\sigma}{}^{\mu}{}_{\mu} - \alpha\nabla_{[\sigma}\left(\partial_{\rho]}\phi\partial_{\nu}\phi\right) = 0,$$

which for equi-affine connections simplifies to

$$\nabla_{[\sigma}R_{\rho]\nu} - \alpha\nabla_{[\sigma}\left(\partial_{\rho]}\phi\partial_{\nu}\phi\right) = 0. \tag{11}$$

Eq. (11) can be integrated once, and the solution takes the familiar form:

$$R_{\mu\nu} - \alpha\partial_{\mu}\phi\partial_{\nu}\phi = \Lambda g_{\mu\nu},$$

where the integration (covariantly) constant tensor, which is invertible and symmetric, has been suggestively denoted by $\Lambda g_{\mu\nu}$. The above equation can be written in the more conventional form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \alpha\left(\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2\right). \tag{12}$$

Thus, we have been able to recover a set of equations, similar to those of general relativity coupled with a free, massless scalar field, but with an arbitrary rank-two, symmetric, covariantly constant tensor playing the role of a metric.

Notice that, from the action in Eq. (10), it is not possible to obtain the field equation for the scalar ϕ, when the vanishing torsion limit is taken. Nonetheless, the second Bianchi identity in Eq. (4) imposes

$$\nabla^{\mu}\partial_{\mu}\phi = 0. \tag{13}$$

This condition is, in the sense argued in Ref. [58], the equation of motion for the scalar field.

5. Finding symmetric ansätze

The usual procedure for solving Einstein's equation is to propose an ansatz for the metric. That ansatz must be compatible with the symmetries we would like to respect in the problem. The formal study of the symmetries of the fields is accomplished via the Lie derivative (for reviews, see [54, 59–61]). Below, we use the Lie derivative for obtaining ansätze for either the metric or the connection.

The form of the Lie derivative for tensors is well known, but the Lie derivative for a connection is not. Thus, for the sake of completeness, we remind the readers that

[4] We have fixed the coefficient $C_1 = 1$.

$$\mathcal{L}_\xi \Gamma^a{}_{bc} = \xi^m \partial_m \Gamma^a{}_{bc} - \Gamma^m{}_{bc}\partial_m \xi^a + \Gamma^a{}_{mc}\partial_b \xi^m + \Gamma^a{}_{bm}\partial_c \xi^m + \frac{\partial^2 \xi^a}{\partial x^b \partial x^c}, \tag{14}$$

where ξ is the vector defining the symmetry flow.

We shall restrict ourselves to the isotropic (spherically symmetric) and homogeneous and isotropic (cosmological symmetry). In the tedious task of calculating the Lie derivative of different objects, we have used the mathematical software SAGE together with its differential geometry package SageManifolds [62, 63].

5.1. Isotropic ansätze

The two-dimensional sphere, S^2, is a maximally symmetric space whose Killing vectors generate an $SO(3)$ symmetry group. The se vectors can be expressed in spherical coordinates as

$$\begin{aligned} J_1 &= (0 \quad 0 \quad -\cos(\varphi) \quad \cot(\theta)\sin(\varphi)), \\ J_2 &= (0 \quad 0 \quad \sin(\varphi) \quad \cos(\varphi)\cot(\theta)), \\ J_3 &= (0 \quad 0 \quad 0 \quad 1). \end{aligned} \tag{15}$$

5.1.1. Isotropic (covariant) two tensor

Let us start by finding the most general isotropic, four-dimensional, covariant rank-two tensor. We shall obtain a generalisation of the famous ansatz for the Schwarzschild metric.

We start from a general rank-two tensor, that is, the 16 components of the tensor depend on all the coordinates. Then, the Lie derivative of the metric along the vector J_3 in Eq. (15) yields

$$\pounds_{J_3} T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial \varphi}, \tag{16}$$

which vanishes only if none of the components of the metric depend on the φ coordinate.

The Lie derivative along the other two generators of the angular momentum yields a nontrivial set of differential equations (not shown here) whose solution fixes a tensor of the form:

$$T = \begin{pmatrix} T_{00}(t,r) & T_{01}(t,r) & 0 & 0 \\ T_{10}(t,r) & T_{11}(t,r) & 0 & 0 \\ 0 & 0 & T_{22}(t,r) & H(t,r)\sin(\theta) \\ 0 & 0 & -H(t,r)\sin(\theta) & T_{22}(t,r)\sin^2(\theta) \end{pmatrix}. \tag{17}$$

This result was found by Papapetrou [64]. Notice that there are six functions of the coordinates t and r, while the θ dependence is fixed by the symmetry. Two out of the six functions vanish whenever one restricts to symmetric tensors, that is, $g(X, Y) = g(Y, X)$, such as the metric tensor. The form of the symmetric, covariant, rank-two tensor is

$$g = \begin{pmatrix} A(t,r) & B(t,r) & 0 & 0 \\ B(t,r) & C(t,r) & 0 & 0 \\ 0 & 0 & D(t,r) & 0 \\ 0 & 0 & 0 & D(t,r)\sin^2(\theta) \end{pmatrix}, \tag{18}$$

which, under a redefinition of the radial and temporal coordinates, takes the standard form:

$$g = \begin{pmatrix} F(t,r) & 0 & 0 & 0 \\ 0 & G(t,r) & 0 & 0 \\ 0 & 0 & D(t,r) & 0 \\ 0 & 0 & 0 & D(t,r)\sin^2(\theta) \end{pmatrix}. \tag{19}$$

It is worth noticing that Eq. (19) is the most general spherical ansatz which is the one wants to solve Einstein's field equations. The static condition is only assured by the Birkhoff theorem [65–68], once the field equations are given.

