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Chapter 4

The Role of Quantumness of Correlations in Entanglement Resource Theory

Tiago Debarba

Additional information is available at the end of the chapter

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Abstract

Quantum correlations: entanglement and quantumness of correlations are main resource for quantum information theory. In this chapter, the scenarios which quantumness of correlations plays an interesting role in entanglement distillation protocol are presented. By means of Koashi-Winter relation, it is discussed that quantumness of correlations are related to the irreversibility of the entanglement distillation protocol. The activation protocol is introduced, and it is proved that quantumness of correlations can create distillable entanglement between the system and the measurement apparatus during a local measurement process.

Keywords: quantumness of correlations, entanglement, quantum information

1. Introduction

Quantum entanglement plays the fundamental role in quantum information and computation [1, 2]. The resource theory of quantum entanglement, entanglement distillation [3] and entanglement cost [4] revealed one of the most fundamental aspects of quantum mechanics. Entanglement distillation protocol consists in converting a number of copies of an entangled state into few copies of maximally entangled states, by means of local operations and classical communication (LOCC) [5]. As maximally entangled states are the main resource of the quantum information, entanglement distillation protocol has many applications in this scenario, as quantum teleport [6], quantum error correction [7] and quantum cryptography [8]. A family of quantum information protocol arises from distillation of quantum entanglement and secret keys [3, 9]

However independently Ollivier and Zurek [10], and Henderson and Vedral [11] found a new quantum property, without counterpart in classical systems. They named it as the *quantumness of correlations*. This new kind of correlation reveals the amount of information destroyed during the local measurement process and goes beyond the quantum entanglement. There are many

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equivalent formulations for characterization and quantification of quantumness of correlations: quantum discord [10, 11], minimum local disturbance [12–14] and geometrical approach [15–17].

This chapter presents in detail two different ways to relate quantum entanglement and quantumness of correlations. The main purpose of this chapter is to discuss that quantumness of correlations plays an interesting role in entanglement distillation protocol. Entanglement and quantumness of correlations connect each other in two different pictures. The relation derived by Koashi and Winter [18] demonstrates the balance between quantumness of correlations and entanglement in the purification process [19]. This balance leads to a formal proof for the irreversibility of the entanglement distillation protocol, in terms of quantumness of correlations [20]. In the named *activation protocol*, the quantumness of correlations of a given composed system can be converted into distillable entanglement with a measurement apparatus during the local measurement process [21, 22].

The chapter is organized as follow. In Section 2, a mathematical overview is presented, and the notation is defined. Section 3 introduces some important concepts about the notion of quantum correlations: entanglement and quantumness of correlations. Section 4 presents the Koashi-Winter relation and its role in the irreversibility of quantum distillation process. Section 5 is intended to the description of the activation protocol, and the demonstration that quantumness of correlation can be activated into distillable entanglement.

2. Mathematical overview

This section introduces some quantum information concepts and defines the notation used in the chapter.

2.1. Density matrix and quantum channels

As the convex combination of positive matrices is also positive, then the space of positive operators forms a convex cone in Hilbert-Schmidt $\mathcal{L}(\mathbb{C}^N)$ [23]. If we restrict the matrices in the positive cone to be trace = 1, we arrive to another set of matrices, that is named the set of density matrices. This set of operators also originates a vector space, this space is denoted as $\mathcal{D}(\mathbb{C}^N)$. Therefore, the matrices that belong to this set, or the vectors in this vector space, are named *density matrices*.

Definition 1. *A linear positive operator $\rho \in \mathcal{D}(\mathbb{C}^N)$ is a density matrix and represents the state of a quantum system, if it satisfies the following properties:*

- *Hermitian:* $\rho = \rho^{\dagger}$
- *Positive semi-definite:* $\rho \geq 0$;
- *Trace one:* $\mathrm{Tr}(\rho) = 1$

As the convex combination of density matrices is a density matrix, the vector space $\mathcal{D}$ is a convex set whose pure states are projectors onto the real numbers. A given density matrix $\rho \in \mathcal{D}(\mathbb{C}^N)$ is a pure state if it satisfies:

$$\rho = \rho^2, \tag{1}$$

then the state ρ is a rank-1 matrix and it can be written as:

$$\rho = |\psi\rangle\langle\psi|. \tag{2}$$

The set of pure states is a $2(N-1)$-dimensional subset of the (N^2-2)-dimensional boundary of $\mathcal{D}(\mathbb{C}^N)$. Every state with at least one eigenvalue equal to zero belongs to the boundary [23]. For two-dimensional systems (it is also named qubit [24]), the boundary is just composed of pure states.

Consider a linear transformation $\Phi : \mathcal{L}(\mathbb{C}^N) \to \mathcal{L}(\mathbb{C}^M)$. This map represents a physical process, if it satisfies some conditions, determined by the mathematical properties of the density matrices. Indeed, to represent a physical process, the transformation must map a quantum state into another quantum state, $\Phi : \mathcal{D}(\mathbb{C}^N) \to \mathcal{D}(\mathbb{C}^M)$. It holds if Φ satisfy the following properties:

- **Linearity:** As a quantum state can be a convex combination of other quantum states, the map must be linear. For two arbitrary operators $\rho, \sigma \in \mathcal{D}(\mathbb{C}^N)$

$$\Phi(\rho + \sigma) = \Phi(\rho) + \Phi(\sigma); \tag{3}$$

- **Trace preserving:** The eigenvalues of the density matrix represent probabilities, and it sum must be one, then a quantum channel must to keep the trace of the density matrix:

$$\mathrm{Tr}[\Phi(\rho)] = 1. \tag{4}$$

- **Completely positive:** Consider a channel $\Phi : \mathcal{D}(\mathbb{C}_A) \to \mathcal{D}(\mathbb{C}_A)$ and a quantum state $\rho, \sigma \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, then

$$\mathbb{I} \otimes \Phi(\rho) \geq 0. \tag{5}$$

The map that satisfies this property is named *completely positive map*. The linear transformations mapping quantum states into quantum states are named completely positive and trace preserve (CPTP) *quantum channels*. The space of quantum channels that maps $N \times N$ density matrices onto $M \times M$ density matrices is denoted as $\mathcal{C}(\mathbb{C}^N, \mathbb{C}^M)$.

2.2. Measurement

Measurement is a classical statistical inference of quantum systems. The measurement process maps a quantum state into a classical probability distribution.

We can define a measurement as a function $\Pi : \Sigma \to \mathcal{P}(\mathbb{C}_\Gamma)$[1], associating an alphabet Σ to positive operators $\{\Pi_x\}_x \subset \mathcal{P}(\mathbb{C}_\Gamma)$. For a given density matrix $\rho \in \mathcal{D}(\mathbb{C}_\Gamma)$, the measurement process consists in to chose an element of Σ randomly. This random choice is represented by a

[1]Just to clarify the notation, when we write a subscript in the complex euclidean vector space, as $\mathbb{C}_\Gamma$, it represents a label to the space, it shall be very useful when we study composed systems. When we write a superscript on it, it represents the dimension of the complex vector space. For example, if $\dim(\mathbb{C}_\Gamma) = N$, we can also represent this space as $\mathbb{C}^N$, the usage of the notation will depend on the context.

probability vector $\vec{p} \in \mathbb{R}_+^N$, with N being the cardinality of the random variable described by $\vec{p}$. The elements of the probability vector $\vec{p}$ are given by:

$$p_x = \mathrm{Tr}\left(\Pi_x \rho\right), \tag{6}$$

where Π_x is the *measurement operator* associated to $x \in \Sigma$. The alphabet Σ is the set of measurement outcomes, and the vector $\vec{p}$ is the classical probability vector associated to the measurement process Π of a given density matrix ρ. As the outcomes are elements of a probability vector, these elements must be positive and sum to one. Which implies that the measurement operators must sum to identity:

$$\sum_x \Pi_x = \mathbb{I}_\Gamma, \tag{7}$$

where $\mathbb{I}_\Gamma$ is the identity matrix in $\mathbb{C}_\Gamma$. It is easy to check that this condition implies $\sum_x p_x = 1$:

$$\sum_x p_x = \sum_x \mathrm{Tr}\left(\Pi_x \rho\right) = \mathrm{Tr}\left(\sum_x \Pi_x \rho\right) = \mathrm{Tr}\left(\rho\right) = 1. \tag{8}$$

For instance, we shall restrict the measurements to a subclass of measurement operators named *projective measurements*. As it is shown later, its generalization can be performed via the Naimark's theorem. For projective measurements, the cardinality of $\vec{p}$ is at least the dimension of ρ, and the measurement operators are projectors:

$$\Pi_x^2 = \Pi_x, \tag{9}$$

for any $x \in \Sigma$. If we consider an orthonormal basis $\{|e_x\rangle\}$, where the vectors $|e_x\rangle$ span $\mathbb{C}_\Gamma$, this set represents a projective measurement for $\Pi_x = |e_x\rangle\langle e_x|$. The output state is described by the expression:

$$\rho_x = \frac{\Pi_x \rho \Pi_x}{\mathrm{Tr}\left(\Pi_x \rho\right)}. \tag{10}$$

The set of operators defines a convex hull in $\mathcal{P}(\mathbb{C}_\Gamma)$, then a measured state represents a pure state in this convex hull. In this way, the post-measurement state can be reconstructed by the convex combination of the output states $\rho = \sum_x p_x \rho_x$.

As physical processes are described by quantum channels, it is possible to describe the classical statistical inference of the quantum measurements as a CPTP channel. A channel that maps a quantum state into a probability vector is the dephasing channel. Therefore, the post-measurement state is the state under the action of the dephasing channel.

Theorem 2. *A given map $\Phi \in \mathcal{C}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ is a measurement if and only if:*

$$\Phi\left(\rho\right) = \sum_x \mathrm{Tr}\left(M_x \rho\right) |e_x\rangle\langle e_x|, \tag{11}$$

where $\rho \in \mathcal{D}(\mathbb{C}_\Gamma)$, $M_x \in \mathcal{P}(\mathbb{C}_\Gamma)$ and $|e_x\rangle \in \mathbb{C}_{\Gamma'}$.

In order to differ the set of measurement channels from a general CPTP channel, this set is represented as $\mathcal{P}$. A given measurement map $\mathcal{M} \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ is a quantum channel that maps a density matrix in a probability vector, $\mathcal{M} : \mathcal{D}(\mathbb{C}_\Gamma) \to \mathbb{R}^+_{\Gamma'}$. This probability vector is described by a diagonal density matrix as in Eq. (11). The dimension of $\mathbb{C}_{\Gamma'}$ is the number of outcomes of the measurement.

For general measurements, described by positive operators valued measure (POVM), the measurement process can be described by a measurement channel $\Phi \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_\Gamma)$. The description performed above can be followed to describe these general measurements, indeed projective measurements are a restriction for a POVM composed by orthogonal operators. Consider a set of positive operators $\{M_x\}_x$, representing a POVM, then $\mathrm{Tr}[M_x\rho] = p_x$ are the elements of a probability vector $\vec{p} \in \mathbb{R}^+_\Gamma$, then the post-measurement state is:

$$\Phi(\rho) = \sum_x p_x |e_x\rangle\langle e_x|. \tag{12}$$

Where $\{|e_x\rangle\}_x$ is an orthonormal basis in $\mathbb{C}_\Gamma$.

Using the Naimark's theorem, the measurement channel is described as a dephasing channel on a state in a enlarged space. In other words, for POVMs whose elements are rank-1 and linearly independent, it is possible to associate a projective measurement on an enlarged space.

