

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

185,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Deformed Phase Space in Cosmology and Black Holes

E.A. Mena-Barboza, L.F. Escamilla-Herrera,
J.C. López-Domínguez and J. Torres-Arenas

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/intechopen.68282>

Abstract

It is well known that one way to study canonical quantum cosmology is through the Wheeler DeWitt (WDW) equation where the quantization is performed on the minisuperspace variables. The original ideas of a deformed minisuperspace were done in connection with noncommutative cosmology, by introducing a deformation into the minisuperspace in order to incorporate an effective noncommutativity. Therefore, studying solutions to Cosmological models through the WDW equation with deformed phase space could be interpreted as studying quantum effects to Cosmology. In this chapter, we make an analysis of scalar field cosmology and conclude that under a phase space transformation and imposed restriction, the effective cosmological constant is positive. On the other hand, obtaining the wave equation for the noncommutativity Kantowski-Sachs model, we are able to derive a modified noncommutative version of the entropy. To that purpose, the Feynman-Hibbs procedure is considered in order to calculate the partition function of the system.

Keywords: noncommutativity, quantum cosmology, thermodynamics of black holes

1. Introduction

Since the initial use of the Hamiltonian formulation to cosmology, different issues have been studied. In particular, thermodynamic properties of black holes, classical and quantum cosmology, dynamics of cosmological scalar fields, and the problem of cosmological constant among others. In this chapter, we present some results in deforming the phase space variables, discussing recent advances on this special topic by presenting three models. In the first model (Section 2), we analyze the effects of the phase space deformations over different scenarios, we start with the noncommutative on Λ cosmological and comment on the possibility that the

origin of the cosmological constant in the $(4 + 1)$ Kaluza-Klein universe is related to the deformation parameter associated to the four-dimensional scale factor and the compact extra dimensions. In Section 3, we study the effects of phase space deformations in late time cosmology. To introduce the deformation, we use the approach given in Refs. [1]. We conclude that for this model an effective cosmological constant Λ_{eff} appears.

In Section 4, the thermodynamic formalism for rotating black holes, characterized by noncommutative and quantum corrections, is constructed. From a fundamental thermodynamic relation, the equations of state are explicitly given, and the effect of noncommutativity and quantum correction is discussed; in this sense, the goal of this section is to explore how these considerations introduced in Bekenstein-Hawking (BH) entropy change the thermodynamic information contained in this new fundamental relation. Under these considerations, Section 4 examines the different thermodynamic equations of state and their behavior when considering the aforementioned modifications to entropy.

In this chapter, we mainly pretend to indulge in recollections of different studies on the noncommutative proposal that has been put forward in the literature by the authors of this chapter [2–4]; in this sense, our guideline has been to concentrate on recent results that still seem likely to be of general interest to those researchers that are interested in this noncommutative subject.

2. Model 1: Kaluza-Klein cosmology with Λ

Let us begin by introducing the model in a classical scenario which is an empty $(4+1)$ theory of gravity with cosmological constant Λ as shown in Eq. (1). In this setup, the action takes the form:

$$I = \int \sqrt{-g} (R - \Lambda) dt d^3 r d\rho, \quad (1)$$

where $\{t, r^i\}$ are the coordinates of the 4-dimensional spacetime and ρ represents the coordinate of the fifth dimension. We are interested in Kaluza-Klein cosmology, so a Friedmann-Robertson-Walker (FRW)-type metric is assumed, which is of the form

$$ds^2 = -dt^2 + \frac{a^2(t) dr^i dr^i}{\left(1 + \frac{\kappa r^2}{4}\right)^2} + \phi^2(t) d\rho^2, \quad (2)$$

where $\kappa = 0, \pm 1$ and $a(t)$, $\phi(t)$ are the scale factors of the universe and the compact dimension, respectively. Substituting this metric into the action Eq. (1) and integrating over the spatial dimensions, we obtain an effective Lagrangian that only depends on (a, ϕ) :

$$L = \frac{1}{2} \left(a\phi\dot{a}^2 + a^2\dot{a}\dot{\phi} - \kappa a\phi + \frac{1}{3} \Lambda a^3 \phi \right). \quad (3)$$

For the purposes of simplicity and calculations, we can rewrite this Lagrangian in a more convenient way:

$$L = \frac{1}{2} \left[(\dot{x}^2 - \omega^2 x^2) - (\dot{y}^2 - \omega^2 y^2) \right], \quad (4)$$

where the new variables were defined as

$$x = \frac{1}{\sqrt{8}} \left(a^2 + a\phi - \frac{3\kappa}{\Lambda} \right), y = \frac{1}{\sqrt{8}} \left(a^2 - a\phi - \frac{3\kappa}{\Lambda} \right), \quad (5)$$

and $\omega = -\frac{2\Lambda}{3}$. The Hamiltonian for the model is calculated as usual and reads

$$H = \left[(p_x^2 + \omega^2 x^2) - (p_y^2 + \omega^2 y^2) \right], \quad (6)$$

which describes an isotropic oscillator-ghost-oscillator system. A full analysis of the quantum behavior of this model is presented in Ref. [1].

2.1. Noncommutative model

As is well known, there are different approaches to introduce noncommutativity in gravity [5]. In particular, to study noncommutative cosmology [6, 7], there exist a well-explored path to introduce noncommutativity into a cosmological setting [6]. In this setup, the noncommutativity is realized in the minisuperspace variables. The deformation of the phase space structure is achieved through the Moyal brackets, which are based on the Moyal product. However, a more appropriate way to introduce the deformation is by means of the Poisson brackets rather than the Moyal ones.

The most conventional way to understand the noncommutativity between the phase space variables (minisuperspace variables) is by replacing the usual product of two arbitrary functions with the Moyal product (or star product) as

$$(f \star g)(x) = \exp \left[\frac{1}{2} \alpha^{ab} \partial_a^{(1)} \partial_b^{(2)} \right] f(x_1) g(x_2) |_{x_1=x_2=x}, \quad (7)$$

such that

$$\alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix}, \quad (8)$$

where the θ and β are 2×2 antisymmetric matrices and represent the noncommutativity in the coordinates and momenta, respectively, and $\sigma = \theta\beta/4$. With this product law, a straightforward calculation gives

$$\{x_i, x_j\} = \theta_{ij}, \quad \{x_i, p_j\} = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\} = \beta_{ij}. \quad (9)$$

The noncommutative deformation has been applied to the minisuperspace variables as well as to the corresponding canonical momenta; this type of noncommutativity can be motivated by

string theory correction to gravity [6, 8]. In the rest of this model, we use for the noncommutative parameters $\theta_{ij} = -\theta\epsilon_{ij}$ and $\beta_{ij} = \beta\epsilon_{ij}$.

