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Chapter 1

Stabilizing Fuzzy Control via Output Feedback

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Additional information is available at the end of the chapter

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Abstract

The chapter presents new conditions suitable in design of stabilizing static as well as dynamic output controllers for a class of continuous-time nonlinear systems represented by Takagi-Sugeno models. Taking into account the affine properties of the TS model structure, and applying the fuzzy control scheme relating to the parallel-distributed output compensators, the sufficient design conditions are outlined in the terms of linear matrix inequalities. Depending on the proposed procedures, the Lyapunov matrix can be decoupled from the system parameter matrices using linear matrix inequality techniques or a fuzzy-relaxed approach can be applied to make closed-loop dynamics faster. Numerical examples illustrate the design procedures and demonstrate the performances of the proposed design methods.

Keywords: continuous-time nonlinear systems, Takagi-Sugeno fuzzy systems, linear matrix inequality approach, parallel-distributed compensation, output feedback

1. Introduction

Contrarily to the linear framework, nonlinear systems are too complex to be represented by unified mathematical resources and so, a generic method has not been developed yet to design a controller valid for all types of nonlinear systems. An alternative to nonlinear system models is Takagi-Sugeno (TS) fuzzy approach [1], which benefits from the advantages of suitable linear approximation of sector nonlinearities. Using the TS fuzzy model, each rule utilizes the local system dynamics by a linear model and the nonlinear system is represented by a collection of fuzzy rules. Recently, TS model-based fuzzy control approaches are being fast and successfully used in nonlinear control frameworks. As a result, a range of stability analysis conditions [2–5] as well as control design methods have been developed for TS fuzzy systems [6–9], relying mostly on the feasibility of an associated set of linear matrix inequalities

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(LMI) [10]. An important fact is that the design problem is a standard feasibility problem with several LMIs, potentially combined with one matrix equality to overcome the problem of bilinearity. In consequence, the state and output feedback control based on fuzzy TS systems models is mostly realized in such structures, which can be designed using numerical techniques based on LMIs.

The TS fuzzy model-based state control is based on an implicit assumption that all states are available for measurement. If it is impossible, TS fuzzy observers are used to estimate the unmeasurable state variables, and the state controller exploits the system state variable estimate values [11–14]. The nonlinear output feedback design is so formulated as two LMI set problem, and treated by the two-stage procedure using the separation principle, that is, dealing with a set of LMIs for the observer parameters at first and then solving another set of LMIs for the controller parameters [15]. Since, the fuzzy output control does not require the measurement of system state variables and can be formulated as a one LMI set problem, such structure of feedback control is preferred, of course, if the system is stabilizable.

From a relatively wide range of problems associated with the fuzzy output feedback control design for the continuous-time nonlinear MIMO systems approximated by a TS model, the chapter deals with the techniques incorporating the slack matrix application and fuzzy membership-relaxed approaches. The central idea of the TS fuzzy model-based control design, that is, to derive control rules so as to compensate each rule of a fuzzy system and construct the control strategy based on the parallel-distributed compensators (PDC), is reflected in the approach of output control. Motivated by the above mentioned observations, the proposed design method respects the results presented in Refs. [16, 17], and is constructed on an enhanced form of quadratic Lyapunov function. Comparing with the approaches based only on quadratic Lyapunov matrix [18], which are particular in the case of large number of rules, that are very conservative as a common symmetric positive definite matrix, is used to verify all Lyapunov inequalities, presented principle naturally extends the affine TS model properties using slack matrix variables to decouple Lyapunov matrix and the system matrices in LMIs, and gives substantial reducing of conservativeness. Moreover, extra quadratic constraints are included to incorporate fuzzy membership functions relaxes [19, 20] and applied for static as well as dynamic TS fuzzy output controllers design. Note, other constraints with respect to, for example, to decay rate and closed-loop pole clustering can be utilized to extend the proposed design procedures.

The remainder of this chapter is organized as follows. In Section 2, the structure of TS model for considered class of nonlinear systems is briefly described, and some of its properties are outlined. The output feedback control design problem for systems with measurable promise variables is given in Section 3, where the design conditions that guarantees global quadratic stability are formulated and proven. To complete the solutions, Section 4 formulate the static decoupling principle in static TS fuzzy output control, and the method is reformulated in Section 5 in defined criteria for TS fuzzy dynamic output feedback control design. Section 6 gives the numerical examples to illustrate the effectiveness of the proposed approach, and to confirm the validity of the control scheme. The last section, Section 7, draws conclusions and some future directions.

Throughout the chapter, the following notations are used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, for a square matrix $\boldsymbol{X} = \boldsymbol{X}^T > 0$ (respectively, $\boldsymbol{X} = \boldsymbol{X}^T < 0$)

means that $\boldsymbol{X}$ is a symmetric positive definite matrix (respectively, negative definite matrix), the symbol $\boldsymbol{I}_n$ represents the n-th order unit matrix, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^{n \times r}$ denotes the set of all $n \times r$ real matrices.

2. Takagi-Sugeno fuzzy models

The systems under consideration are from one class of multi-input and multi-output (MIMO) dynamic systems, which are nonlinear in sectors and represented by TS fuzzy model. Constructing the set of membership functions $h_i(\boldsymbol{\theta}(t))$, $i = 1, 2, \ldots, s$, where

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) & \cdots & \theta_q(t) \end{bmatrix}, \tag{1}$$

is the vector of premise variables, the final states of the systems are inferred in the TS fuzzy system model as follows

$$\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\boldsymbol{A}_i \boldsymbol{q}(t) + \boldsymbol{B}_i \boldsymbol{u}(t)), \tag{2}$$

with the output given by the relation

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t), \tag{3}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, $\boldsymbol{y}(t) \in \mathbb{R}^m$ are vectors of the state, input, and output variables, $\boldsymbol{A}_i \in \mathbb{R}^{n \times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n \times r}$, $\boldsymbol{C} \in \mathbb{R}^{m \times n}$ are real finite values matrix, and where $h_i(\boldsymbol{\theta}(t))$ is the averaging weight for the i-th rule, representing the normalized grade of membership (membership function).