Theorem 3 (Naimark's theorem). *Given a quantum measurement $\mathcal{M} \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$, with POVM elements $\{M_x\}_{x=0}^M$, there exists a projective measurement $\Pi \in \mathcal{P}(\mathbb{C}_{\Gamma'})$, with elements $\{\Pi_y\}_{y=0}^M$ such that:*

$$\mathrm{Tr}(M_x\rho) = \mathrm{Tr}(\Pi_x V \rho V^\dagger), \tag{13}$$

where $V \in \mathcal{U}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ is an isometry.

The action of the isometry on the state ρ, in the Naimark's theorem, is named as embedding operation. In this way, the isometry will be $V = \mathbb{I}_\Gamma \otimes |0\rangle_E$ and the enlarged space $\mathbb{C}_{\Gamma'} = \mathbb{C}_\Gamma \otimes \mathbb{C}_E$. For this simple case, the relation between the POVM elements $\{M_x\}_x$ and the projective measurement on the enlarged space $\{\Pi_x\}_x$:

$$M_x = (\mathbb{I}_\Gamma \otimes \langle 0|_E)\Pi_x(\mathbb{I}_\Gamma \otimes |0\rangle_E). \tag{14}$$

As the measurement can be described by a quantum channel, we can study how quantum measurements can be performed locally.

Definition 4. *Given a N-partite composed system, represented by the state $\rho_{A_1,\ldots,A_N} \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_N})$, we define the measurement on each subsystem applied locally:*

$$\Phi_{A_1} \otimes \cdots \otimes \Phi_{A_N}\left(\rho_{A_1,\ldots,A_N}\right) = \sum_{\vec{k}} \mathrm{Tr}\left[M_{k_1}^{A_1} \otimes \cdots \otimes M_{k_N}^{A_N} \rho_{A_1,\ldots,A_N}\right] \left|\vec{k}\right\rangle\left\langle\vec{k}\right|, \tag{15}$$

where $\left|\vec{k}\right\rangle = |k_1\rangle \otimes \cdots \otimes |k_N\rangle$ and the label $\vec{k}$ in the sum represents the set of indexes $k_1,\ldots, k_N$. $\left\{M_{k_x}^{A_x}\right\}_{k_x}$ are the measurement operators on each subsystem.

Suppose the measurement is performed on some subsystems, the remaining other subsystems are unmeasured. Consider a bipartite system $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and a measurement acting on the system B, then the measurement map will be written as:

$$\mathbb{I}_A \otimes \Phi_B(\rho_{AB}) = \sum_x \mathrm{Tr}_B\left[\mathbb{I}_A \otimes M_x^B \rho_{AB}\right] \otimes |b_x\rangle\langle b_x|. \tag{16}$$

As the measurement is not acting on A, the post-measured state on A will remain the same. If we write $p_x = \mathrm{Tr}_{AB}\left[\mathbb{I}_A \otimes M_x^B \rho_{AB}\right]$ and $\rho_x^A = \frac{\mathrm{Tr}_B\left[\mathbb{I}_A \otimes M_x^B \rho_{AB}\right]}{\mathrm{Tr}_{AB}\left[\mathbb{I}_A \otimes M_x^B \rho_{AB}\right]}$, the post-measured state will be:

$$\mathbb{I}_A \otimes \Phi_B(\rho_{AB}) = \sum_x p_x \rho_x^A \otimes |b_x\rangle\langle b_x|. \tag{17}$$

As the measurement is a classical statistical inference process, the local measurement process destroys the quantum correlations between the systems. Indeed the post-measured state is not a classical probability distribution, although it only has classical correlations.

2.3. Quantum entropy

Consider that one can prepare an ensemble of quantum states $\xi = \{p_x, \rho_x\}_x$, accordingly to some random variable X. Classical information can be extracted from the ensemble of quantum states, in the form of a variable Y, performing measurements on the quantum system. The conditional probability distribution to obtain a value y, given as input the state ρ_x is:

$$p(y|x) = \mathrm{Tr}\left(M_y \rho_x\right), \tag{18}$$

where $\{M_y\}_y$ is a POVM. The joint probability distribution X and Y is given by:

$$p(x,y) = p_x \mathrm{Tr}\left(M_y \rho_x\right). \tag{19}$$

The probability distribution of Y is obtained from the marginal probability distribution:

$$p(y) = \sum_x p(x,y) = \sum_x p_x \mathrm{Tr}\left(M_y \rho_x\right) = \mathrm{Tr}\left(M_y \sum_x p_x \rho_x\right). \tag{20}$$

Considering the Bayes rule:

$$p(x,y) = p_x p(y|x) = p(y) P(x|y), \tag{21}$$

it is possible to obtain the conditional probability distribution with elements:

$$P(x|y) = \frac{p_x p(y|x)}{p(y)}. \tag{22}$$

Even in the case the system is always prepared in the same state, there exists an uncertainty about the measured of an observable. The probability distributions presented above are

evidencing this uncertainty, for the measurement observables of a POVM. These probability distributions are classical probability distributions extracted from the quantum systems, and the Shannon entropy quantifies the degree of surprise related to a given result.

It is also possible to define a quantum analogous to the Shannon entropy. This quantum entropy is named as von Neumann entropy, and in analogy with Shannon entropy, it is defined as the expectation value of the operator $\log_2(\rho)$.

Definition 5 (von Neumann entropy). *Given a density operator $\rho \in \mathcal{D}(\mathbb{C}^N)$, the quantum version of the Shannon entropy is defined as the function:*

$$S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]. \tag{23}$$

The von Neumann entropy can be rewritten as:

$$S(\rho) = -\sum_k \lambda_k \log_2(\lambda_k), \tag{24}$$

where $\{\lambda_k\}_k$ are the eigenvalues of $\rho = \sum_k \lambda_k |k\rangle\langle k|$. The von Neumann entropy has the same interpretation of the Shannon entropy for the probability distribution composed by the eigenvalues of the density matrix. The von Neumann entropy is zero of pure states, and it is maximum for the maximally mixed state $\mathbb{I}/N$, where it is $S(\mathbb{I}/N) = \log_2 N$.

For composed systems, the von Neumann entropy is analogous to the Shannon entropy of the joint probability. For a bipartite state ρ_{AB}, the joint von Neumann entropy is:

$$S(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB} \log_2 \rho_{AB}). \tag{25}$$

Follow some interesting, and useful, properties about von Neumann entropy:

1. (**Pure states)** For a bipartite pure state $|\phi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$, the partitions have the same von Neumann entropy:

$$S(\rho_A) = S(\rho_B), \tag{26}$$

   where $\rho_A = \mathrm{Tr}_B(|\phi\rangle\langle\phi|_{AB})$.

2. (**Additivity)** von Neumann entropy is additive:

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma), \tag{27}$$

   where ρ and σ are density matrices.

3. (**Concavity)** von Neumann entropy is a concave function:

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i), \tag{28}$$

   for a convex combination $\rho = \sum_i p_i \rho_i$.

4. (**Classical-quantum states)** For bipartite state in the form $\rho_{AB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, the von Neumann entropy will be:

$$S\left(\sum_x p_x |x\rangle\langle x| \otimes \rho_x\right) = H(X) + \sum_x p_x S(\rho_x), \tag{29}$$

where $H(X) = -\sum_x p_x \log_2 p_x$

For composed system, it is possible to define a quantum analogous to the mutual information for bipartite states.

Definition 6 (Mutual information). *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the quantum mutual information is defined as:*

$$I(A:B)_{\rho_{AB}} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{30}$$

The quantum mutual information of ρ_{AB} quantifies the correlations in quantum systems. It can be interpreted as the number of qubits that one part must send to another to destroy the correlations between the entire system. As the amount of correlations in a quantum state must be positive, it is possible to conclude that:

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}). \tag{31}$$

From property 2, it is easy to see that mutual information is zero for product state $\rho_{AB} = \rho_A \otimes \rho_B$. The mutual information of pure states will be equal to:

$$I(A:B)_{\psi_{AB}} = 2S(\rho_A) = 2S(\rho_B), \tag{32}$$

where $\psi_{AB} = |\psi\rangle\langle\psi|_{AB}$ is pure state.

The quantum version of the relative entropy quantifies the distinguishability between quantum states.

Definition 7 (Quantum relative entropy). *Given two density matrices $\rho, \sigma \in \mathcal{D}(\mathbb{C}^N)$, the distinguishability between them can be quantified using the quantum relative entropy:*

$$S(\rho||\sigma) = \mathrm{Tr}[\rho \log_2 \rho - \rho \log_2 \sigma]. \tag{33}$$

It will be zero if $\rho = \sigma$.

The quantum relative entropy is a positive function for $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, otherwise it diverges to infinity. The quantum mutual information can also be written as a quantum relative entropy.

Proposition 8. *Consider a bipartite state ρ_{AB}, the following expression holds:*

$$I(A:B)_{\rho_{AB}} = S(\rho_{AB}||\rho_A \otimes \rho_B), \tag{34}$$

where ρ_A and ρ_B are the reduced states of ρ_{AB}.

In contrast with the von Neumman entropy, the relative entropy always decreases under the action of a quantum channel. This property has an operational meaning: two states are always less distinguishable under the action of noise.

Theorem 9. *Given two density matrices $\rho, \sigma \in \mathcal{D}(\mathbb{C}_A)$ and a quantum channel $\Gamma \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B)$, the following inequality holds:*

$$S(\rho||\sigma) \geq S(\Gamma(\rho)||\Gamma(\sigma)) \tag{35}$$

This theorem implies into another property of the quantum mutual information: it decreases monotonically under local CPTP channels. As mutual information quantifies correlations, this means that the amount of correlations reduce under local noise.

Corollary 10. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and quantum channel $\Phi_B \in \mathcal{C}(\mathbb{C}_B, \mathbb{C}_{B'})$, the mutual information satisfies:*

$$I(A:B)_{\rho_{AB}} \geq I(A:B')_{\mathbb{I}\otimes\Phi(\rho_{AB})}. \tag{36}$$

Proof. Given the mutual information:

$$I(A:B)_{\rho_{AB}} = S(\rho_{AB}||\rho_A \otimes \rho_B) \tag{37}$$

using the theorem above:

$$I(A:B)_{\rho_{AB}} \geq S(\mathbb{I}_A \otimes \Phi_B(\rho_{AB})||\rho_A \otimes \Phi_B(\rho_B)) = I(A:B')_{\mathbb{I}\otimes\Phi(\rho_{AB})}. \tag{38}$$

Analogous to the classical conditional entropy, it is possible to define a quantum version of it. For a bipartite system ρ_{AB}, the quantum conditional entropy quantifies the amount of information of A that is available when B is known.

Definition 11 (Conditional entropy). *Consider a bipartite system ρ_{AB}, the quantum conditional entropy is defined as the function:*

$$S(A|B)_{\rho_{AB}} = S(\rho_{AB}) - S(\rho_B). \tag{39}$$

One interesting property of the quantum conditional entropy is that it can be negative. For example, if we consider a bipartite pure state $|\phi\rangle_{AB} = (|00\rangle + |11\rangle)/\sqrt{2}$, von Neumann entropy of the pure state is zero: $S(|\phi\rangle\langle\phi|_{AB}) = 0$. Nonetheless the reduced state is the maximally mixed state: $\rho_B = \mathbb{I}/2$, whose von Neumann entropy is $S(\mathbb{I}/2) = 1$. Therefore, the conditional entropy of this state is negative $S(A|B)_{|\phi\rangle\langle\phi|_{AB}}) = -1$. The negative value of the quantum conditional entropy is defined as the *coherent information*:

$$I(A\rangle B) = -S(A|B). \tag{40}$$

The conditional entropy has an operational meaning in the state merging protocol, where a tripartite pure state is shared by two experimentalists, one will send part of its state through a

quantum channel to the other. The coherent information quantifies the amount of entanglement required to the sender be able to perform the protocol. If it is positive, they cannot use entanglement to perform the state merging, and in the end the amount of entanglement grows [25–27]. The coherent information also quantifies the capacity of a quantum channel, optimizing over all input states ρ_A, the output state is known to be ρ_B. This result is named as *LSD theorem* [28–31].