If we consider the following change of variables in the classical phase space $\{x, y, p_x, p_y\}$

$$\begin{aligned}\hat{y} &= y - \frac{\theta}{2}p_x, \quad \hat{x} = x - \frac{\theta}{2}p_y \\ \hat{p}_y &= p_y + \frac{\beta}{2}x, \quad \hat{p}_x = p_x - \frac{\beta}{2}y,\end{aligned}\tag{10}$$

it can be verified that if $\{x, y, p_x, p_y\}$ obeys the usual Poisson algebra, then

$$\{\hat{y}, \hat{x}\} = \theta, \quad \{\hat{x}, \hat{p}_x\} = \{\hat{y}, \hat{p}_y\} = 1 + \sigma, \quad \{\hat{p}_y, \hat{p}_x\} = \beta.\tag{11}$$

Now that we have defined the deformed phase space, we can see the effects on the proposed cosmological model. From the action Eq. (4), we can obtain the Hamiltonian constraint Eq. (6); inserting relations Eq. (11), a Wheeler DeWitt (WDW) equation can be constructed as:

$$\begin{aligned}H\Psi(\hat{x}, \hat{y}) &= \left\{ \left(\hat{p}_x - \frac{2(\beta - \theta\omega^2)}{4 - \omega^2\theta^2}\hat{y} \right)^2 - \left(\hat{p}_y + \frac{2(\beta - \theta\omega^2)}{4 - \omega^2\theta^2}\hat{x} \right)^2 \right. \\ &\quad \left. + \left(\frac{4(\beta - \theta\omega^2)^2}{(4 - \omega^2\theta^2)^2} + \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2\theta^2} \right)\hat{x}^2 - \left(\frac{4(\beta - \theta\omega^2)^2}{(4 - \omega^2\theta^2)^2} + \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2\theta^2} \right)\hat{y}^2 \right\} \Psi(\hat{x}, \hat{y}) = 0,\end{aligned}\tag{12}$$

By a closer inspection of the equation, it is convenient to make the following definitions:

$$\begin{aligned}\omega'^2 &\equiv \frac{4(\beta - \theta\omega^2)^2}{(4 - \omega^2\theta^2)^2} + \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2\theta^2}, \\ A_{\hat{x}} &\equiv \frac{-2(\beta - \theta\omega^2)}{4 - \omega^2\theta^2}\hat{y}, \quad A_{\hat{y}} \equiv \frac{2(\beta - \theta\omega^2)}{4 - \omega^2\theta^2}\hat{x},\end{aligned}\tag{13}$$

With these definitions, we can rewrite Eq. (12) in a much simpler and suggestive form:

$$H = \left\{ \left[\left(\hat{p}_x - A_{\hat{x}} \right)^2 + \omega'^2\hat{x}^2 \right] - \left[\left(\hat{p}_y - A_{\hat{y}} \right)^2 + \omega'^2\hat{y}^2 \right] \right\},\tag{14}$$

which is a two-dimensional anisotropic ghost-oscillator [1]. From Eq. (14), we can see that the terms $(p_i - A_i)$ can be associated to a minimal coupling term as is done in electromagnetic theory. From this vector potential, we find that $B = \frac{4(\beta - \omega^2\theta)}{4 - \omega^2\theta^2}$ and the vector potential \mathbf{A} can be rewritten as $A_{\hat{x}} = -\frac{B}{2}\hat{y}$ and $A_{\hat{y}} = \frac{B}{2}\hat{x}$. On the other hand, we already know from Eq. (11) that $\{\hat{p}_y, \hat{p}_x\} = \beta$ and if we set $\theta = 0$ in the above equation for B , we can conclude that the deformation of the momentum plays a role analogous to a magnetic field.

2.2. Discussion

We found that ω is defined in terms of the cosmological constant, then modifications to the oscillator frequency will imply modifications to the effective cosmological constant. Here, we have done a deformation of the phase space of the theory by introducing a modification to the momenta and to the minisuperspace coordinates, this gives two new fundamental constants θ and β . As expected, we obtain a different functional dependence for the frequency ω and the magnetic B field as functions of β and θ . With this in mind, we can construct a new frequency $\tilde{\omega}$ in terms of ω'^2 and the cyclotron term $B^2/4$:

$$\tilde{\omega}^2 = \omega'^2 - \frac{B^2}{4} = \frac{4(\omega'^2 - \beta^2/4)}{4 - \omega'^2\theta^2}. \quad (15)$$

This $\tilde{\omega}$ was obtained by a definition of the effective cosmological constant $\tilde{\Lambda}_{eff} = -\frac{3}{2}\tilde{\omega}^2$ as was done in Section 2, to finally get a redefinition of the effective cosmological constant due to noncommutative parameters:

$$\tilde{\Lambda}_{eff} = \frac{4(\Lambda_{eff} + \frac{3}{8}\beta^2)}{4 - \frac{2}{3}\theta^2|\Lambda_{eff}|}. \quad (16)$$

Now if we choose the case $\beta = 0$, this should be equivalent to the noncommutative minisuperspace model, hence we get an effective cosmological constant given by:

$$\tilde{\Lambda}_{eff} = \frac{4\Lambda_{eff}}{(4 - \frac{2}{3}\theta^2|\Lambda_{eff}|)}. \quad (17)$$

We can see from Eq. (17) that the noncommutative parameter θ cannot take the place of the cosmological constant, but depending on the value of θ , the effective cosmological constant $\tilde{\Lambda}_{eff}$ is modified. Equation 17 is in agreement with the results given in Refs. [9, 10].

3. Model 2: Scalar field cosmology

Let us start with a homogeneous and isotropic universe with a flat Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -N^2(t)dt^2 + a^2(t)[dr^2 + r^2d\Omega] \quad (18)$$

As usual, $a(t)$ is the scale factor and $N(t)$ is the lapse function. We use the Einstein-Hilbert action and a scalar field ϕ as the matter content for the model. In units $8\pi G = 1$, the action takes the form:

$$S = \int dt \left\{ -\frac{3a\dot{a}^2}{N} + a^3 \left(\frac{\dot{\phi}^2}{2N} - N\Lambda \right) \right\} \quad (19)$$

Now, we make the following change of variables:

$$x = m^{-1}a^{3/2}\sinh(m\phi), y = m^{-1}a^{3/2}\cosh(m\phi). \quad (20)$$

where $m^{-1} = 2\sqrt{2/3}$. Then the Hamiltonian is

$$H_c = N\left(\frac{1}{2}P_x^2 + \frac{\omega^2}{2}x^2\right) - N\left(\frac{1}{2}P_y^2 + \frac{\omega^2}{2}y^2\right), \quad (21)$$

with $\omega^2 = -\frac{3}{4}\Lambda$. To find the dynamics, we solve the equations of motion; for this model, it can easily be integrated [9].