By definition, the membership functions satisfy the following convex sum properties.

$$0 \le h_i(\boldsymbol{\theta}(t)) \le 1, \quad \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) = 1 \quad \forall i \in \langle 1, \ldots, s \rangle. \tag{4}$$

It is assumed that the premise variable is a system state variable or a measurable external variable, and none of the premise variables depends on the inputs $\boldsymbol{u}(t)$.

It is evident that a general fuzzy model is achieved by fuzzy amalgamation of the linear system models. Using a TS model, the conclusion part of a single rule consists no longer of a fuzzy set [21], but determines a function with state variables as arguments, and the corresponding function is a local function for the fuzzy region that is described by the premise part of the rule. Thus, using linear functions, a system state is described locally (in fuzzy regions) by linear models, and at the boundaries between regions an interpolation is used between the corresponding local models.

Note, the models, Eqs. (2) and (3), are mostly considered for analysis, control, and state estimation of nonlinear systems.

Assumption 1 *Each triplet* ($\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}$) *is locally controllable and observable, the matrix* $\boldsymbol{C}$ *is the same for all local models*.

It is supposed in the next that the aforementioned model does not include parameter uncertainties or external disturbances, and the premise variables are measured.

3. Static fuzzy output controller

In the next, the fuzzy output controller is designed using the concept of parallel-distributed compensation, in which the fuzzy controller shares the same sets of normalized membership functions like the TS fuzzy system model.

Definition 1 *Considering Eqs*. (2) *and* (3), *and using the same set of normalized membership function Eq*. (4), *the fuzzy static output controller is defined as*

$$\boldsymbol{u}(t) = \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\boldsymbol{K}_j\boldsymbol{y}(t) = \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\boldsymbol{K}_j\boldsymbol{C}\boldsymbol{q}(t)\,. \tag{5}$$

Note that the fuzzy controller Eq. (5) is in general nonlinear.

Considering the system, Eqs. (2) and (3), and the control law, Eq. (5), yields

$$\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\big(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\big)\boldsymbol{q}(t) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{A}_{cij}\boldsymbol{q}(t)\,, \tag{6}$$

$$\boldsymbol{A}_{cij} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}, \quad \boldsymbol{A}_{cji} = \boldsymbol{A}_j + \boldsymbol{B}_j\boldsymbol{K}_i\boldsymbol{C}\,. \tag{7}$$

Proposition 1 (*standard design conditions*). *The equilibrium of the fuzzy system Eqs*. (2) *and* (3), *controlled by the fuzzy controller Eq*. (5), *is global asymptotically stable if there exist a positive definite symmetric matrix* $\boldsymbol{W} \in \mathbb{R}^{n\times n}$ *and matrices* $\boldsymbol{Y}_j \in \mathbb{R}^{r\times m}$, $\boldsymbol{H} \in \mathbb{R}^{m\times m}$ *such that*

$$\boldsymbol{W} = \boldsymbol{W}^T > 0\,, \tag{8}$$

$$\boldsymbol{A}_i\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_i^T < 0\,, \tag{9}$$

$$\frac{\boldsymbol{A}_i\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_i^T}{2} + \frac{\boldsymbol{A}_j\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_j^T}{2} + \frac{\boldsymbol{B}_i\boldsymbol{Y}_j\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_j^T\boldsymbol{B}_i^T}{2} + \frac{\boldsymbol{B}_j\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_j^T}{2} < 0\,, \tag{10}$$

$$\boldsymbol{C}\boldsymbol{W} = \boldsymbol{H}\boldsymbol{C} \tag{11}$$

for $i = 1, 2, \ldots, s$ as well as $i = 1, 2, \ldots, s-1$, $j = i+1, i+2, \ldots, s$, and $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$.

When the above conditions hold, the control law gain matrices are given as

$$\boldsymbol{K}_i = \boldsymbol{Y}_i \boldsymbol{H}^{-1}. \tag{12}$$

Proof. (compare, for example, Ref. [16]) Prescribing the Lyapunov function candidate of the form

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) > 0, \tag{13}$$

where $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, the time derivative of Eq. (13) along the system trajectory is

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) < 0. \tag{14}$$

Inserting Eq. (6) into Eq. (14), it has to be satisfied

$$\dot{v}(\boldsymbol{q}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{q}^T(t)\boldsymbol{P}_{cij}\boldsymbol{q}(t) < 0, \tag{15}$$

$$\boldsymbol{P}_{cij} = \boldsymbol{P}\boldsymbol{A}_{cij} + \boldsymbol{A}_{cij}^T\boldsymbol{P}. \tag{16}$$

Since $\boldsymbol{P}$ is positive definite, the state coordinate transform can be defined as

$$\boldsymbol{q}(t) = \boldsymbol{W}\boldsymbol{p}(t), \quad \boldsymbol{W} = \boldsymbol{P}^{-1}, \tag{17}$$

and subsequently, Eqs. (15) and (16) can be rewritten as

$$\dot{v}(\boldsymbol{p}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\boldsymbol{W}_{cij}\boldsymbol{p}(t) < 0, \tag{18}$$

$$\boldsymbol{W}_{cij} = \boldsymbol{A}_{cij}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_{cij}^T. \tag{19}$$

Permuting the subscripts i and j in Eq. (18), also it can write

$$\dot{v}(\boldsymbol{q}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\boldsymbol{W}_{cji}\boldsymbol{p}(t) < 0, \tag{20}$$

$$\boldsymbol{W}_{cji} = \boldsymbol{A}_{cji}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_{cji}^T. \tag{21}$$