3. Quantum correlations

3.1. Entanglement

This section introduces the concept of quantum entanglement, presenting its characterization and quantification.

3.1.1. Separable states

Consider two systems A and B, often named the experimentalists responsible by the systems as Alice and Bob, respectively. The state of the systems A and B is described by a density matrix on a Hilbert space. In this way considering two finite Hilbert spaces $\mathbb{C}_A$ and $\mathbb{C}_B$, and a basis in each one:

$$\{|a_i\rangle\}_{i=0}^{|A|-1} \in \mathbb{C}_A; \tag{41}$$

$$\{|b_k\rangle\}_{k=0}^{|B|-1} \in \mathbb{C}_B, \tag{42}$$

where $|A| = \dim(\mathbb{C}_A)$ and $|B| = \dim(\mathbb{C}_B)$. The global system, composed of A and B, can be obtained through the tensor product between the basis in the Hilbert space of each system:

$$\{|a_i, b_k\rangle\}_{i,j=0}^{|AB|-1} = \{|a_i\rangle \otimes |b_k\rangle\}_{i,k=0}^{|A|-1,\,|B|-1}, \tag{43}$$

hence the dimension of the composed system is the product of the dimension: $|AB| = \dim(\mathbb{C}_{AB}) = \dim(\mathbb{C}_A) \cdot \dim(\mathbb{C}_B)$. The Hilbert space of the composed system is denoted as $\mathbb{C}_{AB} = \mathbb{C}_A \otimes \mathbb{C}_B$. A pure state of the composed system can be decomposed in the basis in Eq. (43):

$$|\psi\rangle_{AB} = \sum_{i,\,k} c_{i,\,k} |a_i\rangle \otimes |b_k\rangle. \tag{44}$$

From this expression, one can realize that: in general a pure state, which describes a composed system, cannot be written as the product of the state of each system. In other words, suppose the system A and B described by the states $|\alpha\rangle_A = \sum_i a_i |a_i\rangle \in \mathbb{C}_A$ and $|\beta\rangle_B = \sum_k b_k |b_k\rangle \in \mathbb{C}_B$, the composed system is described by the state:

$$|\alpha\rangle \otimes |\beta\rangle = \sum_{i,\,k} a_i b_k |a_i\rangle \otimes |b_k\rangle. \tag{45}$$

It is the particular case where the coefficients in Eq. (44) are $c_{i,k} = a_i \cdot b_k$. If a composed system can be written as Eq. (45), it is called a *product state*, and there is no correlations between A and

B. It can be checked easily via the mutual information of the state, which is clearly zero once that the von Neumman entropy of the pure state is zero [32–34].

The concept of product state can be generalized for mixed state. Considering a composed system represented by the state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, it is called a *product state* if can be written as:

$$\rho_{AB} = \rho_A \otimes \rho_B, \tag{46}$$

where $\rho_A \in \mathcal{D}(\mathbb{C}_A)$ and $\rho_B \in \mathcal{D}(\mathbb{C}_B)$ are the states of the systems A and B, respectively. The product state for mixed states is also no correlated, as its mutual information is zero. As the space of quantum states is a convex set, the convex combination of states will also be a quantum state. The convex combination of product states generalizes the notion of product states, that is named as *separable state* [35].

Definition 12 (Separable states). *Considering a composed system described by the state* $\sigma \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, *it is a separable state if and only if can be written as:*

$$\sigma = \sum_{i,j} p_{i,j} \sigma_i^A \otimes \sigma_j^B, \tag{47}$$

where $\sigma_i^A \in \mathcal{D}(\mathbb{C}_A)$ *and* $\sigma_j^B \in \mathcal{D}(\mathbb{C}_B)$.

The set of quantum channels that let separable states invariant is named *local operations and classical communication* (LOCC). The set of separable states form a subspace in the space of density matrices, it can be denoted as $Sep(\mathbb{C}_{AB})$. The separable state can be easily extended to multipartite systems. Considering a *n*-partite system, it is named *m*-separable if it can be decomposed in a convex combination of product states composed by *m* parties.

3.1.2. Entanglement quantification

A measure of entanglement for mixed state can be obtained from the quantification of entanglement for pure states. It is possible to construct a measure of entanglement in this sense calculating the average of entanglement taken on pure states needed to form the state. The most famous measure which follow this idea is named as *entanglement of formation*. The entanglement of formation is interpreted as the minimal pure state entanglement required to build the mixed state [7].

Definition 13. *Considering a quantum state* $\rho \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, *the entanglement of formation is defined as:*

$$E_f(\rho) = \min_{\xi_\rho} \sum_i p_i E(|\psi_i\rangle), \tag{48}$$

where the optimization is performed over all ensembles $\xi_\rho = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^M$, *such that* $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$ *and* $p_i \geq 0$.

The entanglement entropy $E(|\psi_i\rangle)$ is defined as:

$$E\left(|\psi_i\rangle\right) = S\left(\mathrm{Tr}_B\left[|\psi_i\rangle\langle\psi_i|\right]\right), \tag{49}$$

where $S(\mathrm{Tr}_B[|\psi_i\rangle\langle\psi_i|])$ is the von Neumann entropy of the reduced state of $|\psi_i\rangle$. The entanglement of formation is not easy to evaluate. Indeed the minimization process implies in to find an optimal convex hull, in function of a nonlinear function. For two qubits systems, it can be calculated analytically [36].

Quantum entanglement also enables an operational interpretation. This interpretation has two different ways: the resource required to construct a given quantum state and the resource extracted from a quantum system. The resource here refers to the amount of copies of maximally mixed state. Then, one can define the measure of this resource as a measure of entanglement in the limit of many copies.

The number of copies m of maximally entangled states required to construct n copies of a given state ρ, by means of LOCC protocols, is named *entanglement cost* [7]. The entanglement cost can be written as the regularized version of the entanglement of formation [4].

Definition 14 (Entanglement cost). *The number of copies of the maximally entangled states required to build the state ρ is given by:*

$$E_C(\rho) = \lim_{n\to\infty} \frac{E_f\left(\rho^{\otimes n}\right)}{n}, \tag{50}$$

where $E_f(\rho^{\otimes n})$ is the entanglement of formation of the n copies of ρ.

The number of copies m of the maximally entangled state which can be extracted from n copies of a given state ρ, by LOCC, is named as *distillable entanglement* [7].

Definition 15 (Distillable entanglement). *The distillable entanglement of a given state ρ is defined as:*

$$E_D(\rho) = \lim_{n\to\infty} \frac{m}{n}, \tag{51}$$

where m is the number of maximally entangled states that can be extracted from ρ in the limit of many copies.

The distillable entanglement is a very important operational measure of entanglement, because it quantifies how useful is a given quantum state, for the quantum information purpose.

The operational meaning of the entanglement cost and the distillable entanglement compose the research theory of quantum entanglement. The entanglement cost and the distillable entanglement of a given state are not the same. Indeed the cost of entanglement is greater than the distillable entanglement. The point is: it is more expensive to create a state ρ with copies of maximally entangled state than is possible to extract entanglement from ρ. One example is the bound entangled state, even it is entangled it is not possible to extract any maximally entangled state, although it requires an amount of maximally entangled states to build it.

3.2. Quantumness of correlations

This section presents a revision about some basic concepts of quantumness of correlations for distinguishable systems. The notion of classically correlated states and quantum discord is presented.

3.2.1. Classically correlated states

Consider a flip coin game with two distinct events described by the states $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, each with the same probability 1/2. It is known that it is possible to distinguish the faces of the coin, with a null probability of error. The probability of error to distinguish two events, or two probability distributions, depends on the trace distance of the probability vectors of the events:

$$P_E(|0\rangle\langle 0|, |1\rangle\langle 1|) = \frac{1}{2} - \frac{1}{4}\| |0\rangle\langle 0| - |1\rangle\langle 1| \|_1, \tag{52}$$

as the states are orthogonal $\| |0\rangle\langle 0| - |1\rangle\langle 1| \|_1 = 2$, therefore the probability of error $P_E(|0\rangle\langle 0|, |1\rangle\langle 1|) = 0$, as one expected. Now suppose a quantum coin flip, which coherent superposition between the two faces of the coin, described by the events: $\{|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|\}$, with equal probability 1/2, where $|\phi\rangle, |\psi\rangle \in \mathbb{C}^2$. As an example, consider the states $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$. For this case, the overlap is $\langle\phi|\psi\rangle = 1/\sqrt{2}$. The trace distance of these states is simply:

$$\| |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| \|_1 = \sqrt{2},$$

then the probability of error to distinguish the events is not zero. Superposition of states in quantum mechanics creates events that cannot be perfectly distinguished. The distinguishability of quantum or classical events can be quantifier by the Jensen-Shannon divergence. For two probability distributions (or events), it is defined as the symmetric and smoothed version of the Shannon relative entropy, or in the quantum case the von Neumman relative entropy [37, 38].

Definition 16. *The Jensen-Shannon divergence for two arbitrary events $|\psi\rangle, |\phi\rangle$ is defined as:*

$$J(|\psi\rangle, |\phi\rangle) = \frac{1}{2} S\left(\frac{|\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|}{2} \| |\phi\rangle\langle\phi|\right) + \frac{1}{2} S\left(\frac{|\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|}{2} \| |\psi\rangle\langle\psi|\right). \tag{53}$$

For the classical coin flip game, the Jensen-Shannon divergence will be just $J(|0\rangle, |1\rangle) = 1$. On the other hand, for the quantum coin flip with states $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$, it will be $J(|\phi\rangle, |\psi\rangle) = \sqrt{2}$. The Jensen-Shannon divergence is related to the Bures distance and induces a metric for pure quantum states related to the Fisher-Rao metric [39], it is lager for more distinguishable events, and the largest distance characterizes complete distinguishable events. The Jensen-Shannon divergence for two arbitrary events $|\psi\rangle, |\phi\rangle$ is related to the mutual information [37]:

$$J(|\psi\rangle, |\phi\rangle) = I(R:E)_{\rho_{RE}}, \tag{54}$$

where R represents a register, E represents the events and $\rho_{RE} \in \mathcal{D}(\mathbb{C}_R \otimes \mathbb{C}_E)$ characterizes the existence of two distinct events:

$$\rho_{RE} = \frac{1}{2}|0\rangle\langle 0|_R \otimes |\phi\rangle\langle\phi|_E + \frac{1}{2}|1\rangle\langle 1|_R \otimes |\psi\rangle\langle\psi|_E. \tag{55}$$

For the classical coin flip game, it is $\rho^c_{RE} = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$, with mutual information $I(R:E)_{\rho^c_{RE}} = 1$. For the quantum coin, the state will be $\rho^q_{RE} = \frac{1}{2}|0\rangle\langle 0| \otimes |\phi\rangle\langle\phi| + \frac{1}{2}|1\rangle\langle 1| \otimes |\psi\rangle\langle\psi|$, where for $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$, and the mutual information is $I(R:E)_{\rho^q_{RE}} = \sqrt{2}$. As the mutual information is a measure of correlations between two probability distributions, one realizes that there are more correlations between the register and the events for not completely distinguishable registers, in comparison with orthogonal registers. However, two binary classical distributions cannot share more than one bit of information; in other words, their mutual information cannot be greater than one [31]. As the correlations between the quantum coin events and the register are bigger than one, it means that there are correlations beyond the classical case. A quantum state is classically correlated if there exists a local projective measurement such that the state remains the same [10–12]. The state ρ^c_{RE} is an example of *classical-classical state*. In general, these states are defined as:

Definition 17 (classical-classical states). *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, it is strictly classically correlated (or classical-classical state) if there exists a local projective measurement Π_{AB} with elements $\{\Pi^A_l \otimes \Pi^B_k\}_{k,l}$ such that the post-measured state is equal to the input state:*

$$\Pi(\rho_{AB}) = \sum_{k,l} \Pi^A_l \otimes \Pi^B_k \rho_{AB} \Pi^A_l \otimes \Pi^B_k = \rho_{AB}, \tag{56}$$

therefore $\rho_{AB} = \sum_{k,l} p_{k,l} \Pi^A_l \otimes \Pi^B_k$, and $\Pi^Y_x = |e_x\rangle\langle e_x|_Y$ is a projetor in the orthonormal basis $\{|e_x\rangle_Y\}_x \in \mathcal{H}_Y$.