5.1.2. Isotropic affine connection

The strategy used in the previous section can be repeated for an affine connection, and we shall end up with the most general isotropic (affine) connection. For the sake of simplicity, we do not include the differential equations obtained from the calculation of the Lie derivative.[5]

As before, the Lie derivative along the generators of the spherical symmetry fixes the angular dependence of the connection's components. The nonvanishing components of an isotropic $\widehat{\Gamma}^{a}_{bc}$, are

$$\begin{aligned}
&\widehat{\Gamma}^{t}_{tt} = F_{000}(t,r) && \widehat{\Gamma}^{t}_{tr} = F_{001}(t,r) \\
&\widehat{\Gamma}^{t}_{rt} = F_{010}(t,r) && \widehat{\Gamma}^{t}_{rr} = F_{011}(t,r) \\
&\widehat{\Gamma}^{t}_{\theta\theta} = F_{033}(t,r) && \widehat{\Gamma}^{t}_{\theta\phi} = F_{023}(t,r)\sin(\theta) \\
&\widehat{\Gamma}^{t}_{\phi\theta} = -F_{023}(t,r)\sin(\theta) && \widehat{\Gamma}^{t}_{\phi\phi} = F_{033}(t,r)\sin^2(\theta) \\
&\widehat{\Gamma}^{r}_{tt} = F_{100}(t,r) && \widehat{\Gamma}^{r}_{tr} = F_{101}(t,r) \\
&\widehat{\Gamma}^{r}_{rt} = F_{110}(t,r) && \widehat{\Gamma}^{r}_{rr} = F_{111}(t,r) \\
&\widehat{\Gamma}^{r}_{\theta\theta} = F_{133}(t,r) && \widehat{\Gamma}^{r}_{\theta\phi} = F_{123}(t,r)\sin(\theta) \\
&\widehat{\Gamma}^{r}_{\phi\theta} = -F_{123}(t,r)\sin(\theta) && \widehat{\Gamma}^{r}_{\phi\phi} = F_{133}(t,r)\sin^2(\theta)
\end{aligned}$$

[5] Notice that in four dimensions, an affine connection has 64 components. Therefore, there are 192 differential equations to solve for the 3 generators of spherical symmetry.

http://dx.doi.org/10.5772/intechopen.70951

$$
\begin{aligned}
&\widehat{\Gamma}^{\theta}_{t\theta} = F_{303}(t,r) && \widehat{\Gamma}^{\theta}_{t\phi} = -F_{302}(t,r)\sin(\theta)\\
&\widehat{\Gamma}^{\theta}_{r\theta} = F_{313}(t,r) && \widehat{\Gamma}^{\theta}_{r\phi} = -F_{312}(t,r)\sin(\theta)\\
&\widehat{\Gamma}^{\theta}_{\theta t} = F_{330}(t,r) && \widehat{\Gamma}^{\theta}_{\theta r} = F_{331}(t,r)\\
&\widehat{\Gamma}^{\theta}_{\phi t} = -F_{320}(t,r)\sin(\theta) && \widehat{\Gamma}^{\theta}_{\phi r} = -F_{321}(t,r)\sin(\theta)\\
&\widehat{\Gamma}^{\theta}_{\phi\phi} = -\cos(\theta)\sin(\theta) && \widehat{\Gamma}^{\phi}_{t\theta} = \frac{F_{302}(t,r)}{\sin(\theta)}\\
&\widehat{\Gamma}^{\phi}_{t\phi} = F_{303}(t,r) && \widehat{\Gamma}^{\phi}_{r\theta} = \frac{F_{312}(t,r)}{\sin(\theta)}\\
&\widehat{\Gamma}^{\phi}_{r\phi} = F_{313}(t,r) && \widehat{\Gamma}^{\phi}_{\theta t} = \frac{F_{320}(t,r)}{\sin(\theta)}\\
&\widehat{\Gamma}^{\phi}_{\theta\ r} = \frac{F_{321}(t,r)}{\sin(\theta)} && \widehat{\Gamma}^{\phi}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}\\
&\widehat{\Gamma}^{\phi}_{\phi t} = F_{330}(t,r) && \widehat{\Gamma}^{\phi}_{\phi r} = F_{331}(t,r)\\
&\widehat{\Gamma}^{\phi}_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}. &&
\end{aligned}
$$