Thus, adding Eqs. (17) and (19), it yields

$$2\dot{v}(\boldsymbol{p}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\left(\boldsymbol{W}_{cij} + \boldsymbol{W}_{cji}\right)\boldsymbol{p}(t) < 0 \tag{22}$$

and subsequently,

$$\dot{v}(\boldsymbol{p}(t)) = \sum_{i=1}^{s} h_i^2(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\boldsymbol{W}_{cii}\boldsymbol{p}(t) + 2\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\frac{\boldsymbol{W}_{cij}+\boldsymbol{W}_{cji}}{2}\boldsymbol{p}(t) < 0, \quad (23)$$

which leads to the set of inequalities.

$$(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_i\boldsymbol{C})\boldsymbol{W} + \boldsymbol{W}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_i\boldsymbol{C})^T < 0, \quad (24)$$

$$\frac{(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C})\boldsymbol{W}}{2} + \frac{(\boldsymbol{A}_j + \boldsymbol{B}_j\boldsymbol{K}_i\boldsymbol{C})\boldsymbol{W}}{2} + \frac{\boldsymbol{W}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C})^T}{2} + \frac{\boldsymbol{W}(\boldsymbol{A}_j + \boldsymbol{B}_j\boldsymbol{K}_i\boldsymbol{C})^T}{2} < 0 \quad (25)$$

for $i = 1, 2, \ldots, s$ as well as $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$ and $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$.

Thus, setting here

$$\boldsymbol{K}_j\boldsymbol{C}\boldsymbol{W} = \boldsymbol{K}_j\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{C}\boldsymbol{W}, \quad (26)$$

where $\boldsymbol{H}$ is a regular square matrix of appropriate dimension and defining

$$\boldsymbol{H}^{-1}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{W}^{-1}, \quad \boldsymbol{Y}_j = \boldsymbol{K}_j\boldsymbol{H}, \quad (27)$$

the LMI forms of Eqs. (9) and (10) are obtained from Eqs. (24) and (25), respectively, and Eq. (27) implies Eq. (11). This concludes the proof.

Trying to minimize the number of LMIs owing to the limitation of solvers, Proposition 1 is presented in the structure, in which the number of stabilization conditions, used in fuzzy controller design, is equal to $N = (s^2 + s)/2 + 1$. Evidently, the number of stabilization conditions is substantially reduced if s is large.

Proposition 2 (*enhanced design conditions*). *The equilibrium of the fuzzy system Eqs.* (2) *and* (3), *controlled by the fuzzy controller Eq.* (5), *is global asymptotically stable if for given a positive* $\delta \in \mathbb{R}$, *there exist positive definite symmetric matrices* $\boldsymbol{V}, \boldsymbol{S} \in \mathbb{R}^{n \times n}$, *and matrices* $\boldsymbol{Y}_j \in \mathbb{R}^{r \times m}$, $\boldsymbol{H} \in \mathbb{R}^{m \times m}$ *such that*

$$\boldsymbol{S} = \boldsymbol{S}^T > 0, \quad \boldsymbol{V} = \boldsymbol{V}^T > 0, \quad (28)$$

$$\begin{bmatrix} \boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_i^T & * \\ \boldsymbol{V} - \boldsymbol{S} + \delta\boldsymbol{A}_i\boldsymbol{S} + \delta\boldsymbol{B}_i\boldsymbol{Y}_i\boldsymbol{C} & -2\delta\boldsymbol{S} \end{bmatrix} < 0, \quad (29)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{ij} & * \\ \boldsymbol{V} - \boldsymbol{S} + \delta\dfrac{\boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{A}_j\boldsymbol{S}}{2} + \delta\dfrac{\boldsymbol{B}_i\boldsymbol{Y}_j + \boldsymbol{B}_j\boldsymbol{Y}_i}{2}\boldsymbol{C} & -2\delta\boldsymbol{S} \end{bmatrix} < 0, \quad (30)$$

$$\boldsymbol{C}\boldsymbol{S} = \boldsymbol{H}\boldsymbol{C}, \quad (31)$$

for $i = 1, 2, \ldots, s$, *as well as* $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *and*

$$\boldsymbol{\Phi}_{ij} = \frac{\boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_i^T}{2} + \frac{\boldsymbol{A}_j\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_j^T}{2} + \frac{\boldsymbol{B}_i\boldsymbol{Y}_j\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_j^T\boldsymbol{B}_i^T}{2} + \frac{\boldsymbol{B}_j\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_j^T}{2}. \tag{32}$$

When the above conditions hold, the control law gain matrices are given as

$$\boldsymbol{K}_i = \boldsymbol{Y}_i\boldsymbol{H}^{-1}. \tag{33}$$

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Writing Eq. (6) in the form

$$\sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\left(\boldsymbol{A}_{cij}\boldsymbol{q}(t) - \dot{\boldsymbol{q}}(t)\right) = \boldsymbol{0}, \tag{34}$$

then with an arbitrary symmetric positive definite matrix $\boldsymbol{S} \in \mathbb{R}^{n \times n}$ and a positive scalar $\delta \in \mathbb{R}$, it yields

$$\sum_{i=1}^{s}\sum_{j=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\left(\boldsymbol{q}^T(t)\boldsymbol{S} + \delta\dot{\boldsymbol{q}}^T(t)\boldsymbol{S}\right)\left(\boldsymbol{A}_{cij}\boldsymbol{q}(t) - \dot{\boldsymbol{q}}(t)\right) = 0. \tag{35}$$

Since $\boldsymbol{S}$ is positive definite, the new state variable coordinate system can be introduced so that

$$\boldsymbol{p}(t) = \boldsymbol{S}\boldsymbol{q}(t), \quad \dot{\boldsymbol{p}}(t) = \boldsymbol{S}\dot{\boldsymbol{q}}(t), \quad \boldsymbol{V} = \boldsymbol{S}^{-1}\boldsymbol{P}\boldsymbol{S}^{-1}. \tag{36}$$