The state ρ^q_{ER} is an example of a *classical-quantum state*, because there exists a projective measurement, with elements $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, over partition E that keep the state unchanged. On the other hand, there is not a projective measurement over partition R with this property. In general, a state ρ_{AB} is classical-quantum if there exists a projective measurement Π_A with elements $\{\Pi_k\}_k$ such that:

$$\Pi_A \otimes \mathbb{I}_B(\rho_{AB}) = \rho_{AB} = \sum_k p_k \Pi_k \otimes \rho_k. \tag{57}$$

The set o classically correlated states is not convex, once that combination of block diagonal matrices cannot be block diagonal. As the identity matrix is block diagonal, or just diagonal, this set is connected by the maximally mixed state, and it is a *thin set* [40].

3.2.2. Quantum discord

The amount of classical correlations in a quantum state is measured by the capacity to extract information locally [41]. As the measurement process is a classical statistical inference, classical

correlations can be quantified by the amount of correlations that are not destroyed by the local measurement.

Definition 18. *For a bipartite density matrix $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the classical correlations between A and B can be quantified by the amount of correlations that can be extracted via local measurements:*

$$J(A:B)_{\rho_{AB}} = \max_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} = \max_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} \left\{ S(\rho_A) - \sum_x p_x S(\rho_x^A) \right\}, \tag{58}$$

where the optimization is taken over the set of local measurement maps $\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}(\mathcal{H}_{AB},\mathcal{H}_{AX})$, and $\mathbb{I}\otimes\mathcal{B}(\rho_{AB}) = \sum_x p_x \rho_x^A \otimes |b_x\rangle\langle b_x|$ is a quantum-classical state in the space $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_X)$.

Originally, Ollivier and Zurek [10] have defined this expression restricting the optimization to projective measurements. Independently, Henderson and Vedral [11] have defined the optimization of the classical correlations over general POVMs. As the mutual information quantifies the total amount of correlations in the state, it is possible to define a quantifier of quantum correlations as the difference between the total correlations in the system, quantified by mutual information, and the classical correlations, measured by Eq. (58). This measure of quantumness of correlations is named as *quantum discord*:

Definition 19. *The quantum discord $D(A:B)_{\rho_{AB}}$ of a state ρ_{AB} is defined as:*

$$D(A:B)_{\rho_{AB}} = I(A:B)_{\rho_{AB}} - J(A:B)_{\rho_{AB}}, \tag{59}$$

where $I(A:B)_{\rho_{AB}}$ is the von Neumann mutual information.

Quantum discord quantifies the amount of information, that cannot be accessed via local measurements. Therefore, it measures the quantumness shared between A and B that cannot be recovered via a classical statistical inference process. The optimization of quantum discord is a NP-hard problem [42]. A general analytical solution for quantum discord is not known or a criterion for a giving POVM to be optimal. Nonetheless, there are some analytic expressions for some specific states [43–45]. It is a natural generalization of quantum discord for the case the measurement is performed locally on both subsystems.

Definition 20. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ the quantum discord over measurements on both systems is:*

$$D(A:B)_{\rho_{AB}} = \min_{\mathcal{A}\otimes\mathcal{B}\in\mathcal{P}} \left\{ I(A:B)_{\rho_{AB}} - I(A:B)_{\mathcal{A}\otimes\mathcal{B}(\rho_{AB})} \right\}, \tag{60}$$

where $\mathcal{A}\in\mathcal{P}(\mathbb{C}_A,\mathbb{C}_Y)$ and $\mathcal{B}\in\mathcal{P}(\mathbb{C}_B,\mathbb{C}_X)$.

This generalization of quantum discord was first discussed in [46] in the context of the non-local-broadcast theorem. This definition is often named WPM-discord, because it was also studied by Wu et al. [47]. It was also studied restricting to projective measurements by some authors [48, 49].

3.2.3. Relative entropy of quantumness and work deficit

For a given dephasing channel $\Pi \in \mathcal{P}(\mathbb{C}^N)$, acting on any state $\rho \in \mathcal{D}(\mathbb{C}^N)$, the support of the dephased state contains the support of the input state: supp(ρ) ⊆ supp($\Pi[\rho]$); therefore, the measure of quantumness of correlations based on the relative entropy remains finite for every composed state [23, 31].

Suppose Alice and Bob have a common composed system described by the state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, they would like to extract work from this system. To accomplish their task, they can perform the *closed set of local operations and classical communication* (CLOCC). This class of operations is composed of: (i) addition of pure ancillas, (ii) local unitary operations and (iii) local dephasing channels. Classical communication is represented by a local dephasing channel. If Alice and Bob are together in the same laboratory, they can extract work globally from the total system, then the total amount of information that Alice and Bob can extract from ρ_{AB} together is defined as the *total work* [12].

Definition 21. *The work that can be extracted from a quantum system, described by the state* $\rho \in \mathcal{D}(\mathbb{C}^N)$*, is defined as the change in the entropy:*

$$W_t(\rho) = \log_2 N - S(\rho), \tag{61}$$

$\log_2 N$ *is the entropy of the maximally mixed state, and* $S(\rho)$ *is the von Neumann entropy of the state.*

This function can be interpreted as a quantifier of information, such that if the state is a maximally mixed state no information can be extracted from it. Therefore, if the state is a pure state, we have the maximum amount of information [12, 50]. The entropy function represents the amount of information that one *can get to know* about the system; therefore, the function Eq. (61) represents the amount of information that one *already knows*. On the other hand, if Alice and Bob cannot be in the same laboratory, the information that can be extracted from the total state is restricted to be locally accessed. In the same way, it is possible to define the total information, named *local work*. Then, Alice and Bob should perform CLOCC operation in order to obtain the maximal amount of local information [50]:

$$W_l(\rho_{AB}) = \log_2 N - \sup_{\Gamma \in CLOCC} S(\Gamma[\rho_{AB}]), \tag{62}$$

where the state $\Gamma(\rho_{AB})$ is the state after the protocol. As CLOCC consist in sending one part of the state in a dephasing channel, at the end of the protocol, the whole state is with the receiver: $\Gamma(\rho_{AB}) = \rho_{AA}$.

One can be interested in the amount of information that cannot be extracted locally by Alice and Bob. This function is named *work deficit* and it quantifies the amount of work that is not possible to extract locally [12].

Definition 22. *Given a bipartite state* ρ_{AB}*, the information which two parts Alice and Bob cannot access, via CLOCC, is the work deficit:*

$$\Delta(\rho_{AB}) = W_t(\rho_{AB}) - W_l(\rho_{AB}). \tag{63}$$

From the definition of the total work and the local work, we can define the work deficit as the diference of them:

$$\Delta(\rho_{AB}) = \inf_{\Gamma \in CLOCC} \{S(\Gamma[\rho_{AB}]) - S(\rho_{AB})\}. \tag{64}$$

Even though the total and the local work depend explicitly on the dimension of the system, the work deficit should not depend on the dimension of $\Gamma[\rho_{AB}]$. Adding local pure ancillas belongs to the CLOCC cannot change the amount of work deficit. The work deficit can quantify quantum correlations, then it must not change by the simple addition of a uncorrelated system [16, 50].

In the asymptotic limit (the limit of many copies), the work deficit quantifies the amount of pure states that can be extracted locally [51, 52]. However, as a resource cannot be created freely, the addition of pure local ancillas is not allowed, then it is replaced by the addition of maximally mixed states. The set of operations that contains: (i) addition of maximally mixture states, (ii) local unitary operations and (iii) local dephasing channels, is named *noise local operations and classical communication* (NLOCC) [51]. The extraction of local pure states is a protocol, whose goal is to extract resource (coherence). The set of available operations are NLOCC operations, and the set of free resource states is composed only by the maximally mixture state. It is the only state without local purity [53]. It remains an open question if the CLOCC class and the NLOCC class are equivalent classes [50].

In the limit of one copy, the work deficit can quantify quantum correlations present in a given composed system [54]. The scenario where Alice and Bob can perform many steps of classical communication one each other is named *two way*, and the work deficit is named *two-way work deficit*. In this case, they can perform measurements and communicate in each step of the protocol. Mathematically, the two-way work deficit does not have a closed expression [50]. As discussed above, it is possible to activate quantum correlations performing operations on the measured system. Therefore, this many step scenario cannot quantify quantum correlations. Because if Alice and Bob can implement a sequence of non-commuting dephasing channels, the only invariante state is the maximally mixed state. In this way, it is necessary a one round description, where Alice and Bob can communicate at the end of the protocol. Following this idea, it is possible to define the *one-way work deficit*, which just one side can communicate. If Bob communicates to Alice, the state created at the end of the protocol is a quantum-classical state (or a classical-quantum state if Alice communicates at the end of the protocol).

Definition 23 (one-way work deficit). *Given a bipartite state ρ_{AB}, the work deficit with just one side communication is named one-way work deficit* [12]:

$$\Delta^{\rightarrow}(\rho_{AB}) = \min_{\Pi_B \in \mathcal{P}} \{S(\mathbb{I}_A \otimes \Pi_B[\rho_{AB}]) - S(\rho_{AB})\}, \tag{65}$$

where $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$ is a local dephasing on subsystem B. The notation $\Delta^{\rightarrow}(\rho_{AB})$ means that the communication is from A to B and $\Delta^{\leftarrow}(\rho_{AB})$ in the opposite direction.

Another definition for the work deficit is defined when both Alice and Bob communicate at the end of the protocol, this is named *zero-way work deficit*. The state created at the end of the protocol is a classical-classical state.

Definition 24 (zero-way work deficit). *Given a bipartite state ρ_{AB}, the work deficit with no communication until the end of the protocol is named zero work deficit* [12]:

$$\Delta^{\varnothing}(\rho_{AB}) = \min_{\Pi_A \otimes \Pi_B \in \mathcal{P}} \{S(\Pi_A \otimes \Pi_B[\rho_{AB}]) - S(\rho_{AB})\}, \tag{66}$$

where $\Pi_A \otimes \Pi_B \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is a local dephasing on subsystems A and B.

In analogy with the work deficit, Modi et al. proposed a measure of quantumness of correlation defined as the relative entropy of the state and the set of classical correlated states [16]. This measure is named *relative entropy of quantumness*.

Definition 25 (relative entropy of quantumness). *The relative entropy of quantumness $D(\rho_{AB})_{QC}$ for a given state ρ_{AB} is defined as the minimum relative entropy over the set of quantum-classical states* [16]:

$$D(\rho_{AB})_{QC} = \min_{\xi_{AB} \in \Omega_{QC}} S(\rho_{AB} \| \xi_{AB}), \tag{67}$$

where Ω_{QC} is the set of quantum-classical states.

The relative entropy of quantumness for classical-classical states is denoted as $D(\rho_{AB})_{CC}$. It is analogous to Eq. (67) when the optimization is taken over the set of classical-classical states Ω_{CC}:

$$D(\rho_{AB})_{CC} = \min_{\xi_{AB} \in \Omega_{CC}} S(\rho_{AB} \| \xi_{AB}). \tag{68}$$

As discussed previously, in the limit of one copy, the one-way and the zero-way work deficits quantify quantumness of correlations of the system. It is possible to obtain the equivalence between one-way work deficit and relative entropy of quantumness.