To construct the deformed model, we usually follow the canonical quantum cosmology approach, where after canonical quantization [11], one formally obtains the WDW equation. In the deformed phase space approach, the deformation is introduced by the Moyal brackets to get a deformed Poisson algebra. To construct a deformed Poisson algebra, we use the approach given in Refs. [1, 9]. We start with the same transformation on the classical phase space variables $\{x, y, P_x, P_y\}$ that satisfy the usual Poisson algebra as shown in Section 2.1, Eqs. (10) and (11). With this deformed theory in mind, we first calculate the Hamiltonian which is formally analogous to Eq. (21) but constructed with the variables that obey the modified algebra Eq. (11)

$$H = \frac{1}{2} \left[(P_x^2 - P_y^2) - \omega_1^2(xP_y + yP_x) + \omega_2^2(x^2 - y^2) \right]. \quad (22)$$

where we have used the change of variables Eq. (10) and the following definitions:

$$\omega_1^2 = \frac{\beta - \omega^2\theta}{1 - \omega^2\theta^2/4}, \quad \omega_2^2 = \frac{\omega^2 - \beta^2/4}{1 - \omega^2\theta^2/4}. \quad (23)$$

Written in terms of the original variables, the Hamiltonian explicitly has the effects of the phase space deformation. These effects are encoded by the parameters θ and β . In Ref. [9], the late time behavior of this model was studied. From this formulation, two different physical theories arise, one that considers the variables x and y and a different theory based on \hat{x} and \hat{y} . The first theory is interpreted as a “commutative” theory with a modified interaction, and this theory is referred as being realized in the commutative frame “(C-frame)” [12]. The second theory, which privileges the variables \hat{x} and \hat{y} , is a theory with “noncommutative” variables but with the standard interaction and is referred to as realized in the noncommutative frame “(NC-frame).” In the “C-frame,” our deformed model has a very nice interpretation that of a ghost-oscillator in the presence of constant magnetic field and allows us to write the effects of the noncommutative deformation as minimal coupling on the Hamiltonian and write the Hamiltonian in terms of the magnetic B-field [9].

To obtain the dynamics for the model, we derive the equations of motion from the Hamiltonian Eq. (22). The solutions for the variables $x(t)$ and $y(t)$ in the “C-frame” are:

$$\begin{aligned}x(t) &= \eta_0 e^{\frac{-\omega^2}{2}t} \cosh(\omega't + \delta_1) - \zeta_0 e^{\frac{\omega^2}{2}t} \cosh(\omega't + \delta_2), \\y(t) &= \eta_0 e^{\frac{-\omega^2}{2}t} \cosh(\omega't + \delta_1) + \zeta_0 e^{\frac{\omega^2}{2}t} \cosh(\omega't + \delta_2),\end{aligned}\quad (24)$$

where $\omega' = \sqrt{\frac{\beta^2 - 4\omega^2}{4 - \omega^2\theta^2}}$. For $\omega'^2 < 0$, the hyperbolic functions are replaced by harmonic functions. There is a different solution for $\beta = 2\omega$, the solutions in the “C-frame” are:

$$\begin{aligned}x(t) &= (a + bt)e^{\frac{-\omega^2}{2}t} + (c + dt)e^{\frac{\omega^2}{2}t}, \\y(t) &= (a + bt)e^{\frac{-\omega^2}{2}t} - (c + dt)e^{\frac{\omega^2}{2}t}.\end{aligned}\quad (25)$$

To compute the volume of the universe in the “C-frame,” we use Eqs. (24) and (20).

$$a^3(t) = V_0 \cosh^2(\omega't), \quad (26)$$

where we have taken $\delta_1 = \delta_2 = 0$. For the case $\omega'^2 < 0$, the hyperbolic function is replaced by a harmonic function. For the case $\beta = 2\omega$, the volume is given by

$$a^3(t) = V_0 + At + Bt^2, \quad (27)$$

where V_0, A and B are constructed from the integration constants. To develop the dynamics in the “NC-frame,” we start from the “C-frame” solutions and use Eq. (10), we get for the volume

$$\hat{a}^3(t) = \begin{cases} \hat{V}_0 \left[\cosh^2(\omega't) - \frac{\omega'^2\theta^2}{(2 - \omega_1^2\theta)^2} \sinh^2(\omega't) \right] & \text{for } \omega'^2 > 0; \\ \hat{V}_0 + Bt + Ct^2 & \text{for } \omega'^2 = 0; \\ \hat{V}_0 \left[\cos^2(|\omega'|t) - \frac{|\omega'|^2\theta^2}{(2 - \omega_1^2\theta)^2} \sin^2(|\omega'|t) \right] & \text{for } \omega'^2 < 0, \end{cases} \quad (28)$$

where \hat{V}_0 is the initial volume in the “NC-frame.” We can see that for $\theta = 0$, the descriptions in the two frames are the same.

3.1. Discussion

As already discussed, phase space deformation gives two physical descriptions. If we say that both descriptions should be equal, then comparing the late time behavior for the two frames with the scale factor of de Sitter cosmology, an effective positive cosmological constant exists and is given by

$$\Lambda_{eff} = \frac{1}{3} \left(\frac{\beta^2 + 3\Lambda}{1 + \frac{3}{16}\Lambda\theta^2} \right). \quad (29)$$

This result is the same as the one obtained from the WDW formalism of Kaluza-Klein cosmology. Therefore, one can start taking seriously the possibility that noncommutativity can shed light on the cosmological constant problem.

4. Model 3: Thermodynamics of noncommutative quantum Kerr black hole

Thermodynamics of black holes has a long history, focusing mainly on the problem of thermodynamic stability. It is known for a long time that this problem can be extended beyond the asymptotically flat spacetimes [13]. For example, in de Sitter spacetimes, thermodynamic information of black holes exhibit important differences with the previous case [14, 15]. Gibbons and Hawking found that, in analogy with the asymptotically flat space case, such black holes emit radiation with a perfect blackbody spectrum and its temperature is determined by their surface gravity. However, a feature of de Sitter space is that exists a cosmological event horizon, emitting particles with a temperature which is proportional to its surface gravity. The only way to achieve thermal equilibrium is when both surface gravities are equal, which corresponds to a degenerate case [16, 17].

Regarding AdS manifolds, it was shown that thermodynamic stability of black holes in this spacetime can be achieved [18]. In this manifold, gravitational potential produces a confinement for particles with nonzero mass, which acts as an effective cavity of finite volume, containing the black hole. An important feature of black holes in AdS manifolds is that their heat capacity is positive, opposite to the asymptotically flat case; additionally, this positiveness allows a canonical description of the system.

It is also known that thermodynamic stability of black holes is related with dynamical stability of those systems, which brings an additional motivation to study it. For example, in the asymptotically flat spacetime case, it is well known that Schwarzschild black holes are thermodynamically unstable, although they are dynamically stable [19]. For AdS spacetimes, however, it is known that both thermodynamic and dynamical stability are closely related [20, 21].