Therefore, Eq. (14) can be rewritten as

$$\dot{v}(\boldsymbol{p}(t)) = \dot{\boldsymbol{p}}^T(t)\boldsymbol{V}\boldsymbol{p}(t) + \boldsymbol{p}^T(t)\boldsymbol{V}\dot{\boldsymbol{p}}(t) < 0 \tag{37}$$

and Eq. (35) takes the form

$$\sum_{i=1}^{s}\sum_{j=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\left(\boldsymbol{p}^T(t) + \delta\dot{\boldsymbol{p}}^T(t)\right)\left(\boldsymbol{A}_{cij}\boldsymbol{S}\boldsymbol{p}(t) - \boldsymbol{S}\dot{\boldsymbol{p}}(t)\right) = 0. \tag{38}$$

Thus, adding Eq. (38) as well as the transposition of Eq. (38) to Eq. (37), it yields

$$\begin{aligned}\dot{v}(\boldsymbol{p}(i)) &= \dot{\boldsymbol{p}}^T(t)\boldsymbol{V}\boldsymbol{p}(t) + \boldsymbol{p}^T(t)\boldsymbol{V}\dot{\boldsymbol{p}}(t) \\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\left(\boldsymbol{p}^T(t) + \delta\dot{\boldsymbol{p}}^T(t)\right)\left(\boldsymbol{A}_{cij}\boldsymbol{S}\boldsymbol{p}(t) - \boldsymbol{S}\dot{\boldsymbol{p}}(t)\right) \\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\Big(\boldsymbol{A}_{cij}\boldsymbol{S}\boldsymbol{p}(t) - \boldsymbol{S}\dot{\boldsymbol{p}}(t))^T(\boldsymbol{p}(t) + \delta\dot{\boldsymbol{p}}(t)) < 0.\end{aligned} \tag{39}$$

Using the notation

$$\boldsymbol{p}_c^T(t) = \begin{bmatrix} \boldsymbol{p}^T(t) & \dot{\boldsymbol{p}}^T(t) \end{bmatrix}, \tag{40}$$

the inequality Eq. (39) can be written as

$$\dot{v}\left(\boldsymbol{p}_c(t)\right) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\boldsymbol{p}_c^T(t)\boldsymbol{S}_{cij}\boldsymbol{p}_c(t) < 0, \tag{41}$$

$$\boldsymbol{S}_{cij} = \begin{bmatrix} \left(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)\boldsymbol{S} + \boldsymbol{S}\left(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^T & * \\ \boldsymbol{V} - \boldsymbol{S} + \delta\left(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)\boldsymbol{S} & -2\delta\boldsymbol{S} \end{bmatrix} < 0. \tag{42}$$

Permuting the subscripts i and j in Eq. (41), and following the way used above, analogously it can obtain

$$\dot{v}\left(\boldsymbol{p}_c(t)\right) = \sum_{i=1}^{s}\mathrm{h}_i^2(\boldsymbol{\theta}(t))\boldsymbol{p}_c^T(t)\boldsymbol{S}_{cii}\boldsymbol{p}_c(t) + 2\sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\boldsymbol{p}_c^T(t)\frac{\boldsymbol{S}_{cij} + \boldsymbol{S}_{cji}}{2}\boldsymbol{p}_c(t) < 0. \tag{43}$$

Since $r = m$, it is now possible to set

$$\boldsymbol{K}_j\boldsymbol{C}\boldsymbol{S} = \boldsymbol{K}_j\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{C}\boldsymbol{S}, \tag{44}$$

where $\boldsymbol{H}$ is a regular square matrix of appropriate dimension and introducing

$$\boldsymbol{H}^{-1}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{S}^{-1}, \quad \boldsymbol{Y}_j = \boldsymbol{K}_j\boldsymbol{H} \tag{45}$$

then Eqs. (42) and (45) imply Eqs. (29)–(31). This concludes the proof.

Note, Eq. (42) leads to the set of LMIs only if δ is a prescribed constant. (δ can be considered as a tuning parameter). Considering δ as a LMI variable, Eq. (42) represents the set of bilinear matrix inequalities (BMI).

Theorem 1 (*enhanced relaxed design conditions*). *The equilibrium of the fuzzy system Eqs.* (2) *and* (3), *controlled by the fuzzy controller Eq.* (5), *is global asymptotically stable if for given a positive* $\delta \in \mathbb{R}$ *there exist positive definite symmetric matrices* $\boldsymbol{V}, \boldsymbol{S} \in \mathbb{R}^{n \times n}$, *the matrices* $\boldsymbol{X}_{ij} = \boldsymbol{X}_{ji}^T \in \mathbb{R}^{r \times n}$, *and* $\boldsymbol{Y}_j \in \mathbb{R}^{r \times m}$, $\boldsymbol{H} \in \mathbb{R}^{m \times m}$ *such that*

$$\boldsymbol{S} = \boldsymbol{S}^T > 0, \quad \boldsymbol{V} = \boldsymbol{V}^T > 0, \quad \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1s} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{s1} & \boldsymbol{X}_{s2} & \cdots & \boldsymbol{X}_{ss} \end{bmatrix} > 0, \tag{46}$$

$$\begin{bmatrix} \boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_i^T + \boldsymbol{B}_i\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_i^T + \boldsymbol{X}_{ii} & * \\ \boldsymbol{V} - \boldsymbol{S} + \delta\boldsymbol{A}_i\boldsymbol{S} + \delta\boldsymbol{B}_i\boldsymbol{Y}_i\boldsymbol{C} & -2\delta\boldsymbol{S} \end{bmatrix} < 0, \tag{47}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{ij} & * \\ \boldsymbol{V} - \boldsymbol{S} + \delta\dfrac{\boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{A}_j\boldsymbol{S}}{2} + \delta\dfrac{\boldsymbol{B}_i\boldsymbol{Y}_j + \boldsymbol{B}_j\boldsymbol{Y}_i}{2}\boldsymbol{C} & -2\delta\boldsymbol{S} \end{bmatrix} < 0, \tag{48}$$