Theorem 26. *The one-way work deficit is equal to the relative entropy of quantumness for quantum-classical states* [16, 50]:

$$D(\rho_{AB})_{QC} = \Delta^{\rightarrow}(\rho_{AB}), \tag{69}$$

The same equivalence holds for zero-way work deficit and the relative entropy of quantumness of classical-classical states:

$$D(\rho_{AB})_{CC} = \Delta^{\varnothing}(\rho_{AB}). \tag{70}$$

The one-way and zero-way work deficits quantify quantumness correlations beyond the quantum entanglement; therefore, we should be able to compare these two classes of quantum correlations. For the relative entropy, this comparison is natural of the fact that CLOCC is a subclass of LOCC operations, which naturally implies that [12]:

$$\Delta(\rho) \geq E_r(\rho), \tag{71}$$

where $\Delta(\rho)$ is the work deficit and $E_r(\rho)$ is the relative entropy of entanglement. The equality is attached for bipartite pure states: $|\psi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$:

$$\Delta(\Psi_{AB}) = E_r(\Psi_{AB}) = S(\rho_A), \tag{72}$$

where $\Psi_{AB} = |\psi\rangle\langle\psi|_{AB}$. An interesting corollary of this proposition is that the quantum discord is equal to the work deficit for pure states, because it is also equal to the entropy of entanglement for pure states.

In this section, the concept of local disturbance was introduced by the definition of the work deficit. That is the smallest relative entropy between the state and its local disturbed version (obtained performing a local dephasing channel on the state). Indeed there are many other local disturbance quantumness of correlation quantifiers, which can be obtained defining a quantum state discrimination measure, for example, Bures distance [55], Schatten p-norm [17], trace distance [56] and Hilbert-Schmidt distance [15, 57].

4. Monogamy relation: entanglement, classical correlations and quantumness of correlations

Given a bipartite system $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, then it is possible to purify the state in a larger space $\mathbb{C}_{ABE}$ of the dimension: $\dim(\mathbb{C}_{ABE}) = \dim(A) \cdot \dim(B) \cdot \mathrm{rank}(\rho_{AB})$. The purification process creates quantum correlations between the system AB and the purification system E, unless the state is already pure. Intrinsically, there is a restriction in the amount of correlations that can be shared by the systems. This balance between the correlations for tripartite states can be understood by the Koashi-Winter relation.

Given the definition of the classical correlations for a bipartite state ρ_{AB}:

$$J(A:B)_{\rho_{AB}} = \max_{\mathbb{I}\otimes \in \mathcal{P}} I(A:X)_{\mathbb{I}\otimes \boxplus \rho_{AB)}}, \tag{73}$$

where $I(A:X)_{\mathbb{I}\otimes \boxplus \rho_{AB}}$) is the mutual information of the post-measured state $\mathbb{I} \otimes \boxplus \rho_{AB}$), and the optimization is taken over all local POVM measurement maps $\in \mathcal{P}(\mathbb{C}_B, BC_X)$.

Given also the definition of the entanglement of formation of a bipartite state ρ_{AB}:

$$E_f(\rho_{AB}) = \min_{\xi_\rho = \{p_i, |\psi_i\rangle\langle\psi_i|\}_i} \sum_i p_i E(|\psi_i\rangle), \tag{74}$$

where the optimization is taken over all possible convex hull defined by the ensemble $\xi = \{p_i, |\psi_i\rangle\langle\psi_i|\}_i$, such that $\rho_{AB} = \sum_i p_i, |\psi_i\rangle\langle\psi_i|$, and $E(|\psi_i\rangle)$ is the entropy of entanglement of $|\psi_i\rangle$.

Theorem 27 (Koashi-Winter relation). *Considering* $\rho_{ABE} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_E)$ *a pure state then:*

$$J(A:E)_{\rho_{AE}} = S(\rho_A) - E_f(\rho_{AB}), \tag{75}$$

where $\rho_X = Tr_Y[\rho_{YX}]$.

Proof. Suppose $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is the optimum convex combination, such that $E_f(\rho_{AB}) = \sum_i p_i S(Tr_B[|\psi_i\rangle\langle\psi_i|])$. The classical correlations in system AE relates this decomposition with a measurement on the subsystem E. Therefore, there exists a measurement $\left\{M_j^E\right\}$ on system E such that $\rho'_{ABE} = \sum_j Tr_E\left[\rho_{ABE}\left(\mathbb{I}_{AB} \otimes M_j^E\right)\right] \otimes |e_j\rangle\langle e_j|_E$ and $Tr_E[\rho'_{ABE}] = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Tracing over subsystem B, then the post-measurement state will be:

$$\rho'_{AE} = \sum_j p_j Tr_B\left[\left|\psi_j\right\rangle\left\langle\psi_j\right|\right] \otimes |e_j\rangle\langle e_j|, \tag{76}$$

In this way, the mutual information of the post-measurement state:

$$I(A:E)_{\rho'_{AE}} = S(\rho_A) + S(\rho'_E) - S(\rho'_{AE}), \tag{77}$$

$$= S(\rho_A) + H(E) - H(E) - \sum_i p_i S\left(Tr_A\left[\left|\psi_j\right\rangle\left\langle\psi_j\right|\right]\right), \tag{78}$$

$$= S(\rho_A) - \sum_i p_i S\left(Tr_A\left[\left|\psi_j\right\rangle\left\langle\psi_j\right|\right]\right), \tag{79}$$

$$= S(\rho_A) - E_f(\rho_{AB}), \tag{80}$$

It was used as the property of the Shannon entropy for a block diagonal state, where $Tr_B[|\psi_j\rangle\langle\psi_j|] = Tr_A[|\psi_j\rangle\langle\psi_j|]$ and $E_f(\rho_{AB}) = \sum_i p_i S(Tr_B[|\psi_i\rangle\langle\psi_i|])$. By definition $J(A:E)_{\rho_{AE}} \geq I_{(A:E)\rho'_{AE}}$, then

$$J(A:E)_{\rho_{AE}} \geq S(\rho_A) - E_f(\rho_{AB}). \tag{81}$$

Now, it is proved the converse inequality. Given ρ_{AE}, there exists a POVM $\mathcal{M} \in \mathcal{P}(\mathbb{C}_E, \mathbb{C}_{E'})$ with rank-1 elements $\{M_l\}$, such that $Tr_E[M_l \rho_{AE}] = q_l \rho_l^A$ that optimizes the classical correlations $J(\rho_{AE}) = S(\rho_A) - \sum_l q_l S(\rho_l^A)$. As the elements of the POVM are rank-1, $M_l = |\mu_l\rangle\langle\mu_l|$, and the state ρ_{ABE} is pure, the state after local measurement on E will be described by an ensemble of pure states:

$$\rho'_{ABE} = \sum_l Tr_E[\rho_{ABE}(\mathbb{I}_{AB} \otimes |\mu_l\rangle\langle\mu_l|)] \otimes |e_l\rangle\langle e_l| = \sum q_l |\phi_l\rangle\langle\phi_l| \otimes |e_l\rangle\langle e_l|. \tag{82}$$

Once that $\rho_{ABE} = |\kappa\rangle\langle\kappa|_{ABE}$, and the pure state can be written in the bipartite Schmidt decomposition $|\kappa\rangle = \sum_n c_n |n\rangle_{AB} \otimes |n\rangle_E$, if $\langle\mu_l|n\rangle = r_{ln}$, it is easy to see that:

$$Tr_E\left[\rho_{ABE}\left(\mathbb{I}_{AB}\otimes|\mu_l\rangle\langle\mu_l|\right)\right]=\sum_{ij}c_i r_{li}c_j r_{lj}^*|i\rangle\langle j|_{AB}=\left(\sum_i c_i r_{li}|i\rangle_{AB}\right)\left(\sum_j c_j r_{lj}^*\langle j|_{AB}\right)=q_l|\phi_l\rangle\langle\phi_l|. \tag{83}$$

Calculating the mutual information of $\rho'_{AE}=Tr_B\left[\rho'_{ABE}\right]$:

$$I(A:E)_{\rho'_{AE}}=S\left(\rho_A\right)-\sum_l q_l S\left(Tr_B\left[|\phi_l\rangle\langle\phi_l|\right]\right), \tag{84}$$

As the POVM $\mathcal{M}$ is the optimal measurement in the calculation of the classical correlations, it implies $I(A:E)_{\rho'_{AE}}=J(A:E)_{\rho_{AE}}$. By definition, the entanglement of formation satisfies: $E_f(\rho_{AB})\leq\sum_l q_l S(Tr_B[|\phi_l\rangle\langle\phi_l|])$ for any decomposition $\{p_l,|\phi_l\rangle\langle\phi_l|\}$. Substituting the mutual information in Eq. (84):

$$J(A:E)_{\rho_{AE}}\leq S\left(\rho_A\right)-E_f\left(\rho_{AB}\right). \tag{85}$$

Given Eqs. (81) and (85), it proves the theorem.

The Koashi-Winter equation quantifies the amount of entanglement among A and B, considering that the former is classically correlated with another system C. This property is interesting once that it is related to the monogamy of entanglement [58], where the amount of entanglement shared by three parts is limited, and this limitation is given by the amount of classical correlations among the parties. This limitation holds for any tripartite state as stated in the following corollary:

Corollary 28. *For any tripartite state $\rho_{ABC}\in\mathcal{D}(\mathbb{C}_A\otimes\mathbb{C}_B\otimes\mathbb{C}_C)$, it follows:*

$$E_f\left(\rho_{AB}\right)+J(A:C)_{\rho_{AC}}\leq S\left(\rho_A\right). \tag{86}$$

The equality holds for ρ_{ABC} pure.

Proof. If ρ_{ABC} is not a pure state, there exists a purification ρ_{ABCE}, then $\mathbb{C}_A\otimes\mathbb{C}_B\otimes\mathbb{C}_{CE}$, followed by the last theorem:

$$J(A:CE)_{\rho_{ACE}}+E_f\left(\rho_{AB}\right)=S\left(\rho_A\right), \tag{87}$$

Therefore, as the classical correlations are monotonic under local maps, then taking the trace over the system E we have $J(A:CE)_{\rho_{ACE}}\geq J(A:C)_{\rho_{AC}}$.

As the Shannon entropy of ρ_A represents the effective size of A in qubits [24], this size can be approached as the capacity of the system A makes correlations with other systems B and C [18]. In other words, this means that the existence of the quantum or classical correlations between A and another system B is enough to restrict the amount of quantum or classical correlations which A can make with a third system C.

Summing the mutual information $I(A:E)_{\rho_{AE}}$ on both sides of the Koashi-Winter relation, Eq. (75), it is possible to obtain a monogamy expression for the entanglement of formation of the state ρ_{AB} in function of the quantum discord [19]:

$$D(A:E)_{\rho_{AE}} = E_f(\rho_{AB}) - S(A|E)_{\rho_{AE}}, \tag{88}$$

where $D(A:E)_{\rho_{AE}}$ is the quantum discord of the state ρ_{AE} with local measurement on the subsystem E and $S(A|E)_{\rho_{AE}} = S(AE) - S(E)$ is the conditional entropy. As the label in the states is arbitrary, we can rewrite this expression changing the labels $E \to B$ and vice versa to obtain $D(A:B)_{\rho_{AB}} = S(A|B)_{\rho_{AB}} - E_f(\rho_{AE})$, taking the sum between this and Eq. (88):

$$D(A:E)_{\rho_{AE}} + D(A:B)_{\rho_{AB}} = E_f(\rho_{AE}) + E_f(\rho_{AB}), \tag{89}$$

as the total state is pure $S(A|E)_{\rho_{AE}} = -S(A|B)_{\rho_{AB}}$. This expression means that the sum of total amount of entanglement that A shares with B and E is equal to the sum of the amount of quantum discord shared with B and E [19].