This general isotropic connection depends on 20 functions of the coordinates t and r, and has nonvanishing torsion and nonmetricity. Therefore, for our purposes within this paper, we can restrict even further to a torsion-free connection, whose components are

$$
\begin{aligned}
&\Gamma^{t}_{tt} = F_{000}(t,r) && \Gamma^{t}_{tr} = F_{001}(t,r)\\
&\Gamma^{t}_{rr} = F_{011}(t,r) && \Gamma^{t}_{\theta\ \theta} = F_{033}(t,r)\\
&\Gamma^{t}_{\phi\phi} = F_{033}(t,r)\sin^2(\theta) && \Gamma^{r}_{tt} = F_{100}(t,r)\\
&\Gamma^{r}_{tr} = F_{101}(t,r) && \Gamma^{r}_{rr} = F_{111}(t,r)\\
&\Gamma^{r}_{\theta\theta} = F_{133}(t,r) && \Gamma^{r}_{\phi\phi} = F_{133}(t,r)\sin^2(\theta)\\
&\Gamma^{\theta}_{t\ \theta} = F_{303}(t,r) && \Gamma^{\theta}_{t\phi} = -F_{302}(t,r)\sin(\theta)\\
&\Gamma^{\theta}_{r\theta} = F_{313}(t,r) && \Gamma^{\theta}_{r\phi} = -F_{312}(t,r)\sin(\theta)\\
&\Gamma^{\theta}_{\phi\phi} = -\cos(\theta)\sin(\theta) && \Gamma^{\phi}_{t\ \theta} = \frac{F_{302}(t,r)}{\sin(\theta)}\\
&\Gamma^{\phi}_{t\phi} = F_{303}(t,r) && \Gamma^{\phi}_{r\theta} = \frac{F_{312}(t,r)}{\sin(\theta)}\\
&\Gamma^{\phi}_{r\phi} = F_{313}(t,r) && \Gamma^{\phi}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}.
\end{aligned}
\tag{20}
$$

The last connection depends on 12 functions of t and r, and these are the functions to be fixed by solving the field Eq. (4).

Using Eq. (20), we calculated the Ricci tensor and noticed that it is not symmetric, because the connection is not equi-affine. In order for the Ricci to be symmetric, the following conditions must hold:

$$\frac{\partial F_{000}}{\partial r} - \frac{\partial F_{001}}{\partial t} + 2\frac{\partial F_{303}}{\partial r} = 0, \tag{21}$$

and

$$\frac{\partial F_{101}}{\partial r} - \frac{\partial F_{101}}{\partial t} + 2\frac{\partial F_{313}}{\partial r} = 0. \tag{22}$$

There are several solutions to these conditions, and each of them could (in principle) provide a solution to the field Eq. (4). It is worth mentioning that if we do not demand the connection to be equi-affine, the field equations to solve would be Eq. (3), which has an extra term which cannot be obtained from the Yang-Mills-like effective action in Eq. (8).

5.2. Cosmological ansätze

The Lorentian isotropic and homogeneous spaces in four dimensions have isometry group either $SO(4)$, $SO(3,1)$ or $ISO(3)$, and their algebra can be obtained from the algebra $\mathfrak{so}(4)$ through a 3+1 decomposition, that is, $J_{AB}=\{J_{ab}, J_{a*}\}$,[6] where the extra dimension has been denoted by an asterisk. In terms of these new generators, the algebra reads

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} + \delta_{ad}J_{bc} - \delta_{bd}J_{ac}, \\ [J_{ab}, J_{c*}] &= \delta_{bc}J_{a*} - \delta_{ac}J_{b*}, \\ [J_{a*}, J_{c*}] &= -\kappa J_{ac}, \end{aligned} \tag{23}$$

with[7]

$$\kappa = \begin{cases} 1 & SO(4) \\ 0 & ISO(3) \\ -1 & SO(3,1) \end{cases}. \tag{24}$$

One can express the Killing vectors in spherical coordinates, and in addition to those obtained in Eq. (15), we get

[6] We shall use the standard (Euclidean) three-dimensional correspondence $J_a = \frac{1}{2}\varepsilon_{abc}J^{bc}$, in order to match notations with the previous case.

[7] The inhomogeneous algebra of $ISO(n)$ can be obtained from the ones of $SO(n+1)$ or $SO(n,1)$ through the Inönü Wigner contraction [69].

$$
\begin{aligned}
P_1 = J_{1*} &= \sqrt{1-\kappa r^2}\begin{pmatrix} 0 & \cos(\varphi)\sin(\theta) & \dfrac{\cos(\varphi)\cos(\theta)}{r} & -\dfrac{\sin(\varphi)}{r\sin(\theta)} \end{pmatrix},\\
P_2 = J_{2*} &= \sqrt{1-\kappa r^2}\begin{pmatrix} 0 & \sin(\varphi)\sin(\theta) & \dfrac{\cos(\theta)\sin(\varphi)}{r} & \dfrac{\cos(\varphi)}{r\sin(\theta)} \end{pmatrix},\\
P_3 = J_{3*} &= \sqrt{1-\kappa r^2}\begin{pmatrix} 0 & \cos(\theta) & -\dfrac{\sin(\theta)}{r} & 0 \end{pmatrix}.
\end{aligned}
\tag{25}
$$