In this study, we study black holes in asymptotically flat spacetime, whereby it seems very legitimate to ask whether corrections like the above discussed noncommutativity or even semiclassical ones can modify thermodynamic properties of black holes in order to have thermodynamic stable systems.

In a number of studies [22–24], black hole entropy proposed by Bekenstein and Hawking is postulated to be the fundamental thermodynamic relation for black holes, which contains all thermodynamic information of the system. Under this assumption, corresponding classical thermodynamic formalism is constructed, finding that its thermodynamic structure resembles ordinary magnetic systems instead of fluids.

4.1. Schwarzschild and Kerr black holes

As previously discussed, it is well known that for an asymptotically flat spacetime, temperature of black holes is proportional to its surface gravity κ , as $T = \kappa\hbar/2\pi k_B$, which is commonly

known as Hawking temperature [25]; this semiclassical result, along with Bekenstein bound for entropy, leads to the Bekenstein-Hawking entropy,

$$S_{BH} = \frac{c^3}{4G\hbar} A. \quad (30)$$

Where A stands for the area of the event horizon of the black hole. The Kerr metric, which describes a rotating black hole, can be written as:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra \sin^2\theta}{\Sigma} dt d\theta + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{B \sin^2\theta}{\Sigma} d\phi^2; \quad (31)$$

where, $\Sigma = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, $B = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$ and $a = J/Mc$. The area of the event horizon of a black hole is given by $A = \int_s \sqrt{\det|g_{\mu\nu}|} ds$. Applying for the elements of the metric tensor given in Eq. (31), the resulting area is:

$$A = 8\pi G^2 M^2 c^{-4} \left[1 + \sqrt{1 - \frac{c^2 J^2}{G^2 M^4}} \right]. \quad (32)$$

Assumed thermodynamic fundamental relation for Kerr black holes is found substituting the above result in Eq. (30); where $U = Mc^2$ is the internal energy of the system and J is its angular momentum. This relation can be written as [22]:

$$S_{BH}(U, J) = \frac{2\pi k_B}{\hbar c} \left(\frac{GU^2}{c^4} + \sqrt{\frac{G^2 U^4}{c^8} - c^2 J^2} \right); \quad (33)$$

where the following constants appear: G is the universal gravitational constant, c is the speed of light, \hbar is the reduced Planck constant, and k_B is the Boltzmann constant. In recent years, in the search of suitable candidates of quantum gravity, that is, in the quest to understand microscopic states of black holes [26, 27], a number of quantum corrections to Bekenstein-Hawking (BH) entropy S_{BH} have arisen. We are interested not only in the possible thermodynamic implications of quantum corrections to this entropy but also in the consequences of introducing noncommutativity as proposed by Obregon et al. [28], considering that coordinates of minisuperspace are noncommutative. From a variety of approaches that have emerged in recent years to correct S_{BH} , logarithmic ones are a popular choice among those. These corrections arise from quantum corrections to the string theory partition function [29] and are related to infrared or low-energy properties of gravity. They are also independent of high-energy or ultraviolet properties of the theory [26, 29–31]. We will denote the selected expression for quantum and noncommutative corrected entropy as S^* , which is obtained by following the ideas presented in [28]. The starting point is the diffeomorphism between the Kantowski-Sachs cosmological model, describing a homogeneous but anisotropic universe [32], and the Schwarzschild interior solution, whose line element for $r < 2M$ is given by:

$$ds^2 = -\left(\frac{2M}{t} - 1\right)^{-1} dt^2 + \left(\frac{2M}{t} - 1\right) dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2); \quad (34)$$

where the role of temporal t and the spatial r coordinates is swapped, that is, transformation $t \leftrightarrow r$ is performed, leading to a change on the causal structure of spacetime; considering the Misner parametrization of the Kantowski-Sachs metric it follows:

$$ds^2 = -N^2 dt^2 + e^{(2\sqrt{3}\gamma)} dr^2 + e^{(-2\sqrt{3}\gamma)} e^{(-2\sqrt{3}\lambda)} (d\theta^2 + \sin^2\theta d\phi^2). \quad (35)$$

Parameters λ and γ play the role of the *cartesian coordinates* in the Kantowski-Sachs minisuperspace. If Eqs. (34) and (35) are compared, it is straightforward to notice correspondence between components of the metric tensor, which allows us to identify the functions N , γ , and λ as:

$$N^2 = \left(\frac{2M}{t} - 1\right)^{-1}, \quad e^{(-2\sqrt{3}\gamma)} = \frac{2M}{t} - 1, \quad e^{(-2\sqrt{3}\gamma)} e^{(-2\sqrt{3}\lambda)} = t^2.$$

Next, the Wheeler DeWitt (WDW) equation for Kantowski-Sachs metric with the above parametrization of the Schwarzschild interior solution is found, along with the corresponding Hamiltonian of the system H through the Arnowitt-Deser-Misner (ADM) formalism. This Hamiltonian is introduced into the quantum wave equation $H\Psi = 0$, where $\Psi(\gamma, \lambda)$ is the wave function. This process leads to the WDW equation whose solution can be found by separation of variables.

However, we are not interested in the usual case, rather our point of interest is the solution that can be found when the symplectic structure of minisuperspace is modified by the inclusion of a noncommutativity parameter between the coordinates λ and γ , that is, the following commutation relation is obeyed: $[\lambda, \gamma] = i\theta$, where θ is the noncommutative parameter; this relation strongly resembles noncommutative quantum mechanics. It is also possible to introduce the aforementioned deformation in terms of a Moyal product [7], which modifies the original phase space, similarly to noncommutative quantum mechanics [33]:

These modifications allow us to redefine the coordinates of minisuperspace in order to obtain a noncommutative version of the WDW equation:

$$\left[\frac{\partial^2}{\partial \gamma^2} - \frac{\partial^2}{\partial \lambda^2} + 48e^{(-2\sqrt{3}\lambda + \sqrt{3}\theta P_\gamma)} \right] \Psi(\lambda, \gamma) = 0; \quad (36)$$

where P_γ is the momentum on coordinate γ . The above equation can be solved by separation of variables to obtain the corresponding wave function [6]:

$$\Psi(\lambda, \gamma) = e^{i\sqrt{3}\nu\gamma} K_{i\nu} [4e^{(-\sqrt{3}(\lambda + \sqrt{3}\nu\theta/2))}]; \quad (37)$$

where ν is the separation constant and $K_{i\nu}$ are the modified Bessel functions. We can see in Eq. (37) that the wave function has the form $\Psi(\lambda, \gamma) = e^{i\sqrt{3}\nu\gamma} \Phi(\lambda)$; therefore, dependence on the coordinate γ is the one of a plane wave. It is worth mentioning that this contribution vanishes when thermodynamic observables are calculated.