From Eq. 88, it is possible to calculate an interesting expression, which relates the irreversibility of the entanglement distillation protocol and quantum discord [20]. As discussed, the entanglement cost is larger than the distillable entanglement. Given the entanglement cost defined as the regularization of the entanglement of formation [4]:

Definition 29. *For a mixed state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the regularization of the entanglement of formation $E_f(\rho_{AB})$ results in the entanglement cost:*

$$E_C(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} E_f(\rho_{AB}^{\otimes n}). \tag{90}$$

The Hashing inequality says that the distillable entanglement of ρ_{AB} is lower bounded by the coherent information $I(A\rangle B)_{\rho_{AB}} = -S(A|B)$ [3]. As the coherent information can increase under LOCC, it is possible to optimize it under LOCC attaining the distillable entanglement [3].

Definition 30. *The regularized coherent information after optimization over LOCC for a mixed state ρ_{AB} gives the distillable entanglement:*

$$E_D(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} I(A\rangle B)_{(V_n \otimes \mathbb{I})\rho_{AB}^{\otimes n}}, \tag{91}$$

where $V_n \otimes \mathbb{I}$ acts locally on the n copies of ρ_{AB}.

It is also possible to define the regularized quantum discord:

Definition 31. *The regularized quantum discord can be defined as the quantum discord of a state ρ_{AB} in the limit of many copies:*

$$D^{\infty}(A:B)_{\rho_{AB}} = \lim_{n\to\infty} \frac{1}{n} D(A:B)_{\rho_{AB}^{\otimes n}}. \tag{92}$$

Therefore, similarly to Eq. (88) in the limit of many copies:

$$D(A:E)_{\rho_{AE}^{\otimes n}} = E_f\left(\rho_{AB}^{\otimes n}\right) - S(A|E)_{\rho_{AE}^{\otimes n}}, \tag{93}$$

taking the regularization we have:

$$D^{\infty}(A:E)_{\rho_{AE}} = E_C\left(\rho_{AB}\right) - S(A|E)_{\rho_{AE}}, \tag{94}$$

as the conditional entropy is additive $S(A|E)_{\rho_{AE}^{\otimes n}} = nS(A|E)_{\rho_{AE}}$. Therefore, the following theorem comes from Eq. (88).

Theorem 32 (Cornelio et al. [20]). *For every mixed entangled state ρ_{AB}, if*

$$E_D\left(\rho_{AB}\right) = \frac{1}{n} I(A\rangle B)_{(V_n \otimes \mathbb{I})\rho_{AB}^{\otimes n}} \tag{95}$$

$$E_C\left(\rho_{AB}\right) = \frac{1}{k} E_F\left(\rho_{AB}^{\otimes n}\right), \tag{96}$$

for a finite number of n and k, the entanglement is irreversible $E_C(\rho_{AB}) > E_D(\rho_{AB})$.

Taking the limit of many copies, the equation can be rewritten as:

$$D^{\infty}(A:E)_{\sigma_{AE}} = E_C(\sigma_{AB}) - E_D(\sigma_{AB}), \tag{97}$$

where $\sigma_{AB} = (V_k \otimes \mathbb{I})\rho_{AB}$ and $E_D(\sigma_{AB}) = kE_D(\rho_{AB})$. The quantum discord $D^{\infty}(A:E)_{\sigma_{AE}}$ in this context can be viewed as the minimal amount of entanglement lost in the distillation protocol, for states belonging to the class described in the theorem [20]. This expression has an operational interpretation for quantum discord, where the quantum discord between the system and the purification system restricts the amount of e-bits lost in the distillation process. A consequence of this result is expressed by the state merging protocol [27], Alice (A), Bob (B) and the Environment (E) share a pure tripartite state ρ_{ABE}, she would like to send her state to Bob, keeping the coherence with the system E. They can perform this protocol consuming an amount of entanglement in the process; the amount of entanglement is the regularized quantum discord $D^{\infty}(A:E)_{\rho_{AE}}$ [25, 26].

In addition to the above relations, some upper and lower bounds between quantum discord and entanglement of formation have been calculated via the Koashi-Winter relation and the properties of entropy [59–62]. Equation (88) was also used to calculate the quantum discord and the entanglement of formation analytically for systems with rank-2 and dimension $2 \otimes n$ [41, 63, 64]. Experimental investigations of Eq. (88) were performed in the characterization of the information flow between system and environment of a non-Markovian process [65].

5. Activation protocol

Physically, a measurement process can be described as an interaction between the measurement apparatus and the system, followed by a projective measurement on the apparatus. Consider a state $\rho_{\mathcal{S}} = \sum_k \lambda_k |k\rangle\langle k| \in \mathcal{D}(\mathbb{C}_{\mathcal{S}})$. The input state is described as $\rho_{\mathcal{S}:\mathcal{M}} = \rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{M}}$, by coupling a pure ancilla, that represents the measurement apparatus. The interaction between the system and the ancillary state is performed by a unitary evolution: $U_{\mathcal{S}:\mathcal{M}} \in \mathcal{U}(\mathbb{C}_{\mathcal{S}} \otimes \mathbb{C}_{\mathcal{M}})$, such that $Tr_{\mathcal{M}}\left[U_{\mathcal{S}:\mathcal{M}} \rho_{\mathcal{S}:\mathcal{M}} U_{\mathcal{S}:\mathcal{M}}^{\dagger}\right] = \sum_l \Pi_l \rho_{\mathcal{S}} \Pi_l^{\dagger}$. A unitary operation satisfying this condition is given by:

$$U_{\mathcal{S}:\mathcal{M}} |k\rangle_{\mathcal{S}} |0\rangle_{\mathcal{M}} = |k\rangle_{\mathcal{S}} |k\rangle_{\mathcal{M}}, \tag{98}$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathbb{C}_{\mathcal{S}}$. If the orthogonal basis $\{|k\rangle\langle k|\}$ is the canonical basis, this interaction is a Cnot gate [1]. Therefore, after the interaction, the state will be:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}:\mathcal{M}}) U_{\mathcal{S}:\mathcal{M}}^{\dagger} = \sum_k \lambda_k |k\rangle\langle k|_{\mathcal{S}} \otimes |k\rangle\langle k|_{\mathcal{M}}. \tag{99}$$

The interaction between the system and the measurement apparatus results in a classically correlated state between the system and the apparatus. Hence performing a projective measurement on the state of the apparatus, the state of the system can be recovered.

Suppose now that the state of the system is composed, for example a bipartite system $\mathbb{C}_{\mathcal{S}} = \mathbb{C}_A \otimes \mathbb{C}_B$. The measurements are performed locally in each system; therefore, the ancilla is also a bipartite system $\mathbb{C}_{\mathcal{M}} = \mathbb{C}_{\mathcal{M}_A} \otimes \mathbb{C}_{\mathcal{M}_B}$. The unitary operator representing the interaction between the system and the measurement apparatus is $U_{\mathcal{S}:\mathcal{M}} = U_{A:\mathcal{M}_A} \otimes U_{B:\mathcal{M}_B}$. Then, the post-measured state is:

$$\tilde{\rho}_{\mathcal{S}} = Tr_{\mathcal{M}}\left[U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|) U_{\mathcal{S}:\mathcal{M}}^{\dagger}\right] = \sum_{k,l} \Pi_k^A \otimes \Pi_l^B \rho_{AB} \Pi_k^{\dagger A} \otimes \Pi_l^{\dagger B}. \tag{100}$$

As aforementioned, the measurement process consists in interacting the system with an ancilla, which represents the measurement apparatus, and then perform a projective measurement over the ancilla. However, as the dimension of the ancilla is arbitrary, to represent a general measurement (POVM), it is necessary to couple another ancilla with the same size of the state: $\rho_{\mathcal{S}':\mathcal{M}} = \rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}} \otimes |0\rangle\langle 0|_{\mathcal{M}}$, where $|0\rangle\langle 0|_{\mathcal{E}}$ is an ancillary state on space $\mathbb{C}_{\mathcal{E}}$. Then, the interaction with the apparatus, given by a unitary evolution $U_{\mathcal{S}':\mathcal{M}}$, results in the post-measured state

$$\tilde{\rho}_{\mathcal{S}} = Tr_{\mathcal{M}}\left[U_{\mathcal{S}':\mathcal{M}} \rho_{\mathcal{S}':\mathcal{M}} U_{\mathcal{S}':\mathcal{M}}^{\dagger}\right] = \sum_l \Pi_l \left(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}}\right) \Pi_l. \tag{101}$$

By the Naimark's theorem $Tr\left[\Pi_l\left(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}}\right)\right] = Tr\left[E_l \rho_{\mathcal{S}}\right]$, where $E_l = (\mathbb{I} \otimes \langle 0|)\Pi_l(\mathbb{I} \otimes |0\rangle)$ is a element of a POVM.

A general bipartite state can be written as $\rho = \sum_{i,j} |i\rangle\langle j| \otimes O_{i,j}$, where $O_{i,j}$ is an Hermitian operator with trace different from zero. Then if the measurement is performed only on the subsystem A, the state $\tilde{\rho}_{\mathcal{S}:\mathcal{M}}$ after the interaction with the measurement apparatus will be:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}:\mathcal{M}})U^{\dagger}_{\mathcal{S}:\mathcal{M}} \tag{102}$$

$$= U_{A:\mathcal{M}_A} \otimes \mathbb{I}_B \left(\sum_{i,j} |i\rangle\langle j|_A \otimes |0\rangle\langle 0|_{\mathcal{M}_A} \otimes O^B_{i,j} \right) U^{\dagger}_{A:\mathcal{M}_A} \otimes \mathbb{I}_B \tag{103}$$

$$= \sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_{\mathcal{M}_A} \otimes O^B_{i,j}. \tag{104}$$

Differently of the global measurement process, for local measurements, entanglement can be created during the measurement process. For example, if $O_{ij} = \frac{1}{2}|i\rangle\langle j|$, the interaction with the measurement apparatus creates a maximally entangle state. different from the case where the measurement is performed on the A quantum state cannot create quantum entanglement with the measurement apparatus, if it is classically correlated. As proved in the following theorem.

Theorem 33 ([21, 22]). *A state is classically correlated (has no quantumness of correlations), if and only if there exists an unitary operation such that the post interaction state is separable with respect to system and measurement apparatus.*

Proof. The proof is performed for the general case, for measurements on both systems.

If: If the state is classically correlated:

$$\rho_{\mathcal{S}} = \sum_{k,j} p_{k,j} |a_k, b_j\rangle\langle a_k, b_j|_{\mathcal{S}}, \tag{105}$$

the state after the interaction with the measurement apparatus represented by the unitary operation $U_{A:\mathcal{M}_A} \otimes U_{B:\mathcal{M}_B}$ will be:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = \sum_{k,j} p_{k,j} |a_k, b_j\rangle\langle a_k, b_j|_{\mathcal{S}} \otimes |a_k, b_j\rangle\langle a_k, b_j|_{\mathcal{M}}, \tag{106}$$

which is clearly separable.