$$\boldsymbol{CS} = \boldsymbol{HC}, \tag{49}$$

for $i = 1, 2, \ldots, s$, *as well as* $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$ *and*

$$\boldsymbol{\Phi}_{ij} = \frac{\boldsymbol{A}_i\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_i^T}{2} + \frac{\boldsymbol{A}_j\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_j^T}{2} + \frac{\boldsymbol{B}_i\boldsymbol{Y}_j\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_j^T\boldsymbol{B}_i^T}{2} + \frac{\boldsymbol{B}_j\boldsymbol{Y}_i\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}_i^T\boldsymbol{B}_j^T}{2} + \frac{\boldsymbol{X}_{ij} + \boldsymbol{X}_{ji}}{2}. \tag{50}$$

When the above conditions hold, the control law gain matrices are given as

$$\boldsymbol{K}_i = \boldsymbol{Y}_i\boldsymbol{H}^{-1}. \tag{51}$$

Proof. Introducing the positive real term

$$v_v(\boldsymbol{\theta}(t)) = \boldsymbol{q}^T(t)\boldsymbol{Z}(\boldsymbol{\theta}(t))\boldsymbol{q}(t) > 0, \tag{52}$$

$$\boldsymbol{Z}(\boldsymbol{\theta}(t)) = \boldsymbol{Z}^T(\boldsymbol{\theta}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\boldsymbol{Z}_{ij} > 0, \tag{53}$$

where $\boldsymbol{Z}_{ij} = \boldsymbol{Z}_{ji}^T \in \mathbb{R}^{n\times n}$, $i, j = 1, 2, \ldots, s$ is the set of associated matrices and using the state coordinate transform Eq. (36), then Eq. (53) can be rewritten as

$$v_v(\boldsymbol{p}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\boldsymbol{X}_{ij}\boldsymbol{p}(t) > 0, \quad \boldsymbol{X}_{ij} = \boldsymbol{S}^{-1}\boldsymbol{Z}_{ij}\boldsymbol{S}^{-1} = \boldsymbol{X}_{ji}^T, \tag{54}$$

where

$$\boldsymbol{Z}(\boldsymbol{\theta}(t)) = [\,\mathrm{h}_1(\boldsymbol{\theta}(t)\boldsymbol{p}(t) \quad \mathrm{h}_2(\boldsymbol{\theta}(t)\boldsymbol{p}(t) \quad \cdots \quad \mathrm{h}_s(\boldsymbol{\theta}(t)\boldsymbol{p}(t)\,]\begin{bmatrix}\boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1s}\\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{X}_{s1} & \boldsymbol{X}_{s2} & \cdots & \boldsymbol{X}_{ss}\end{bmatrix}\begin{bmatrix}\mathrm{h}_1(\theta(t)p(t)\\ \mathrm{h}_2(\theta(t)p(t)\\ \vdots\\ \mathrm{h}_s(\theta(t)p(t)\end{bmatrix} \tag{55}$$

is symmetric, an positive definite if Eq. (46) is satisfied. Then, in the sense of the Krasovskii theorem (see, for example, Ref. [22]), it can be set up in Eq. (39)

$$\begin{aligned}\dot{v}(\boldsymbol{p}(i)) &= \dot{\boldsymbol{p}}^T(t)\boldsymbol{V}\boldsymbol{p}(t) + \boldsymbol{p}^T(t)\boldsymbol{V}\dot{\boldsymbol{p}}(t)\\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\left(\boldsymbol{p}^T(t) + \delta\dot{\boldsymbol{p}}^T(t)\right)\left(\boldsymbol{A}_{cij}\boldsymbol{S}\boldsymbol{p}(t) - \boldsymbol{S}\dot{\boldsymbol{p}}(t)\right)\\ &+ \sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))(\boldsymbol{A}_{cij}\boldsymbol{S}\boldsymbol{p}(t) - \boldsymbol{S}\dot{\boldsymbol{p}}(t))^T(\boldsymbol{p}(t) + \delta\dot{\boldsymbol{p}}(t))\\ &< -\sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}(t))\mathrm{h}_j(\boldsymbol{\theta}(t))\boldsymbol{p}^T(t)\boldsymbol{X}_{ij}\boldsymbol{p}(t)\\ &< 0,\end{aligned} \tag{56}$$

which in the consequence, modifies Eq. (42) as follows

$$S_{cij} = \begin{bmatrix} (A_i + B_i K_j C)S + S(A_i + B_i K_j C)^T + X_{ij} & * \\ V - S + \delta(A_i + B_i K_j C)S & -2\delta S \end{bmatrix} < 0. \tag{57}$$

Following the same way as in the proof of Proposition 2, then Eqs. (47) and (48) can be derived from Eq. (57), while Eq. (55) implies Eq. (46). This concludes the proof.

This principle naturally exploits the affine TS model properties. Introducing the slack matrix variable S into the LMIs, the system matrices are decoupled from the equivalent Lyapunov matrix V. Note, to respect the conditions $X_{1j} = X_{ji}^T$, the set of inequalities Eqs. (47) and (48) have to be constructed. In the opposite case, constructing a set on s^2 LMIs, the constraint conditions have to be set as $X_{1j} = X_{ij}^T > 0$, that is, the weighting matrices have to be symmetric positive definite.