5.2.1. Isotropic and homogeneous (covariant)-two tensor

In order to find the most general isotropic and homogeneous covariant two tensor, we can start from the result in Eq. (17) and impose now the symmetries from the extra generators. The equations are[8]

$$
\begin{aligned}
&\pounds_{P_3}T_{tt} : \sqrt{1-\kappa r^2}\cos(\theta)\frac{\partial T_{00}}{\partial r} = 0, \quad \pounds_{P_3}T_{tr} : \left(\kappa r T_{01} + \left(\kappa r^2 - 1\right)\frac{\partial T_{01}}{\partial r}\right) = 0,\\
&\pounds_{P_3}T_{t\theta} : T_{01} = 0, \quad \pounds_{P_3}T_{rt} : \left(\kappa r T_{10} + \left(\kappa r^2 - 1\right)\frac{\partial T_{10}}{\partial r}\right) = 0,\\
&\pounds_{P_3}T_{rr} : \left(2\kappa r T_{11} + \left(\kappa r^2 - 1\right)\frac{\partial T_{11}}{\partial r}\right) = 0, \quad \pounds_{P_3}T_{r\theta} : \left(\kappa r^4 - r^2\right)T_{11} + T_{22} = 0,\\
&\pounds_{P_3}T_{r\varphi} : H = 0, \quad \pounds_{P_3}T_{\theta t} : T_{10} = 0,\\
&\pounds_{P_3}T_{\theta r} : \left(\kappa r^4 - r^2\right)T_{11} + T_{22} = 0, \quad \pounds_{P_3}T_{\theta\theta} : \left(r\frac{\partial T_{22}}{\partial r} - 2T_{22}\right) = 0,\\
&\pounds_{P_3}T_{\theta\varphi} : \left(r\frac{\partial H}{\partial r} - 2H\right) = 0, \quad \pounds_{P_3}T_{\varphi r} : H = 0,\\
&\pounds_{P_3}T_{\varphi\theta} : \left(r\frac{\partial H}{\partial r} - 2H\right) = 0, \quad \pounds_{P_3}T_{\varphi\varphi} : \left(r\frac{\partial T_{22}}{\partial r} - 2T_{22}\right) = 0.
\end{aligned}
\tag{26}
$$

The above equations are solved for a tensor of the form:

$$
T = G_{00}(t)\mathrm{d}t\otimes\mathrm{d}t + \frac{G_{11}(t)}{1-\kappa r^2}\mathrm{d}r\otimes\mathrm{d}r + r^2 G_{11}(t)\mathrm{d}\theta\otimes\mathrm{d}\theta + r^2 G_{11}(t)\sin^2(\theta)\mathrm{d}\varphi\otimes\mathrm{d}\varphi. \tag{27}
$$

Under a redefinition of the "time" coordinate, the tensor is nothing but the standard ansatz for the Friedman-Robertson-Walker metric.

5.2.2. Isotropic and homogeneous affine connection

Without further details, we present the nonvanishing components of an isotropic and homogeneous affine connection. It is given by

[8] Due to the isotropy, one needs to solve just for one of these extra generators.