With the wave function presented in Eq. (37) for the noncommutative Kantowski-Sachs cosmological model, a modified noncommutative version of the entropy can be obtained. In order to calculate the partition function of the system, the Feynman-Hibbs procedure is considered [34]. Starting with the separated differential equation for λ :

$$\left[-\frac{d^2}{d\lambda^2} + 48e^{-2\sqrt{3}\lambda+3\nu\theta} \right] \Phi(\lambda) = 3\nu^2 \Phi(\lambda); \quad (38)$$

In this equation, the exponential in the potential term $V(\lambda) = 48 \exp[-2\sqrt{3}\lambda + 3\nu\theta]$ is expanded up to second order in λ and if a change of variables is considered, resulting differential equation can be compared with a one-dimensional quantum harmonic oscillator, which is a non-degenerate quantum system. In the Feynman-Hibbs procedure, the potential under study is modified by quantum effects, for the harmonic oscillator is given by:

$$U(x) = V(\bar{x}) + \frac{\beta \hbar^2}{24m} V''(\bar{x});$$

where \bar{x} is the mean value of x and $V''(\bar{x})$ stands for the second derivative of the potential. For the considered change of variables, the noncommutative quantum-corrected potential can be written as:

$$U(x) = \frac{3}{4\pi} \frac{E_p}{l_p^2} e^{3\nu\theta} \left[x^2 + \frac{\beta l_p^2 E_p}{12} \right]. \quad (39)$$

The above potential allows us to calculate the canonical partition function of the system:

$$Z(\beta) = C \int_{-\infty}^{\infty} e^{-\beta U(x)} dx; \quad (40)$$

where β^{-1} is proportional to the Bekenstein-Hawking temperature and $C = [2\pi l_p^2 E_p \beta]^{-1/2}$ is a constant. Substituting $U(x)$ into Eq. (40) and performing the integral over x , the partition function is given by:

$$Z(\beta) = \sqrt{\frac{2\pi}{3}} \frac{e^{3\nu\theta/2}}{E_p \beta} \exp \left[-\frac{\beta^2 E_p^2}{16\pi} e^{3\nu\theta} \right]; \quad (41)$$

This partition function allows us to calculate any desired thermodynamic observable by means of the thermodynamic connection of the Helmholtz free energy $A = -k_B T \ln Z(\beta)$, with the internal energy and the Legendre transformation:

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta); \quad \frac{S}{k_B} = \ln Z(\beta) + \beta \langle E \rangle.$$

With this equation for $\langle E \rangle$, the value of β can be determined as a function of the Hawking temperature $\beta_H = 8\pi M c^2 / E_p$, obtaining:

$$\beta = \beta_H e^{-3\nu\theta} \left[1 - \frac{1}{\beta_H e^{-3\nu\theta}} \frac{1}{Mc^2} \right]; \quad (42)$$

With the aid of this relation and the Legendre transformation for Helmholtz free energy presented above, an expression for the noncommutative quantum-corrected black hole entropy can be found:

$$S^* = S_{BH} e^{-3\nu\theta} - \frac{1}{2} k_B \ln \left[\frac{S_{BH}}{k_B} e^{-3\nu\theta} \right] + \mathcal{O}(S_{BH}^{-1} e^{-3\nu\theta}). \quad (43)$$

Functional form of S^* is basically the same than quantum-corrected commutative case, besides the addition of multiplicative factor $e^{-3\nu\theta}$ to Bekenstein-Hawking entropy. From now on, we will denote the noncommutative term in this expression, for the sake of simplicity, as:

$$\Gamma = \exp[-3\nu\theta].$$

Likewise, natural units, $G = \hbar = k_B = c = 1$, will be considered through the rest of this chapter. In this section, the previous result found in Eq. (43) for the Schwarzschild noncommutative black hole is extended to the rotating case, that is, the Kerr black hole. This is not straightforward as an analog expression for the noncommutative entropy of the rotating black hole is required, implying the application of a similar procedure to the one presented above: A diffeomorphism between the Kerr metric and some appropriated cosmological model and the procedure is presented in Ref. [28]. To our knowledge, the implementation of this procedure has not been yet reported. However, we are interested to have an expression to study not only the static case but also the effect of angular momentum over the physical properties of the system. Our proposal to have an *approximated* relation for the extended Kerr black hole entropy starts with the assumption that for entropy found in Eq. (43), Bekenstein-Hawking entropy for Schwarzschild in this relation S_{BH} can be also substituted for its Kerr counterpart given in Eq. (33). As the noncommutative relation for quantum Schwarzschild black hole entropy is correct, it is clear that our proposal to the quantum noncommutative Kerr black hole entropy will be a good approximation for small values of J when compared to the values of U^2 , whatever be the exact expression for the rotating case. For our proposal, in the vicinity of small values of angular momentum, λ and γ , the coordinates of the minisuperspace are the same than in the Schwarzschild case. Therefore, the corrected entropy that will be analyzed is:

$$S^* = 2\pi \Gamma \left(U^2 + \sqrt{U^4 - J^2} \right) - \frac{1}{2} \ln \left[2\pi \Gamma \left(U^2 + \sqrt{U^4 - J^2} \right) \right]. \quad (44)$$

A clarification must be made that Eq. (44) is not a unique valid generalization for the quantum-corrected noncommutative entropy of a rotating black hole in the neighborhood of small J . However, we claim that this is the most natural extension from the Schwarzschild case to the Kerr one. Although, to our knowledge, there is no general argument to support that Eq. (43) remains valid for any other black hole besides the Schwarzschild one. However, there is some evidence that for the case of charged black holes, the functional form of Eq. (43) is maintained, at least partially [35].

Through the rest of this section, all thermodynamic expressions with superindex \star will stand for noncommutative quantum-corrected quantities derived from Eq. (44), meanwhile, all thermodynamic functions without subindexes or superindexes will represent the corresponding noncommutative Bekenstein-Hawking counterparts. It is known that noncommutativity parameter θ in spacetime is small, from observational evidence [36, 37]; although in this study, noncommutativity on the coordinates of minisuperspace is considered instead, it is expected such parameter to be also small [38]; nonetheless, its actual bounds are not well known yet. We will consider that parameter Γ is bounded in the interval $0 < \Gamma \leq 1$. As previously mentioned for the non-corrected Kerr black hole, Eq. (44) is now assumed to be a fundamental thermodynamic relation for the rotating black hole, when noncommutative and quantum corrections are considered. It is well known from classical thermodynamics that fundamental equations contain all the thermodynamic information of the considered system [39], and, as a consequence, modifications introduced by corrections to entropy (which imply modifications to thermodynamic information) are carried through all thermodynamic quantities.

In **Figure 1**, plots for both Bekenstein-Hawking entropy and its quantum-corrected counterpart are presented for $\Gamma = 1$. **Figure 1a** shows plots for $S = S(U)$ and $S^\star = S^\star(U)$; Bekenstein-Hawking entropy is above the quantum-corrected one, in all its dominion, even in the region of low masses, where entropy is thermodynamically stable [22, 24]. **Figure 1b** presents the same curves as function of angular momentum instead, for $U = 1$; a similar behavior can be noticed in this case. If this analysis is performed over the noncommutative relation, it is found that for small values of θ , differences between both S_{BH} and S^\star are negligible.