Corollary 1 *Prescribing $S = V$ and using the Schur complement property, then Eq.* (57) *implies*

$$A_{cij}S + SA_{cij}^T + X_{ij} + 0.5\delta SA_{cij}^T \delta^{-1} S^{-1} \delta A_{cij} S < 0 \tag{58}$$

and for $\delta = 0$ evidently, it has to be

$$A_{cij}S + SA_{cij}^T + X_{ij} < 0. \tag{59}$$

Evidently, then Eqs. (47) and (48) imply

$$S(A_i + B_i K_i C)^T + (A_i + B_i K_i C)S + X_{ii} < 0, \tag{60}$$

$$\frac{(A_i + B_i K_j C)S}{2} + \frac{(A_j + B_j K_i C)S}{2} + \frac{S(A_i + B_i K_j C)^T}{2} + \frac{S(A_j + B_j K_i C)^T}{2} + \frac{X_{ij} + X_{ji}}{2} < 0. \tag{61}$$

Considering $S = W$ and comparing with Eqs. (23) and (24), then Eqs. (60) and (61) are the extended set of inequalities Eqs. (23) and (24). The result is that the equilibrium of the fuzzy system Eqs. (2) and (3), controlled by the fuzzy controller Eq. (5), is global asymptotically stable if there exist a positive definite symmetric matrices $S \in \mathbb{R}^{n \times n}$, the matrices $X_{1j} = X_{ji}^T \in \mathbb{R}^{r \times n}$, and $Y_j \in \mathbb{R}^{r \times n}$, $H \in \mathbb{R}^{m \times m}$ such that

$$S = S^T > 0, \quad \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1s} \\ X_{21} & X_{22} & \cdots & X_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ X_{s1} & X_{s2} & \cdots & X_{ss} \end{bmatrix} > 0, \tag{62}$$

$$A_i S + SA_i^T + B_i Y_i C + C^T Y_i^T B_i^T + X_{ii} < 0, \tag{63}$$

$$\frac{A_i S + SA_i^T}{2} + \frac{A_j S + SA_j^T}{2} + \frac{B_i Y_j C + C^T Y_j^T B_i^T}{2} + \frac{B_j Y_i C + C^T Y_i^T B_j^T}{2} \frac{X_{ij} + X_{ji}}{2} < 0, \tag{64}$$

$$CS = HC, \tag{65}$$

for $i = 1, 2, \ldots, s$, as well as $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$, and $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$. Subsequently, if this set of LMIs is satisfied, the set of control law gain matrices is given as

$$\boldsymbol{K}_i = \boldsymbol{Y}_i \boldsymbol{H}^{-1}. \tag{66}$$

These LMIs form relaxed design conditions.

Note the derived results are linked to some existing finding when the design problem involves additive performance requirements and the relaxed quadratic stability conditions of fuzzy control systems (see, e.g., Refs. [11, 19]) are equivalently steered.

4. Forced mode in static output control

In practice, the plant with $r = m$ (square plants) is often encountered, since in this case, it is possible to associate with each output signal as a reference signal, which is expected to influence this wanted output. Such mode, reflecting nonzero set working points, is called the forced regime.

Definition 2 *A forced regime for the TS fuzzy system Eqs.* (2) *and* (3) *with the TS fuzzy static output controller Eq.* (5) *is foisted by the control policy*

$$\boldsymbol{u}(t) = \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\boldsymbol{K}_j \boldsymbol{y}(t) + \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{W}_{ij}\boldsymbol{w}(t), \tag{67}$$

where $r = m$, $\boldsymbol{w}(i) \in \mathbb{R}^m$ is desired output signal vector, and $\boldsymbol{W}_{ij} \in \mathbb{R}^{m \times m}$, $i, j = 1, 2, \ldots s$ is the set of the signal gain matrices.

Lemma 1. *The static decoupling challenge is solvable if* $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ *is stabilizable and*

$$rank\begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m. \tag{68}$$

Proof. If $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is stabilizable, it is possible to find $\boldsymbol{K}_j$ such that matrices $\boldsymbol{A}_{cij} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}$ are Hurwitz. Assuming that for such $\boldsymbol{K}_j$, it yields

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{K}_j\boldsymbol{C} & \boldsymbol{I}_m \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C} & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}, \tag{69}$$

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C} & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ -\boldsymbol{C}\left(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1} & \boldsymbol{I}_m \end{bmatrix}\begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C} & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}, \tag{70}$$

respectively, then

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C} & \boldsymbol{B}_i \\ \boldsymbol{0} & -\boldsymbol{C}\left(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i \end{bmatrix} = n + m, \tag{71}$$

since $\operatorname{rank}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}) = n$, and $\operatorname{rank}\boldsymbol{B}_i = m$.

Thus, evidently, it has to be satisfied

$$\operatorname{rank}\left(\boldsymbol{C}\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\right)=m\,. \tag{72}$$

This concludes the proof.

Theorem 2. *To reach a forced regime for the TS fuzzy system Eqs.* (2) *and* (3) *with the TS fuzzy control policy Eq.* (67), *the signal gain matrices have to take the forms*

$$\boldsymbol{W}_{ij}=\left(\boldsymbol{C}\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\right)^{-1}, \tag{73}$$

where $\boldsymbol{W}_{ij}\in\mathbb{R}^{m\times m}$, $i, j = 1, 2, \ldots s$.

Proof. In a steady state, which corresponds to $\dot{\boldsymbol{q}}(t)=\boldsymbol{0}$, the equality $\boldsymbol{y}_o = \boldsymbol{w}_o$ must hold, where $\boldsymbol{q}_o\in\mathbb{R}^n$, $\boldsymbol{\theta}_o\in\mathbb{R}^q$, $\boldsymbol{y}_o$, $\boldsymbol{w}_o\in\mathbb{R}^m$ are the vectors of steady state values of $\boldsymbol{q}(t)$, $\boldsymbol{\theta}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{w}(t)$, respectively.