$$
\begin{aligned}
\widehat{\Gamma}^{t}_{tt} &= G_{000}(t) & \widehat{\Gamma}^{t}_{rr} &= \frac{G_{011}(t)}{1-\kappa r^2}\\
\widehat{\Gamma}^{t}_{\theta\theta} &= r^2 G_{011}(t) & \widehat{\Gamma}^{t}_{\varphi\varphi} &= r^2 G_{011}(t)\sin^2(\theta)\\
\widehat{\Gamma}^{r}_{tr} &= G_{101}(t) & \widehat{\Gamma}^{r}_{rt} &= G_{110}(t)\\
\widehat{\Gamma}^{r}_{rr} &= \frac{\kappa r}{1-\kappa r^2} & \widehat{\Gamma}^{r}_{\theta\theta} &= \kappa r^3 - r\\
\widehat{\Gamma}^{r}_{\theta\varphi} &= \sqrt{1-\kappa r^2} r^2 G_{123}(t)\sin(\theta) & \widehat{\Gamma}^{r}_{\varphi\theta} &= -\sqrt{1-\kappa r^2} r^2 G_{123}(t)\sin(\theta)\\
\widehat{\Gamma}^{r}_{\varphi\varphi} &= \left(\kappa r^3 - r\right)\sin^2(\theta) & \widehat{\Gamma}^{\theta}_{t\theta} &= G_{101}(t)\\
\widehat{\Gamma}^{\theta}_{r\theta} &= \frac{1}{r} & \widehat{\Gamma}^{\theta}_{r\varphi} &= -\frac{G_{123}(t)\sin(\theta)}{\sqrt{1-\kappa r^2}}\\
\widehat{\Gamma}^{\theta}_{\theta t} &= G_{110}(t) & \widehat{\Gamma}^{\theta}_{\theta r} &= \frac{1}{r}\\
\widehat{\Gamma}^{\theta}_{\varphi r} &= \frac{G_{123}(t)\sin(\theta)}{\sqrt{1-\kappa r^2}} & \widehat{\Gamma}^{\theta}_{\varphi\varphi} &= -\cos(\theta)\sin(\theta)\\
\widehat{\Gamma}^{\varphi}_{t\varphi} &= G_{101}(t) & \widehat{\Gamma}^{\varphi}_{r\theta} &= \frac{G_{123}(t)}{\sqrt{1-\kappa r^2}\sin(\theta)}\\
\widehat{\Gamma}^{\varphi}_{r\varphi} &= \frac{1}{r} & \widehat{\Gamma}^{\varphi}_{\theta r} &= -\frac{G_{123}(t)}{\sqrt{1-\kappa r^2}\sin(\theta)}\\
\widehat{\Gamma}^{\varphi}_{\theta\varphi} &= \frac{\cos(\theta)}{\sin(\theta)} & \widehat{\Gamma}^{\varphi}_{\varphi t} &= G_{110}(t)\\
\widehat{\Gamma}^{\varphi}_{\varphi r} &= \frac{1}{r} & \widehat{\Gamma}^{\varphi}_{\varphi\theta} &= \frac{\cos(\theta)}{\sin(\theta)},
\end{aligned}
\tag{28}
$$

which is determined by five independent functions, but this affine connection still possess torsion. The imposition of vanishing torsion kills two of the above functions, and the remaining components of the connection are

$$
\begin{aligned}
\Gamma^{t}_{tt} &= G_{000}(t) & \Gamma^{t}_{rr} &= \frac{G_{011}(t)}{1-\kappa r^2}\\
\Gamma^{t}_{\theta\theta} &= r^2 G_{011}(t) & \Gamma^{t}_{\varphi\varphi} &= r^2 G_{011}(t)\sin^2(\theta)\\
\Gamma^{r}_{tr} &= G_{101}(t) & \Gamma^{r}_{rr} &= \frac{\kappa r}{1-\kappa r^2}\\
\Gamma^{r}_{\theta\theta} &= \kappa r^3 - r & \Gamma^{r}_{\varphi\varphi} &= \left(\kappa r^3 - r\right)\sin^2(\theta)\\
\Gamma^{\theta}_{t\theta} &= G_{101}(t) & \Gamma^{\theta}_{r\theta} &= \frac{1}{r}\\
\Gamma^{\theta}_{\varphi\varphi} &= -\cos(\theta)\sin(\theta) & \Gamma^{\varphi}_{t\varphi} &= G_{101}(t)\\
\Gamma^{\varphi}_{r\varphi} &= \frac{1}{r} & \Gamma^{\varphi}_{\theta\varphi} &= \frac{\cos(\theta)}{\sin(\theta)}.
\end{aligned}
\tag{29}
$$

6. Towards the solution of the field equations

6.1. Cosmological solutions

Using the connection in Eq. (29), the Ricci is calculated and yields

$$\begin{aligned} R_{tt} &= 3G_{000}(t)G_{101}(t) - 3G_{101}(t)^2 - 3\frac{\partial G_{101}}{\partial t} \\ R_{rr} &= \frac{G_{000}(t)G_{011}(t) + G_{011}(t)G_{101}(t) + 2\kappa + \frac{\partial G_{011}}{\partial t}}{1-\kappa r^2} \\ R_{\theta\theta} &= \left(G_{000}(t)G_{011}(t) + G_{011}(t)G_{101}(t) + 2\kappa + \frac{\partial G_{011}}{\partial t}\right) r^2 \\ R_{\varphi\varphi} &= \left(G_{000}(t)G_{011}(t) + G_{011}(t)G_{101}(t) + 2\kappa + \frac{\partial G_{011}}{\partial t}\right) r^2 \sin^2(\theta). \end{aligned} \tag{30}$$

Notice that the Ricci tensor is symmetric, determined by only three independent functions, and as expected by the symmetry, just two of their components behave differently.