4.2. Equations of state

Working in entropic representation, fundamental Bekenstein-Hawking thermodynamic relation for a Kerr black hole has the form $S_{BH} = S_{BH}(U, J)$. For these systems, partial derivatives of S_{BH} $T \equiv (\partial_S U)_J$ and $\Omega \equiv (\partial_J U)_S$ play the role of thermodynamic equations of state; here, T

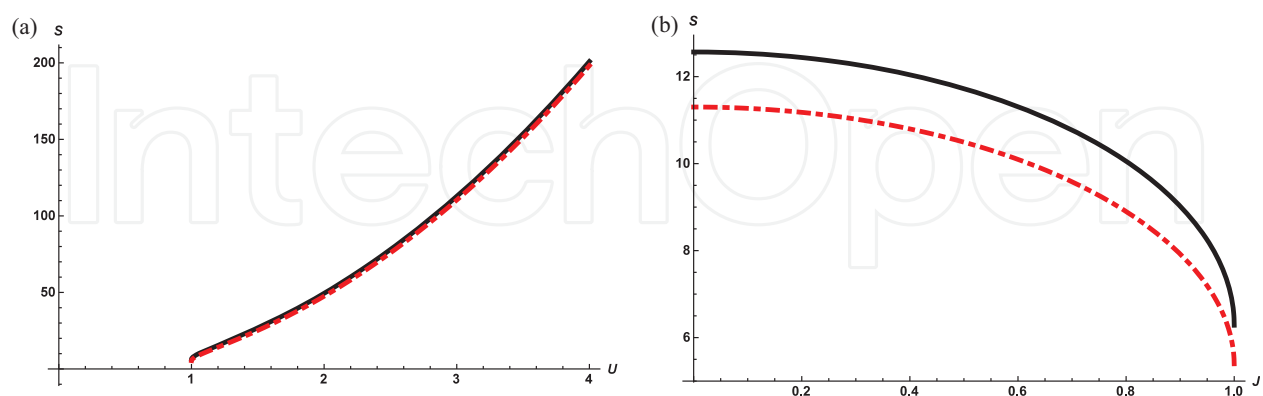


Figure 1. A comparison between Bekenstein-Hawking entropy (solid line) and its quantum-corrected counterpart (dash-dot line) is presented; both relations exhibit a region where entropy is a concave function, implying the existence of metastable states. (a) Entropy as a function of internal energy, $J = 1$. (b) Entropy as a function of angular momentum for $U = 1$, $S = S(1, J)$.

stands for Hawking temperature and Ω is the angular velocity. In entropic representation, equations of state are defined by:

$$\frac{1}{T} \equiv \left(\frac{\partial S_{BH}}{\partial U} \right)_J; \quad \frac{\Omega}{T} \equiv - \left(\frac{\partial S_{BH}}{\partial J} \right)_U. \quad (45)$$

For the entropy of the quantum-corrected entropy S^* , the above relations remain valid. In entropic representation, T and Ω for the noncommutative quantum-corrected entropy are given by:

$$\frac{1}{T^*} = \frac{U \left(4\pi \Gamma \sqrt{U^4 - J^2} + 4\pi \Gamma U^2 - 1 \right)}{\sqrt{U^4 - J^2}}, \quad (46a)$$

$$\frac{\Omega^*}{T^*} = \frac{1}{2} \frac{J \left(4\pi \Gamma \sqrt{U^4 - J^2} + 4\pi \Gamma U^2 - 1 \right)}{\sqrt{U^4 - J^2} \left(U^2 + \sqrt{U^4 - J^2} \right)}. \quad (46b)$$

The same relations for noncommutative Bekenstein-Hawking entropy are calculated as:

$$\frac{1}{T} = \frac{4\pi \Gamma U (U^2 + \sqrt{U^4 - J^2})^2}{\sqrt{U^4 - J^2}}, \quad (47a)$$

$$\frac{\Omega}{T} = \frac{2\pi \Gamma J}{\sqrt{U^4 - J^2}}. \quad (47b)$$

When the overall effect over T and T^* of noncommutativity was analyzed, different values of parameter Γ were tested, including $\Gamma = 1$ (commutative case). The corresponding curves present a noticeable effect by the presence of Γ ; nonetheless, functional behavior either of T or T^* is not modified. A comparison of the plots of both temperature is presented in **Figure 2** for $\Gamma = 1$, in order to illustrate how quantum corrections introduced in entropy affect thermodynamic properties of black holes. Resulting curves of T and T^* are very similar, although the latter one is slightly higher than $T(U, J)$, an opposite result to the one obtained when entropy was studied; it indicates that for a given change in its internal energy, variations of entropy are greater for quantum-corrected entropy when compared to the Bekenstein-Hawking one.

As previously mentioned, when values in the vicinity of $\Gamma = 1$ are considered, temperature is minimally affected by noncommutativity. We also tested smaller values of noncommutativity parameter, it was found that the maximum values that T and T^* are able to reach are noticeably increased. However, the shape of both curves is not modified by changing the value of Γ .

An interesting result is obtained for angular velocity Ω , this property seems to be independent of both quantum and noncommutative corrections to entropy, namely:

$$\Omega = \Omega^* = \frac{J}{2U \left(U^2 + \sqrt{U^4 - J^2} \right)}. \quad (48)$$

In **Figure 3**, plots for angular velocity are presented. As this equation of state is not modified by any of the considered corrections, only one curve per graphic appears; first, in **Figure 3a**, Ω as a function of the black hole internal energy is presented, as can be noticed, angular velocity steadily decreases as black hole mass is increased, asymptotically going to zero. **Figure 3b** considers instead the case where the black hole mass is fixed at $U = 10$, for which Ω grows until it reaches a maximum value determined by the square root that appears in the denominator of Eq. (48), beyond this value angular velocity becomes complex.