Substituting Eq. (67) in Eq. (2) yields the expression

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}_o)\mathrm{h}_j(\boldsymbol{\theta}_o)\left(\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)\boldsymbol{q}_o+\boldsymbol{B}_i\boldsymbol{W}_{ij}\boldsymbol{w}_o\right)=\boldsymbol{0}, \tag{74}$$

$$-\sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}_o)\mathrm{h}_j(\boldsymbol{\theta}_o)\boldsymbol{q}_o=-\boldsymbol{q}_o=\sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}_o)\mathrm{h}_j(\boldsymbol{\theta}_o)\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\boldsymbol{W}_{ij}\boldsymbol{w}_o\,, \tag{75}$$

respectively, and it can be set

$$\boldsymbol{y}_o=\boldsymbol{C}\boldsymbol{q}_o=-\sum_{i=1}^{s}\sum_{j=1}^{s}\mathrm{h}_i(\boldsymbol{\theta}_o)\mathrm{h}_j(\boldsymbol{\theta}_o)\boldsymbol{C}\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\boldsymbol{W}_{ij}\boldsymbol{w}_o=\boldsymbol{I}_m\boldsymbol{w}_o\,. \tag{76}$$

Thus, Eq. (76) gives the solution

$$\boldsymbol{W}_{ij}^{-1}=-\boldsymbol{C}\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\,, \tag{77}$$

which implies Eq. (68). Hence, declaredly,

$$\operatorname{rank}\boldsymbol{W}_j=\operatorname{rank}\left(\boldsymbol{C}\left(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{K}_j\boldsymbol{C}\right)^{-1}\boldsymbol{B}_i\right)=m\,. \tag{78}$$

This concludes the proof.

The forced regime is basically designed for constant references and is very closely related to shift of origin. If the command value $\boldsymbol{w}(t)$ is changed "slowly enough," the above scheme can do a reasonable job of tracking, that is, making $\boldsymbol{y}(t)$ follow $\boldsymbol{w}(t)$ [23].

5. Bi-proper dynamic output controller

The full order biproper dynamic output controller is defined by the equation

$$\dot{\boldsymbol{p}}(t) = \sum_{j=1}^{s} \mathrm{h}_j(\boldsymbol{\theta}(t)) \left(\boldsymbol{J}_j \boldsymbol{p}(t) + \boldsymbol{L}_j \boldsymbol{y}(t) \right), \tag{79}$$

$$\boldsymbol{u}(t) = \sum_{j=1}^{s} \mathrm{h}_j(\boldsymbol{\theta}(t)) \left(\boldsymbol{M}_j \boldsymbol{p}(t) + \boldsymbol{N}_j \boldsymbol{y}(t) \right), \tag{80}$$

where $\boldsymbol{p}(t) \in \mathbb{R}^h$ is the vector of the controller state variables and the parameter matrix

$$\boldsymbol{K}_j^{\circ} = \begin{bmatrix} \boldsymbol{J}_j & \boldsymbol{L}_j \\ \boldsymbol{M}_j & \boldsymbol{N}_j \end{bmatrix}, \tag{81}$$

$\boldsymbol{K}_j^{\circ} \in \mathbb{R}^{(n+r)\times(h+m)}$, is considered in this block matrix structure with respect to the matrices $\boldsymbol{J}_j \in \mathbb{R}^{h \times h}$, $\boldsymbol{L}_j \in \mathbb{R}^{h \times m}$, $\boldsymbol{M}_j \in \mathbb{R}^{r \times h}$, and $\boldsymbol{N}_j \in \mathbb{R}^{r \times m}$. For simplicity, the full order $p = n$ controller is considered in the following.

To analyze the stability of the closed-loop system structure with the dynamic output controller, the closed-loop system description implies the following form

$$\dot{\boldsymbol{q}}^{\circ}(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \boldsymbol{A}_{cij}^{\circ} \boldsymbol{q}^{\circ}(t), \tag{82}$$

$$\boldsymbol{y}^{\circ}(t) = \boldsymbol{I}^{\circ} \boldsymbol{C}^{\circ} \boldsymbol{q}^{\circ}(t), \tag{83}$$

where

$$\boldsymbol{q}^{\circ T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{p}^T(t) \end{bmatrix}, \tag{84}$$

$$\boldsymbol{A}_{cij}^{\circ} = \begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{N}_j \boldsymbol{C} & \boldsymbol{B}_i \boldsymbol{M}_j \\ \boldsymbol{L}_j \boldsymbol{C} & \boldsymbol{N}_j \end{bmatrix}, \quad \boldsymbol{I}^{\circ} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}, \quad \boldsymbol{C}^{\circ} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \tag{85}$$

and $\boldsymbol{A}_{cij}^{\circ} \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{I}^{\circ} \in \mathbb{R}^{m \times (n+m)}$, $\boldsymbol{C}^{\circ} \in \mathbb{R}^{(n+m) \times 2n}$.

Introducing the notations

$$\boldsymbol{A}_i^{\circ} = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{B}_i^{\circ} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_i \\ \boldsymbol{I}_n & \boldsymbol{0} \end{bmatrix}, \tag{86}$$

where $\boldsymbol{A}_i^{\circ} \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{B}_i^{\circ} \in \mathbb{R}^{2n\times(n+r)}$, the closed-loop system matrices take the equivalent forms

$$A_{cij}^{\circ} = A_i^{\circ} + B_i^{\circ} K_j^{\circ} C^{\circ}. \tag{87}$$

In the sequel, it is supposed that $(A_i^{\circ}, B_i^{\circ})$ is stabilizable, $(A_i^{\circ}, C_i^{\circ})$ is detectable [24].

Note this kind of controllers can be preferred in fault tolerant control (FTC) structures with virtual actuators [25].