A first kind of solutions can be found by solving the system of equations determined by vanishing Ricci. However, this strategy requires the fixing of one of the unknown functions. A solution inspired in the components of the connection for Friedmann-Robertson-Walker gives

$$G_{000} = 0, G_{101} = \frac{1}{t - C_1}, G_{011} = -2\kappa t + C_2. \tag{31}$$

A second class of solutions can be found by solving the parallel Ricci equation, $\nabla_\lambda R_{\mu\nu} = 0$, which surprisingly yields three independent field equations:

$$\nabla_t R_{tt} = -6\, G_{000}^2 G_{101} + 6\, G_{000} G_{101}^2 + 3\, G_{101}\frac{\partial G_{000}}{\partial t} + 3\,(3\, G_{000} - 2\, G_{101})\frac{\partial G_{101}}{\partial t} - 3\,\frac{\partial^2 G_{101}}{\partial t^2}, \tag{32}$$

$$\nabla_t R_{ii} \sim 2\, G_{011} G_{101}^2 + 2\,(G_{000}G_{011} + 2\,\kappa)G_{101} - G_{011}\frac{\partial G_{000}}{\partial t}, \tag{33}$$

$$-(G_{000} - G_{101})\frac{\partial G_{011}}{\partial t} - G_{011}\frac{\partial G_{101}}{\partial t} - \frac{\partial^2 G_{011}}{\partial t^2}$$

$$\nabla_i R_{ti} = 2\, G_{011} G_{101}^2 - 2\,(2\, G_{000}G_{011} + \kappa)G_{101} - G_{101}\frac{\partial G_{011}}{\partial t} + 3\, G_{011}\frac{\partial G_{101}}{\partial t}. \tag{34}$$

However, the system of equations is complicated enough to avoid an analytic solution.[9] Despite the complication, we can try a couple of assumptions that simplify the system of equations, for example:

[9] Of course one can propose a formal solution in terms of power series. However, this approach that has been used in a paper is still under developed.

- Again, inspired in the Friedmann-Robertson-Walker results, we choose $G_{000}=0$, and solve for the other functions (see [70]).
- We could choose $G_{000}=G_{101}$, to eliminate the nonlinear terms in Eq. (32).
- We can solve both Eqs. (10) and (34) by setting $G_{011}=0$, $\kappa\neq 0$ and (say) $G_{101}=$ const., and solve Eq. (32) which turns to be a known ordinary differential equation.

Finally, the third class of solutions is that of Eq. (4). The set of equations degenerate and yield a single independent field equation:

$$4\,G_{011}G_{101}^{2}-2\,(G_{000}G_{011}-\kappa)G_{101}-G_{011}\frac{\partial G_{000}}{\partial t}-G_{000}\frac{\partial G_{011}}{\partial t}+2\,G_{011}\frac{\partial G_{101}}{\partial t}-\frac{\partial^{2}\,G_{011}}{\partial t^{2}}=0. \quad (35)$$

Therefore, we need to set two out of the three unknown functions to be able to solve the connection.

7. Conclusions

We have presented a short review of the polynomial affine gravity, whose field equations (in the torsion-free sector) generalise those of the standard general relativity. In the mentioned approximation, the field equations coincide with (part of) those of the gravitational Yang-Mills theory of gravity, known in the literature as the SKY model.

Among the features of the polynomial affine gravity, we highlighted the following:

- Although our spacetime could be metric, the metric plays no role in the model building.
- The nonrelativistic limit of the model yields a Keplerian potential, even with the contributions of torsion and nonmetricity.
- In the torsion-free sector, all the vacuum solutions to the Einstein gravity are solutions of the polynomial affine gravity.
- Scalar matter can be coupled to the polynomial affine gravity through a symmetric, $\binom{2}{0}$-tensor density. The coupled field equations can be written in a similar form to those of general relativity, with the subtlety that wherever the metric appears in Einstein's equations, we just need a covariantly constant, symmetric, $\binom{0}{2}$-tensor.
- Although the model is built up without the necessity of a metric, one can still assume that the connection is a metric potential. Such consideration yields to obtain new solutions to Eq. (4), which are not solutions of the standard Einstein's equations.[10]

[10] Some solutions will be presented in [70].

- We found the general ansatz for the connection, compatible with isotropic and cosmological symmetries. Additionally, we have sketched the road to solving Eq. (4), for the cosmological connection ansatz. A thorough analysis will be presented in [70].

We would like to finish commenting about the necessity of coupling other forms of matters and a formal counting of degrees of freedoms.

Acknowledgements

O.C-F. wants to thank to the ICTP-SAIFR and IFT-UNESP (Sao Paulo, Brazil) for the hospitality while finishing this article. This work has been partially founded by the CONICYT (Chile) project PAI-79140040. The Centro Científico Tecnológico de Valparaíso (CCTVal) is funded by the Chilean government through the Centres of Excellence Basal Financing Program FB0821 of CONICYT.

Author details

Oscar Castillo-Felisola[1,2*]

*Address all correspondence to: o.castillo.felisola@gmail.com

1 Centro Científico Tecnológico de Valparaíso, Valparaíso, Chile

2 Departamento de Física, Universidad Técnica Federico Santa María, Valparaíso, Chile

References

[1] Einstein A. Zur allgemeinen relativitätstheorie. Sitzungsberichte der Preussischen Akademie der Wissenschaften. 1915;**1**:778

[2] Hilbert D. Die grundlagen der physik.(erste mitteilung). Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse. 1915;**1915**:395

[3] Hooft G 't. An algorithm for the poles at dimension four in the dimensional regularization procedure. Nuclear Physics B. 1973;**62**:444