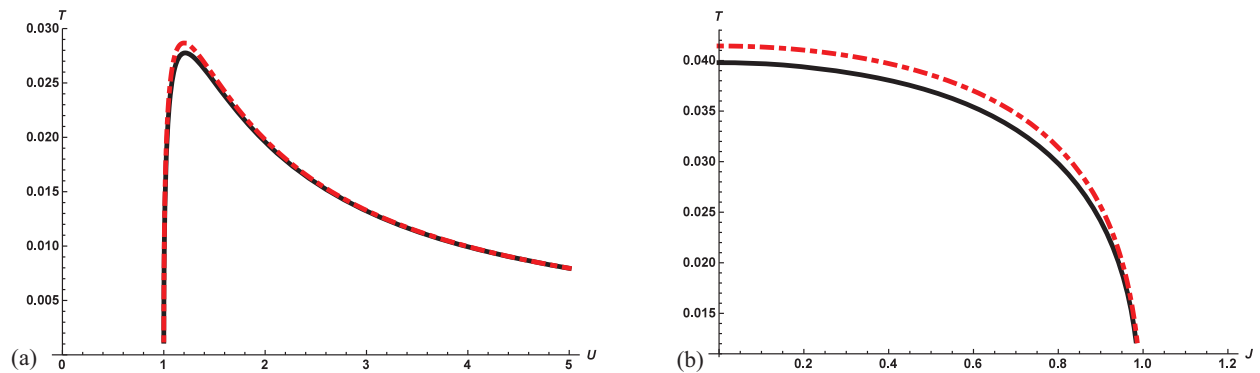


Figure 2. Temperature in the commutative case $\Gamma = 1$ for Bekenstein-Hawking usual entropy and its quantum-corrected counterpart. (a) Plots of $T(U, 1)$ (solid line) versus $T^*(U, 1)$ (dash-dot line) as a function of internal energy for a fixed value of angular momentum $J = 1$. (b) The same curves, considering instead for variations in J at a fixed $U = 1$.

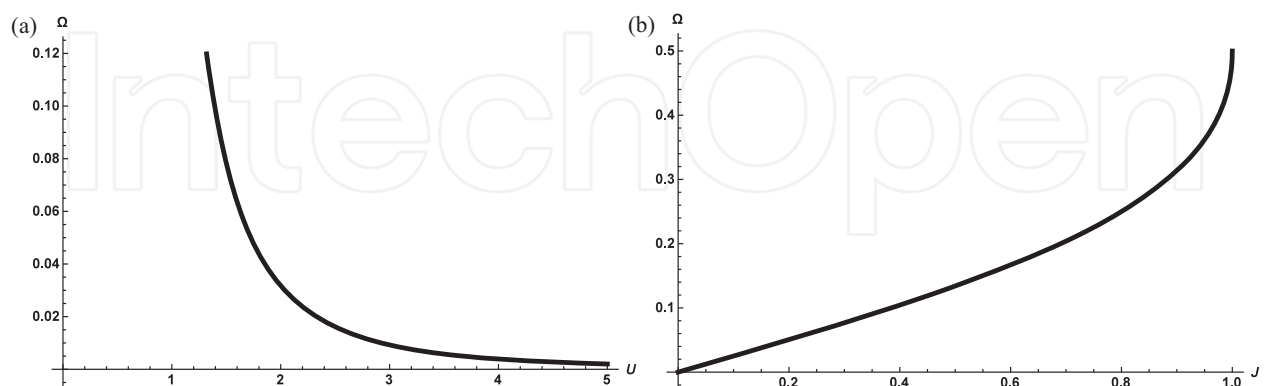


Figure 3. Angular velocity for Bekenstein-Hawking entropy and the quantum-corrected version are presented in Eq. (44). (a) Ω as a function of internal energy considering a fixed value of angular momentum ($J = 1$). (b) Angular velocity as a function of angular momentum considering $U = 10$.

5. Conclusions

In section 2, if we turn our attention to the case where there is no deformation on the coordinates. Taking the noncommutative parameter $\theta = 0$, we have that the frequency and the effective cosmological constant are given by:

$$\tilde{\omega}^2 = \omega^2 - \frac{\beta^2}{4}, \quad \text{and} \quad \tilde{\Lambda}_{eff} = \Lambda_{eff} + \frac{3\beta^2}{8}. \quad (49)$$

From the last equation, we get the most interesting result of this section. We can see that noncommutative parameter β and Λ_{eff} compete to give the effective cosmological constant $\tilde{\Lambda}_{eff}$. If we consider the case of a flat universe with a vanishing Λ_{eff} , we see that $\tilde{\Lambda}_{eff} = \frac{3\beta^2}{8}$. This shows the relationship between the cosmological constant and the deformed parameter. Recently, some evidence on the possibility that the effects of the phase space deformation could be related to the late time acceleration of the universe as well as to the cosmological constant were presented [8]. Interestingly, in the particular case of $\beta = \omega^2\theta$, we find that frequency reduces to $\tilde{\omega}^2 = \omega^2$ and we have that $\tilde{\Lambda}_{eff} = \Lambda_{eff}$. In this case, even as we have done a deformation on the minisuperspace of the theory, the effects cancel out and the resulting theory behaves as in the commutative theory. The results are similar for model 2 (Section 3), where under a totally classical regime, we find the same functional relationship between the cosmological constant and the deformation parameter β . Therefore, we conclude that noncommutative phase space deformations can hold the answer to the cosmological constant problem.

Then, in Section 4, an analysis on the thermodynamic properties of noncommutative quantum-corrected Kerr black holes using an approximate relation was presented. Although the resulting expressions are mathematically more complicated, the thermodynamic properties still retain the same functional behavior with respect to those calculated through Bekenstein-Hawking entropy. It can be proved that Kerr black holes do not pass through a first-order phase transition [4]; since the local criteria to find the critical point are not fulfilled for any value in the domain, corresponding isotherms do not exhibit van der Waals loops, and the Maxwell construction cannot be obtained; all of which are characteristic of this kind of transition. Regarding the effective noncommutativity incorporated in the coordinates of minisuperspace, outside the vicinity where $\Gamma \approx 1$, changes introduced by this parameter over the thermodynamic information of the system are relevant. For a complete analysis using this phase deformations, for example, thermodynamic response functions, thermodynamic stability, and phase transitions for Kerr black holes, see Ref. [4].

Acknowledgements

Eri Atahualpa Mena Barboza thanks the financial support from C.U.CI., U. de G. project Desarrollo de la investigación y fortalecimiento del posgrado 235506.

Author details

E.A. Mena-Barboza^{1*}, L.F. Escamilla-Herrera², J.C. López-Domínguez³ and J. Torres-Arenas⁴