Only if: Given a general separable state between the system and the measurement apparatus:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = \sum_{\alpha} p_{\alpha} |\phi_{\alpha}\rangle\langle\phi_{\alpha}|_{\mathcal{S}} \otimes |\psi_{\alpha}\rangle\langle\psi_{\alpha}|_{\mathcal{M}}, \tag{107}$$

and the fact that the interaction is unitary, there is a convex combination such that $\rho_{\mathcal{S}} = \sum_{\alpha} p_{\alpha} |\kappa_{\alpha}\rangle\langle\kappa_{\alpha}|$; therefore, the interaction must act in the following way:

$$U_{\mathcal{S}:\mathcal{M}} |\kappa_{\alpha}\rangle|0\rangle = |\phi_{\alpha}\rangle|\psi_{\alpha}\rangle. \tag{108}$$

On the other hand, as the state $\rho_{\mathcal{S}}$ is bipartite, the pure states $\{|\kappa_\alpha\rangle\}$ can be written in general as: $|\kappa_\alpha\rangle = \sum_{l,i} c_{l,i}^{\alpha} |a_l^{\alpha}\rangle |b_i^{\alpha}\rangle$, and after the interaction, the states will be:

$$U_{\mathcal{S}:\mathcal{M}}|\kappa_\alpha\rangle|0\rangle = \sum_{l,j} c_{l,j}^{\alpha} \left|a_l^{\alpha}, b_j^{\alpha}\right\rangle_{\mathcal{S}} \otimes \left|a_l^{\alpha}, b_j^{\alpha}\right\rangle_{\mathcal{M}}. \tag{109}$$

As the state in Eq. (109) must be separable, it implies that the coefficients must satisfy:

$$c_{i,j}^{\alpha} = c_{f(\alpha)}\delta_{i,j;f(\alpha)} \quad \text{and} \quad |c_{f(\alpha)}| = 1 \tag{110}$$

where $f(\alpha) \in \mathbb{N}^2$. As $f(\alpha)$ are orthogonal, it proves the theorem.

If the state of the system has quantum correlations, the local measurement process creates entanglement between the system and the measurement apparatus, for a every unitary interaction. Then, it is possible to fix the base of the ancilla and change the base of the system. Then, rewriting the evolution as $U_{\mathcal{S}:\mathcal{M}} = C_{\mathcal{S}:\mathcal{M}}(U_{\mathcal{S}} \otimes \mathbb{I}_{\mathcal{M}})$, where for bipartite systems $U_{\mathcal{M}} = U_A \otimes U_B$ is a local unitary operation and $C_{\mathcal{S}:\mathcal{M}} = C_{A:\mathcal{M}_A} \otimes C_{B:\mathcal{M}_B}$ is a Cnot gate acting on the system as the control, and the apparatus as the target. It is possible to quantify the amount of quantum correlation in a given system starting on the amount of entanglement created with the measurement apparatus.

Definition 34 ([21, 22]). *Each measure of entanglement used to quantify the entanglement between the system and the apparatus will result in a measure of quantumness of correlations.*

$$Q_E(\rho_{\mathcal{S}}) = \min_{U_{\mathcal{S}}} E_Q(\rho_{\mathcal{S}:\mathcal{M}}). \tag{111}$$

Different entanglement measures will lead, in principle, to different quantifiers for the quantumness of correlations. The only requirement is that the entanglement measure must be an entanglement monotone [21, 22, 66]. Some quantifiers of quantumness of correlations can be recovered with the activation protocol: the quantum discord [22], one-way work deficit [22], zero-way work deficit [21] and the geometrical measure of discord via trace norm [66], are some examples. Taking the distillable entanglement in Eq. (111) is quite simple to see that it results in zero-way work deficit. As shown in Eq. (106), the interaction with the measurement apparatus results in the state

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = \sum_{k,j} p_{k,j} |a_k, b_j\rangle\langle a_k, b_j|_{\mathcal{S}} \otimes |a_k, b_j\rangle\langle a_k, b_j|_{\mathcal{M}}. \tag{112}$$

That is named *maximally correlated state*, and as showed in Ref.[67], the distillable entanglement of this state attach the Hashing inequality [68]:

$$E_D(\tilde{\rho}_{\mathcal{S}:\mathcal{M}}) = -S(\mathcal{S}|\mathcal{M}), \tag{113}$$

where $S(\mathcal{S}|\mathcal{M}) = S(\tilde{\rho}_{\mathcal{S}}) - S(\tilde{\rho}_{\mathcal{S}:\mathcal{M}})$ is conditional entropy of $\tilde{\rho}_{\mathcal{S}:\mathcal{M}}$. On the other hand, the zero-way work deficit for $\rho_{\mathcal{S}}$ is:

$$\Delta^{\varnothing}(\rho_{\mathcal{S}}) = \min_{\Pi_{\mathcal{S}_A} \otimes \Pi_{\mathcal{S}_B} \in \mathcal{P}} \{S(\Pi_{\mathcal{S}_A} \otimes \Pi_{\mathcal{S}_B}[\rho_{\mathcal{S}}]) - S(\rho_{\mathcal{S}})\}, \tag{114}$$

where $\Pi_{\mathcal{S}_A} \otimes \Pi_{\mathcal{S}_B} \in \mathcal{P}(\mathbb{C}_{\mathcal{S}_A} \otimes \mathbb{C}_{\mathcal{S}_B})$ is a local dephasing on subsystem A and B. As $\tilde{\rho}_{\mathcal{S}}$ is the measured state of the system and $\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = U_{\mathcal{S}:\mathcal{M}} \rho_{\mathcal{S}:\mathcal{M}} U^{\dagger}_{\mathcal{S}:\mathcal{M}}$, then:

$$S(\tilde{\rho}_{\mathcal{S}}) - S(\tilde{\rho}_{\mathcal{S}:\mathcal{M}}) = S(\Pi_{\mathcal{S}_A} \otimes \Pi_{\mathcal{S}_B}[\rho_{\mathcal{S}}]) - S(\rho_{\mathcal{S}}).$$

Therefore:

$$\Delta^{\varnothing}(\rho_{\mathcal{S}}) = \min_{U_{\mathcal{M}}} E_D(\tilde{\rho}_{\mathcal{S}:\mathcal{M}}). \tag{115}$$

This equation means that the activation protocol creates distillable entanglement between the system and the measurement apparatus during a local measurement. In other words, quantumness of correlations of the system can be converted resource for quantum information protocol, and this conversion is ruled by the activation protocol.

From Eq. (111), it is possible to show that quantum entanglement is a lower bound for quantumness of correlations.

Proposition 35 (Piani and Adesso [66]). *For* $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$:

$$Q_E(\rho_{AB}) \geq E_Q(\rho_{AB}), \tag{116}$$

where Q_E *and* E_Q *are related by Eq.* (111).

To compare two measures of different quantities as quantumness of correlation and quantum entanglement, it is necessary a common rule. The activation protocol gives the rule to compare these two quantities and this rule says that the measures of quantumness of correlations and quantum entanglement must be related from Eq. (111). Entanglement is a lower bound for quantumness of correlations also in the geometrical approach [17, 56].

Activation protocol determines that a composed state is classically correlated if and only if it cannot create entanglement during the measurement process, for a given unitary interaction [21, 22, 66]. This result provides an important tool for characterization of quantum correlations in identical particle systems (bosons and fermions), once that system and apparatus are distinguishable partitions, even if the particles in the system are identical. This approach have been applied to identical particles systems to prove how are the classically correlated states of bosons and fermions [69]. The activation protocol device also allows to determine the class of classically correlated states of the modes of a fermionic system and its relation to the correlations of the fermions [70].

The entanglement generation by means of quantumness of correlations, as stated by the activation protocol, was experimentally evidenced using programmable quantum measurement [71]. In the experiment setup, the optimization on the unitary operations was performed by a set of programable quantum measurements in different local basis. As quantumness of

correlation can be generated by local operations [10], activation protocol was explored experimentally in the generation of distillable entanglement via local operations on the measured partition of the system [72].

6. Conclusion

This chapter leads to the fundamental aspects of quantum correlations: entanglement and quantumness of correlations. The purpose of this chapter is to demonstrate that quantumness of correlations plays an important role in entanglement resource theory and by consequence in quantum information theory. It was presented that entanglement and quantumness of correlations connect each other in two different pictures. The relation derived by Koashi and Winter demonstrates the balance between quantumness of correlations and entanglement in the purification process. This balance leads to a formal proof for the irreversibility of the entanglement distillation protocol, in terms of quantumness of correlations. Indeed in this fashion quantumness of correlations revealed to play the main role in the state merging protocol, quantifying the amount of entanglement consumed during the protocol. In the named *activation protocol*, the quantumness of correlations of a given composed system can be converted into distillable entanglement with a measurement apparatus during the local measurement process. In resume, the entanglement created by the interaction between the system and the measurement apparatus is limited below by the amount of quantumness of correlations of the system.

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Author details

Tiago Debarba

Address all correspondence to: debarba@utfpr.edu.br

Universidade Tecnológica Federal do Paraná (UTFPR), Campus Cornélio Procópio, Cornélio Procópio, Paraná, Brazil

References

[1] Nielsen M, Chuang I. Quantum Computation and Quantum Information. Cambridge UK: Cambridge University Press; 2000. DOI: 10.1017/CBO9780511976667.001

[2] Horodecki R, Horodecki P, Horodecki M, Horodecki K. Quantum entanglement. Reviews of Modern Physics. 2009;**81**:865. DOI: 10.1103/RevModPhys.81.865

[3] Devetak I, Winter A. Distillation of secret key and entanglement from quantum states. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science. 2005a;**461**:207. DOI: 10.1098/rspa.2004.1372

[4] Hayden PM, Horodecki M, Terhal BM. The asymptotic entanglement cost of preparing a quantum state. Journal of Physics A: Mathematical and General. 2001;**34**:6891. DOI: 10.1088/0305-4470/34/35/314

[5] Bennett CH, Brassard G, Popescu S, Schumacher B, Smolin JA, Wootters WK. Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels. Physical Review Letters. 1996a;**76**:722. DOI: 10.1103/PhysRevLett.76.722

[6] Bennett CH, Brassard G, Crépeau C, Jozsa R, Peres A, Wootters WK. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Physical Review Letters. 1993;**70**:1895. DOI: 10.1103/PhysRevLett.70.1895

[7] Bennett CH, DiVincenzo DP, Smolin JA, Wootters WK. Mixed-state entanglement and quantum error correction. Physical Review A. 1996b;**54**:3824. DOI: 10.1103/PhysRevA.54.3824

[8] Bennett CH, Brassard G. Theoretical Computer Science. 2014;**560**,Part 1:7. ISSN 0304-3975, theoretical Aspects of Quantum Cryptography – celebrating 30 years of {BB84}. DOI: 10.1016/j.tcs.2014.05.025

[9] Abeyesinghe A, Devetak I, Hayden P, Winter A. The mother of all protocols: restructuring quantum information's family tree. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science. 2009;**465**:2537. DOI: 10.1098/rspa.2009.0202

[10] Ollivier H, Zurek WH. Quantum Discord: A Measure of the Quantumness of Correlations. Physical Review Letters. 2001;**88**:017901. DOI: 10.1103/PhysRevLett.88.017901

[11] Henderson L, Vedral V. Classical, quantum and total correlations. Journal of Physics A: Mathematical and General. 2001;**34**:6899. DOI: 10.1088/0305-4470/34/35/315

[12] Oppenheim J, Horodecki M, Horodecki P, Horodecki R. A Thermodynamical Approach to Quantifying Quantum Correlations. Physical Review Letters. 2002;**89**:180402. DOI: 10.1103/PhysRevLett.89.180402

[13] Luo S. Using measurement-induced disturbance to characterize correlations as classical or quantum. Physical Review A. 2008;**77**:022301. DOI: 10.1103/PhysRevA.77.022301

[14] Nakano T, Piani M, Adesso G. Negativity of quantumness and its interpretations. Physical Review A. 2013;**88**:012117. DOI: 10.1103/PhysRevA.88.012117

[15] Luo S, Fu S. Geometric measure of quantum discord. Physical Review A. 2010;**82**:034302. DOI: 10.1103/PhysRevA.82.034302