[4] Hooft G 't, Veltman MJG. One loop divergencies in the theory of gravitation. Annales de l'Institut Henri Poincare, Section A: Physique Theorique. 1974;**20**:69

[5] Deser S, van Nieuwenhuizen P. One loop divergences of quantized Einstein-Maxwell fields. Physical Review D. 1974;**10**:401

[6] Deser S, van Nieuwenhuizen P. Nonrenormalizability of the quantized Dirac-Einstein system. Physical Review D. 1974;**10**:411

[7] DeWitt BS. Quantum theory of gravity. 1. The canonical theory. Physics Review. 1967; **160**:1113

[8] DeWitt BS. Quantum theory of gravity. 2. The manifestly covariant theory. Physics Review. 1967;**162**:1195

[9] DeWitt BS. Quantum theory of gravity. 3. Applications of the covariant theory. Physics Review. 1967;**162**:1239

[10] Rovelli C. Notes for a brief history of quantum gravity. https://arxiv.org/abs/gr-qc/0006061. 2000

[11] Eddington AS. The Mathematical Theory of Relativity. Cambridge: Cambridge University Press. 1923

[12] Schrödinger E. Space-Time Structure. Cambridge University Press; 1950

[13] Cartan E. Sur une généralisation de la notion de courbure de riemann et les espaces à torsion. Comptes Rendus de l'Acad emie des Sciences. Paris. 1922;**174**:593

[14] Cartan E. Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie). Annales Scientifiques de l'Ecole Normale Superieure. 1923;**40**:325

[15] Cartan E. Sur les variétés à connexion affine, et la théorie de la relativité généralisée (première partie) (suite). Annales Scientifiques de l'Ecole Normale Superieure. 1924;**41**:1

[16] Cartan E. Sur les variétés à connexion affine et la théorie de la relativité généralisée, (deuxième partie). Annales Scientifiques de l'Ecole Normale Superieure. 1925;**42**:17

[17] Kibble TWB. Lorentz invariance and the gravitational field. Journal of Mathematical Physics. 1961;**2**:212-221

[18] Sciama DW. The physical structure of general relativity. Reviews of Modern Physics. 1964;**36**:463

[19] Krasnov K. Non-metric gravity: A status report. Modern Physics Letters A. 2007;**22**:3013-3026

[20] Plebanski JF. On the separation of Einsteinian substructures. Journal of Mathematical Physics. 1977;**18**:2511

[21] Kijowski J. On a new variational principle in general relativity and the energy of the gravitational field. General Relativity and Gravitation. 1978;**9**(10):857

[22] Ferraris M, Kijowski J. General relativity is a gauge type theory. Letters in Mathematical Physics. 1981;**5**(2):127

[23] Ferraris M, Kijowski J. On the equivalence of the relativistic theories of gravitation. General Relativity and Gravitation. 1982;**14**(2):165

[24] Krasnov K. Non-metric gravity. I. Field equations. Classical and Quantum Gravity. 2008; **25**:025001

[25] Krasnov K, Shtanov Y. Non-metric gravity. II. Spherically symmetric solution, missing mass and redshifts of quasars. Classical and Quantum Gravity. 2008;**25**:025002

[26] Krasnov K. Pure connection action principle for general relativity. Physical Review Letters. 2011;**106**:251103

[27] Nikodem J. Popławski. On the nonsymmetric purely affine gravity. Modern Physics Letters A. 2007;**22**(36):2701

[28] Nikodem J. Popławski. Affine theory of gravitation. General Relativity and Gravitation, **46**:1625, 2014

[29] Castillo-Felisola O, Skirzewski A. A polynomial model of purely affine Gravity. Revista Mexicana de Fisica. 2015;**61**:421

[30] Castillo-Felisola O and Skirzewski A. Einstein's gravity from an affine model. https://arxiv.org/abs/1505.04634. 2015

[31] Hehl FW, Dermott McCrea J, Mielke EW, Yuval N'e. Metric affine gauge theory of gravity: Field equations, noether identities, world spinors, and breaking of dilation invariance. Physics Reports. 1995;**258**:1-171

[32] Schouten JA. Ricci-calculus: An introduction to tensor analysis and its geometrical applications. Vol. 10. Berlin: Springer. 2013

[33] Ortín T. Gravity and Strings, Cambridge Monographs on Mathematical Physics. 2nd ed. Vol. 5. Cambridge: Cambridge University Press. 2015

[34] Nikodem J. Popławski. Cosmological constant from quarks and torsion. Annals of Physics, **523**:291, 2011

[35] Nikodem J. Popławski. Four-fermion interaction from torsion as dark energy. General Relativity and Gravitation. 2012;**44**:491–499

[36] Nikodem J. Popławski. Cosmology with torsion—An alternative to cosmic inflation. Physics Letters B. 2010;**694**:181

[37] Castillo-Felisola O, Corral C, del Pino S, Ramírez F. Kaluza–Klein cosmology from five-dimensional Lovelock–Cartan theory. Physical Review D. 2016;**94**(12):124020