*Address all correspondence to: emena@cuci.udg.mx

1 Centro Universitario de la Ciénega, Universidad de Guadalajara Ave. Universidad, Ocotlán, Jalisco, México

2 Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Ciudad de, México

3 Unidad Académica de Física, Universidad Autónoma de Zacatecas, Calzada Solidaridad, Zacatecas, México

4 Departamento de Física de la Universidad de Guanajuato, León, Guanajuato, México

References

- [1] Rezaei-Aghdam A, Darabi F, Rastkar AR. Physics Letters B. 2005;**615**:141-145. [gr-qc/0412089]
- [2] Pérez-Payán S, Sabido M, Mena E. Available from: <http://arxiv.org/abs/1111.6137v2>
- [3] Sinuhe Pérez-Payán, Sabido M, Mena E, Yee-Romero C. Analysis of scalar field cosmology with phase space deformations. Advances in High Energy Physics. 2014. Article ID 958137, 5 pages. DOI:10.1155/2014/958137
- [4] Escamilla-Herrera LF, Mena-Barboza E, Torres-Arenas J. Entropy. 2016;**18**(11):406. DOI:10.3390/e18110406
- [5] García-Compeán H, Obregón O, Ramírez C, Sabido M. Physical Review D. 2003;**68**:044015; García-Compeán H, Obregón O, Ramírez C, Sabido M. Physical Review D. 2003;**68**:045010; Aschieri P, Dimitrijevic M, Meyer F, Wess J. Classical and Quantum Gravity. 2006;**23**:1883; Calmet X, Kobakhidze A. Physical Review D. 2005;**72**:045010; Alvarez-Gaume L, Meyer F, Vazquez-Mozo MA. Nuclear Physics B. 2006;**753**:92; Estrada-Jimenez S, García-Compeán H, Obregón O, Ramírez C. Physical Review D. 2008;**78**:124008
- [6] García-Compeán H, Obregón O, Ramírez C. Noncommutative quantum cosmology. Physical Review Letters. 2002;**88**:161301
- [7] Barbosa GD, Pinto-Neto N. Physical Review D. 2004;**70**:103512; Guzman W, Sabido M, Socorro J. Physical Review D. 2007;**76**:087302, International Journal of Modern Physics D. 2007;**16**:1625; Vakili B, Khosravi N, Sepangi HR. Classical and Quantum Gravity. 2007;**24**:931; Aguero, Aguilar JA, Ortiz SC, Sabido M, Socorro J. International Journal of

- Theoretical Physics. 2007;**46**:2928; Ortiz C, Mena-Barboza E, Sabido M, Socorro J. International Journal of Theoretical Physics. 2008;**47**:1240; Guzman W, Ortiz C, Sabido M, Socorro J, Aguero MA. International Journal of Modern Physics D. 2007;**16**:1625; Obregon O, Quiros I. Physical Review. 2011;**D84**:044005; Guzman W, Sabido M, Socorro J, Physics Letters. 2011;**B697**:271-274.
- [8] Vakili B, Pedram P, Jalalzadeh S. Physics Letters B. 2010;**687**:119-123
 - [9] Pérez-Payán S, Sabido M, Yee-Romero C. Physical Review D. 2013;**88**:027503
 - [10] Sabido M, Yee-Romero C. Physics Letters B. 2016;**757**:57
 - [11] Cordero R, Garcia-Compean H, Turrubiates FJ. Physical Review D. 2011;**83**:125030
 - [12] Barbosa GD. Physical Review D. 2005;**71**:063511
 - [13] Gibbons GW, Hawking SW. Cosmological event horizons, thermodynamics, and particle creation. Physical Review D. 1977;**15**:2738-2751
 - [14] Davies PCW. Thermodynamic phase transitions of Kerr-Newman black holes in de Sitter space. Classical and Quantum Gravity. 1989;**6**:1909
 - [15] Chamblin A, Emparan R, Johnson CV, Myers RC. Charged AdS black holes and catastrophic holography. Physical Review D. 1999;**60**:064018
 - [16] Nariai H. On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. Science Reports of the Tohoku Imperial University. 1950;**34**:160-167
 - [17] Nariai H. On a new cosmological solution of Einstein's field equations of gravitation. Science Reports of the Tohoku Imperial University. 1950;**35**:62-67
 - [18] Hawking SW, Page DN. Thermodynamics of black holes in anti-de Sitter space. Communications in Mathematical Physics. 1983;**87**:577-588
 - [19] Price RH. Nonspherical perturbations of relativistic gravitational collapse. 1. Scalar and gravitational perturbations. Physical Review D. 1972;**5**:2419
 - [20] Gubser SS, Mitra I. The evolution of unstable black holes in anti-de Sitter space. Journal of High Energy Physics. 2001;**0108**:018
 - [21] Hollands S, Wald RM. Stability of black holes and black branes. Communications in Mathematical Physics. 2013;**321**:629-680
 - [22] Davies PCW. The thermodynamics of black holes. Proceedings of the Royal Society A. 1977;**353**:499
 - [23] Martinez EA. The postulates of gravitational thermodynamics. Physical Review D. 1996;**54**:6302
 - [24] Escamilla L, Torres-Arenas J. Thermodynamic response functions and Maxwell relations for a Kerr black hole. Revista Mexicana de Fisica. 2014;**60**:59-68
 - [25] Hawking SW. Gravitational radiation from colliding black holes. Physical Review Letters. 1971;**26**:1344

- [26] Keeler C, Larsen F, Lisboa P. Logarithmic corrections to $\mathcal{N} \geq 2$ black hole entropy. *Physical Review D*. 2014;**90**:043011
- [27] Sen A. Logarithmic corrections to Schwarzschild and other non-extremal black hole entropy in different dimensions. *Journal of High Energy Physics*. 2013;**04**:156
- [28] López-Domínguez JC, Obregón O, Sabido M, Ramírez C. Towards noncommutative quantum black holes. *Physical Review D*. 2006;**74**:084024
- [29] Sen A. Logarithmic corrections to $\mathcal{N} \geq 2$ black hole entropy: An infrared window into the microstates. *General Relativity and Gravitation*. 2012;**44**:1207-1266
- [30] Banerjee S, Gupta RK, Mandal I, Sen A. Logarithmic corrections to and black hole entropy: A one loop test of quantum gravity. *Journal of High Energy Physics*. 2011;**03**:143
- [31] Carlip S. Logarithmic corrections to black hole entropy, from the Cardy formula. *Classical and Quantum Gravity*. 2000;**17**:4175
- [32] Kantowski R, Sachs RK. Some spatially homogeneous anisotropic relativistic cosmological models. *Journal of Mathematical Physics*. 1966;**7**:443
- [33] Gamboa J, Loewe M, Rojas JC. Noncommutative quantum mechanics. *Physical Review D*. 2011;**64**:067901
- [34] Feynman RP, Hibbs AR. *Quantum Mechanics and Path Integrals*. 1st ed. New York: McGraw-Hill; 1965
- [35] Zacarias S, Escamilla-Herrera LF. To be published.
- [36] Aoki H, Ishibashi N, Iso S, Kawai H, Kitazawa Y, Tada T. Noncommutative Yang-Mills in IIB matrix model. *Nuclear Physics B*. 2000;**565**:176
- [37] Joby PK, Chingangbam P, Das S. Constraint on noncommutative spacetime from PLANCK data. *Physical Review D*. 2015;**91**:083503
- [38] Guzmán W, Sabido M, Socorro J. On noncommutative minisuperspace and the Friedmann equations. *Physics Letters B*. 2011;**697**:271
- [39] Callen HB. *Thermodynamics and an Introduction to Thermo-statistics*. 2nd ed. Singapore: John Willey & Sons; 1985. pp. 35, 183, 215