[16] Modi K, Paterek T, Son W, Vedral V, Williamson M. Unified View of Quantum and Classical Correlations. Physical Review Letters. 2010;**104**:080501. DOI: 10.1103/PhysRevLett.104.080501

[17] Debarba T, Maciel TO, Vianna RO. Witnessed entanglement and the geometric measure of quantum discord. Physical Review A. 2012;**86**:024302. DOI: 10.1103/PhysRevA.86.024302

[18] Koashi M, Winter A. Monogamy of entanglement and other correlations. Physical Review A. 2004;**69**:022309. DOI:10.1103/PhysRevA.69.022309

[19] Fanchini FF, Cornelio MF, de Oliveira MC, Caldeira AO. Conservation law for distributed entanglement of formation and quantum discord. Physical Review A. 2011;**84**:012313. DOI: 10.1103/PhysRevA.84.012313

[20] Cornelio MF, de Oliveira MC, Fanchini FF. Entanglement Irreversibility from Quantum Discord and Quantum Deficit. Physical Review Letters. 2011;**107**:020502. DOI: 10.1103/PhysRevLett.107.020502

[21] Piani M, Gharibian S, Adesso G, Calsamiglia J, Horodecki P, Winter A. All Nonclassical Correlations Can Be Activated into Distillable Entanglement. Physical Review Letters. 2011;**106**:220403. DOI: 10.1103/PhysRevLett.106.220403

[22] Streltsov A, Kampermann H, Bruß D. Linking Quantum Discord to Entanglement in a Measurement. Physical Review Letters. 2011;**106**:160401. DOI: 10.1103/PhysRevLett.106.160401

[23] Bengtsson I, Zyczkowski K. Geometry of Quantum States. Cambridge University Press; 2006. DOI: 10.1017/CBO9780511535048

[24] Schumacher B. Quantum coding. Physical Review A. 1995;**51**:2738. DOI: 10.1103/PhysevA.51.2738

[25] Cavalcanti D, Aolita L, Boixo S, Modi K, Piani M, Winter A. Operational interpretations of quantum discord. Physical Review A. 2011;**83**:032324. DOI: 10.1103/PhysRevA.83.032324

[26] Madhok V, Datta A. Interpreting quantum discord through quantum state merging. Physical Review A. 2011;**83**:032323. DOI: 10.1103/PhysRevA.83.032323

[27] Horodecki M, Oppenheim J, Winter A. Partial quantum information. Nature. 2005; **436**:673. DOI: 10.1038/nature03909

[28] Devetak I. The private classical capacity and quantum capacity of a quantum channel. Information Theory, IEEE Transactions on. 2005;**51**:44. DOI: 10.1109/TIT.2004.839515

[29] Lloyd S. Capacity of the noisy quantum channel. Physical Review A. 1997;**55**:1613. DOI: 10.1103/PhysRevA.55.1613

[30] Shor PW. The quantum channel capacity and coherent information. lecture notes, MSRI Workshop on Quantum Computation, 2002. Avaliable at http://www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/

[31] Wilde MM. Quantum Information Theory. Cambridge University Press; 2013. ISBN: 9781107034259

[32] Popescu S. Bell's inequalities versus teleportation: What is nonlocality? Physical Review Letters. 1994;**72**:797. DOI: 10.1103/PhysRevLett.72.797

[33] Popescu S. Bell's inequalities and density matrices. Revealing hidden nonlocality. Physical Review Letters. 1995;**74**:2619. DOI: 10.1103/PhysRevLett.74.2619

[34] Życzkowski K, Horodecki P, Sanpera A, Lewenstein M. Volume of the set of separable states. Physical Review A. 1998;**58**:883. DOI: 10.1103/PhysRevA.58.883

[35] Werner RF. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Physical Review A. 1989;**40**:4277. DOI: 10.1103/PhysRevA.40.4277

[36] Wootters WK. Entanglement of Formation of an Arbitrary State of Two Qubits. Physical Review Letters. 1998;**80**:2245. DOI: 10.1103/PhysRevLett.80.2245

[37] Majtey A, Lamberti P, Prato D. Jensen-Shannon divergence as a measure of distinguishability between mixed quantum states. Physical Review A. 2005;**72**:052310. DOI: 10.1103/PhysRevA.72.052310

[38] Lin J. Divergence measures based on the Shannon entropy. Information Theory, IEEE Transactions on. 1991;**37**:145. DOI: 10.1109/18.61115

[39] Lamberti P, Majtey A, Borras A, Casas M, Plastino A. On the metric character of the quantum Jensen-Shannon divergence. Physical Review A. 2008;**77**:052311. DOI: 10.1103/PhysRevA.77.052311

[40] Ferraro A, Aolita L, Cavalcanti D, Cucchietti FM, Acin A. Almost all quantum states have nonclassical correlations. Physical Review A. 2010;**81**:052318. DOI: 10.1103/PhysRevA.81.052318

[41] Fanchini FF, Castelano LK, Cornelio MF, de Oliveira MC. Locally inaccessible information as a fundamental ingredient to quantum information. New Journal of Physics. 2012;**14**:013027. DOI: 10.1088/1367-2630/14/1/013027

[42] Huang Y. Computing quantum discord is NP-complete. New Journal of Physics. 2014;**16**:033027. DOI: 10.1088/1367-2630/16/3/033027

[43] Ali M, Rau ARP, Alber G. Quantum discord for two-qubit X states. Physical Review A. 2010;**81**:042105. DOI: 10.1103/PhysRevA.81.042105

[44] Girolami D, Adesso G. Quantum discord for general two–qubit states: Analytical progress. Physical Review A. 2011;**83**:052108. DOI: 10.1103/PhysRevA.83.052108

[45] Lu X-M, Ma J, Xi Z, Wang X. Optimal measurements to access classical correlations of two-qubit states. Physical Review A. 2011;**83**:012327. DOI: 10.1103/PhysRevA.83.012327

[46] Piani M, Horodecki P, Horodecki R. No-Local-Broadcasting Theorem for Multipartite Quantum Correlations. Physical Review Letters. 2008;**100**:090502. DOI: 10.1103/PhysRevLett.100.090502

[47] Wu S, Poulsen UV, Mølmer K. Correlations in local measurements on a quantum state, and complementarity as an explanation of nonclassicality. Physical Review A. 2009;**80**: 032319. DOI: 10.1103/PhysRevA.80.032319

[48] Girolami D, Paternostro M, Adesso G. Faithful nonclassicality indicators and extremal quantum correlations in two-qubit states. Journal of Physics A: Mathematical and Theoretical. 2011;**44**:352002. DOI: 10.1088/1751-8113/44/35/352002

[49] Rulli C, Sarandy M. Global quantum discord in multipartite systems. Physical Review A. 2011;**84**:042109. DOI: 10.1103/PhysRevA.84.042109

[50] Horodecki M, Horodecki P, Horodecki R, Oppenheim J, De A, Sen U, Synak-Radtke B. Local versus non-local information in quantum information theory: formalism and phenomena. Physical Review A. 2005;**71**:062307. DOI: 10.1103/PhysRevA.71.062307

[51] Horodecki M, Horodecki K, Horodecki P, Horodecki R, Oppenheim J, Sen(De) A, Sen U. Local Information as a Resource in Distributed Quantum Systems. Physical Review Letters. 2003;**90**:100402. DOI: 10.1103/PhysRevLett.90.100402

[52] Devetak I. Distillation of local purity from quantum states. Physical Review A. 2005b;**71**:062303. DOI: 10.1103/PhysRevA.71.062303

[53] Horodecki M, Oppenheim J. (Quantumness in the context of) Resource Theories. International Journal of Modern Physics B. 2013;**27**:1345019. DOI: 10.1142/S0217979213450197

[54] Oppenheim J, Horodecki K, Horodecki M, Horodecki P, Horodecki R. A new type of complementarity between quantum and classical information. Physical Review A. 2003;**68**:022307. DOI: 10.1103/PhysRevA.68.022307

[55] Spehner D, Orszag M. Geometric quantum discord with Bures distance. New Journal of Physics. 2013;**15**:103001. DOI: 10.1088/1367-2630/15/10/103001

[56] Debarba T, Maciel TO, Vianna RO. Reply to "Comment on 'Witnessed entanglement and the geometric measure of quantum discord'" Physical Review A. 2013;**87**:046301. DOI: 10.1103/PhysRevA.87.046301

[57] Piani M. Problem with geometric discord. Physical Review A. 2012;**86**:034101. DOI: 10.1103/PhysRevA.86.034101

[58] Coffman V, Kundu J, Wootters WK. Distributed Entanglement. Physical Review A. 2000;**61**:052306. DOI: 10.1103/PhysRevA.61.052306

[59] Yu S, Zhang C, Chen Q, Oh C. Tight bounds for the quantum discord. arXiv:1102.1301v2. 2012;**85**:032109. URL: http://arxiv.org/abs/1102.1301v2

[60] Xi Z, Lu X-M, Wang X, Li Y. Necessary and sufficient condition for saturating the upper bound of quantum discord. Physical Review A. 2012;**85**:032109. DOI: 10.1103/PhysRevA.85.032109

[61] Xi Z, Lu X-M, Wang X, Li Y. The upper bound and continuity of quantum discord. Journal of Physics A: Mathematical and Theoretical. 2011;**44**:375301. DOI: 10.1088/1751-8113/44/37/375301

[62] Zhang C, Yu S, Chen Q, Oh CH. Observable estimation of entanglement of formation and quantum discord for bipartite mixed quantum states. Physical Review A. 2011;**84**:052112. DOI: 10.1103/PhysRevA.84.052112

[63] Chi DP, Lee S. Entanglement for a two-parameter class of states. Journal of Physics A: Mathematical and General. 2003;**36**:11503. DOI: 10.1088/0305-4470/36/45/010

[64] Cen LX, Li X-Q, Shao J, Yan Y. Quantifying quantum discord and entanglement of formation via unified purifications. Physical Review A. 2011;**83**:054101. DOI: 10.1103/PhysRevA.83.054101

[65] Fanchini FF, Karpat G, Akmak B, Castelano LK, Aguilar GH, Jimnez Faras O, Walborn SP, Souto Ribeiro PH, de Oliveira MC. Non-Markovianity through accessible information. Physical Review Letters. 2014;**112**:210402. DOI: 10.1103/PhysRevLett.112.210402

[66] Piani M, Adesso G. Quantumness of correlations revealed in local measurements exceeds entanglement. Physical Review A. 2012a;**85**:040301. DOI: 10.1103/PhysRevA.85.040301

[67] Hiroshima T, Hayashi M. Finding a maximally correlated state: Simultaneous Schmidt decomposition of bipartite pure states. Physical Review A. 2004;**70**:030302. DOI: 10.1103/PhysRevA.70.030302

[68] Devetak I, Winter A. Distillation of secret key and entanglement from quantum states. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science. 2005b;**461**:207. DOI: 10.1098/rspa.2004.1372

[69] Iemini F, Debarba T, Vianna RO. Quantumness of correlations in indistinguishable particles. Physical Review A. 2014;**89**:032324. DOI: 10.1103/PhysRevA.89.032324

[70] Debarba T, Vianna RO, Iemini F. Quantumness of correlations in fermionic systems. Physical Review A. 2017;**95**:022325. DOI: 10.1103/PhysRevA.95.022325

[71] Adesso G, D'Ambrosio V, Nagali E, Piani M, Sciarrino F. Experimental entanglement activation from discord in a programmable quantum measurement. Physical Review Letters. 2014;**112**:140501. DOI: 10.1103/PhysRevLett.112.140501

[72] Orieux A, Ciampini MA, Mataloni P, Bru D, Rossi M, Macchiavello C. Experimental Generation of Robust Entanglement from Classical Correlations via Local Dissipation. Physical Review Letters. 2015;**115**:160503. DOI: 10.1103/PhysRevLett.115.160503

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