[38] Belyaev AS, Shapiro IL. Torsion action and its possible observables. Nuclear Physics B. 1999;**543**:20

[39] Belyaev AS, Shapiro IL, do Vale MAB. Torsion phenomenology at the LHC. Physical Review D. 2007;**75**:034014

[40] Fabbri L, Vignolo S. A modified theory of gravity with torsion and its applications to cosmology and particle physics. International Journal of Theoretical Physics. 2012;**51**: 3186-3207

[41] Fabbri L. A Torsional Model of Leptons. Modern Physics Letters A. 2012;**A27**:1250199

[42] Capozziello S, Fabbri L, Vignolo S. Weak Forces and neutrino oscillations under the standards of hybrid gravity with torsion. Modern Physics Letters A. 2013;**28**(35):1350155

[43] Castillo-Felisola O, Corral C, Villavicencio C, Zerwekh AR. Fermion masses through condensation in spacetimes with torsion. Physical Review D. 2013;**88**:124022

[44] Castillo-Felisola O, Corral C, Schmidt I, Zerwekh AR. Updated limits on extra dimensions through torsion and LHC data. Modern Physics Letters A. 2014;**29**:1450081

[45] Castillo-Felisola O, Corral C, Kovalenko S, Schmidt I. Torsion in extra dimensions and one-loop observables. Physical Review D. 2014;**90**:024005

[46] Castillo-Felisola O, Corral C, Kovalenko S, Schmidt I, Lyubovitskij VE. Axions in gravity with torsion. Physical Review D. 2015;**91**(8):085017

[47] Curtright T. Generalized gauge fields. Physics Letters B. 1985;**165**:304

[48] McGady DA, Rodina L. Higher-spin massless *S*-matrices in four-dimensions. Physical Review D. 2014;**90**(8):084048

[49] Camanho XO, Edelstein JD, Maldacena J, Zhiboedov A. Causality constraints on corrections to the graviton three-point coupling. Journal of High Energy Physics. 2016 Feb 1; **2016**(2):20

[50] Nomizu K, Sasaki T. Affine differential geometry. Cambridge: Cambridge University Press. 1994

[51] Robert L. Bryant. Symmetries of non-Riemannian curvature tensor (answer). MathOverflow. Available from: http://mathoverflow.net/a/212794/25356 (Accessed: 3 August 2015)

[52] Bourguignon J-P. Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'einstein. Inventiones Mathematicae. 1981;**63**(2):263

[53] Bourguignon J-P, Blaine Lawson H Jr. Yang-Mills theory: Its physical origins and differential geometric aspects. In: Seminar on differential Geometry, Annals of Mathematics Studies. Vol. 102. 1982. p. 395

[54] Nakahara M. Geometry, Topology and Physics. London: Institute of Physics. 2005

[55] Stephenson G. Quadratic lagrangians and general relativity. Nuovo Cimento. 1958;**9**(2): 263-269

[56] Kilmister CW, Newman DJ. The use of algebraic structures in physics. In: Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 57. Cambridge: Cambridge University Press. 1961. p. 851

[57] Yang CN. Integral formalism for gauge fields. Physical Review Letters. Aug 1974;**33**:445

[58] Bekenstein JD, Majhi BR. Is the principle of least action a must? Nuclear Physics B. 2015;**892**:337

[59] Yano K. The theory of Lie derivatives and its applications, volume 3. Amsterdam: North-Holland. 1957

[60] Choquet-Bruhat Y, DeWitt-Morette C, Dillard-Bleick M. Analysis, manifolds and physics. Vol. 1 & 2. Amsterdam: North-Holland. 1989

[61] McInerney A. First Steps in Differential Geometry. New York: Springer. 2013

[62] Stein WA, et al. Sage Mathematics Software (Version 7.6). The Sage Development Team; 2017

[63] Gourgoulhon E, Bejger M, et al. SageManifolds (Version 1.0.1). SageManifolds Development Team; 2017

[64] Papapetrou A. Static spherically symmetric solutions in the unitary field theory. Proceedings of the Royal Irish Academy A. 1948;**52**:69

[65] Jebsen JT. On the General Spherically Symmetric Solutions of Einstein's Gravitational Equations in Vacuo. Arkiv for Matematik, Astronomi och Fysik. 1921;**15**:18

[66] Birkhoff GD. Relativity and Modern Physics. Boston: Harvard University Press. 1923

[67] Alexandrow W. Über den kugelsymmetrischen Vakuumvorgang in der Einsteinschen Gravitationstheorie. Annalen der Physik. 1923;**377**(18):141

[68] Eiesland J. The group of motions of an Einstein space. Transactions of the American Mathematical Society. 1925;**27**(2):213

[69] Gilmore R. Lie Groups, Lie Algebras, and Some of their Applications. New York: Dover. 2005

[70] Oscar Castillo-Felisola, Oscar Orellana, and Aureliano Skirzewski. Metric and non-metric solutions to the polynomial affine gravity (in preparation)

IntechOpen

IntechOpen

IntechOpen