Theorem 3 (*relaxed design conditions*). *The equilibrium of the fuzzy system Eqs.* (2) *and* (3) *controlled by the fuzzy dynamic output controller Eqs.* (79) *and* (80) *is global asymptotically stable if there exist a positive definite symmetric matrix* $S^{\circ} \in \mathbb{R}^{2n \times 2n}$, *symmetric matrices* $X_{ij}^{\circ} = X_{ji}^{\circ} \in \mathbb{R}^{2n \times 2n}$, *a regular matrix* $H^{\circ} \in \mathbb{R}^{(n+m) \times (n+m)}$, *and matrices* $Y_j^{\circ} \in \mathbb{R}^{(n+r) \times (n+m)}$ *such that*

$$S^{\circ} = S^{\circ T} > 0, \quad \begin{bmatrix} X_{11}^{\circ} & X_{12}^{\circ} & \cdots & X_{1s}^{\circ} \\ X_{21}^{\circ} & X_{22}^{\circ} & \cdots & X_{2s}^{\circ} \\ \vdots & \vdots & \ddots & \vdots \\ X_{s1}^{\circ} & X_{s2}^{\circ} & \cdots & X_{ss}^{\circ} \end{bmatrix} > 0, \tag{88}$$

$$A_i^{\circ} S^{\circ} + S^{\circ} A_i^{\circ T} + B_i^{\circ} Y_i^{\circ} C^{\circ} + C^{\circ T} Y_i^{\circ T} B_i^{\circ T} + X_{ii}^{\circ} < 0, \tag{89}$$

$$\frac{A_i^{\circ} + A_j^{\circ}}{2} S^{\circ} + S^{\circ} \frac{A_i^{\circ T} + A_j^{\circ T}}{2} + \frac{B_i^{\circ} Y_j^{\circ} + B_j^{\circ} Y_i^{\circ}}{2} C^{\circ} + C^{\circ T} \frac{Y_j^{\circ T} B_i^{\circ T} + Y_i^{\circ T} B_j^{\circ T}}{2} + \frac{X_{ij}^{\circ} + X_{ji}^{\circ}}{2} < 0, \tag{90}$$

$$C^{\circ} S^{\circ} = H^{\circ} C^{\circ}, \tag{91}$$

for all $i \in \langle 1, 2, \ldots s \rangle$, $i < j \leq s$, $i, j \in \langle 1, 2, \ldots s \rangle$, *respectively, and* $h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \neq 0$.

When the above conditions hold, the set of control law gain matrices are given as

$$K_j^{\circ} = Y_j^{\circ} (H^{\circ})^{-1}, \quad j = 1, 2, \ldots, s \tag{92}$$

Proof. Defining the Lyapunov function as follows

$$v(q^{\circ}(t)) = q^{\circ T}(t) P^{\circ} q^{\circ}(t) > 0, \tag{93}$$

where $P^{\circ} \in \mathbb{R}^{2n \times 2n}$ is a positive definite matrix, then the time derivative of $v(q(t))$ along a closed-loop system trajectory is

$$\dot{v}(q^{\circ}(t)) = \dot{q}^{\circ T}(t) P^{\circ} q^{\circ}(t) + q^{\circ T}(t) P^{\circ} \dot{q}^{\circ}(t) < 0. \tag{94}$$

Substituting Eq. (87), then Eq. (94) implies

$$\dot{v}(q^{\circ}(t)) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) q^{\circ T}(t) P_{cij}^{\circ} q^{\circ}(t) < 0, \tag{95}$$

$$P_{cij}^{\circ} = P^{\circ} A_{cij}^{\circ} + A_{cij}^{\circ T} P^{\circ}. \tag{96}$$

Since P° is positive definite, the state coordinate transform can now be defined as

$$\boldsymbol{q}^{\circ}(t) = \boldsymbol{S}^{\circ}\boldsymbol{p}^{\circ}(t), \quad \boldsymbol{S}^{\circ} = (\boldsymbol{P}^{\circ})^{-1}, \tag{97}$$

and subsequently Eqs. (95) and (96) can be rewritten as

$$\dot{v}(\boldsymbol{p}^{\circ}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^{\circ T}(t)\boldsymbol{S}^{\diamond}_{cij}\boldsymbol{p}^{\circ}(t) < 0, \tag{98}$$

$$\boldsymbol{S}^{\diamond}_{cij} = \boldsymbol{A}^{\circ}_{cij}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}^{\circ T}_{cij}. \tag{99}$$

Introducing, analogously to Eqs. (54) and (55), the positive term

$$v_v(\boldsymbol{p}^{\circ}(t)) = \boldsymbol{p}^{\circ T}(t)\boldsymbol{Z}^{\circ}(\boldsymbol{\theta}(t))\boldsymbol{p}^{\circ}(t) > 0, \tag{100}$$

defined by the set of matrices $\left\{\boldsymbol{X}^{\circ}_{ij} = \boldsymbol{X}^{\circ T}_{ji} \in \mathbb{R}^{n\times n}, \;\; i,j = 1,2,\ldots,s\right\}$ in the structure Eq. (88) such that

$$\boldsymbol{Z}^{\circ}(\boldsymbol{\theta}(t)) = \boldsymbol{Z}^{\circ T}(\boldsymbol{\theta}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{X}^{\circ}_{ij} > 0, \tag{101}$$

then, in the sense of Krasovskii theorem, it can be set up

$$\dot{v}(\boldsymbol{p}^{\circ}(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^{\circ T}(t)\boldsymbol{S}^{\circ}_{cij}\boldsymbol{p}^{\circ}(t) < 0, \tag{102}$$

where

$$\boldsymbol{S}^{\circ}_{cij} = \boldsymbol{A}^{\circ}_{cij}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}^{\circ T}_{cij} + \boldsymbol{X}^{\circ}_{ij}. \tag{103}$$

Therefore, Eq. (102) can be factorized as follows

$$\dot{v}(\boldsymbol{p}^{\circ}(t)) = \sum_{i=1}^{s} h_i^2(\boldsymbol{\theta}(t))\boldsymbol{p}^{\circ T}(t)\boldsymbol{S}^{\circ}_{cii}\boldsymbol{p}^{\circ}(t) + 2\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\boldsymbol{p}^{\circ T}(t)\frac{\boldsymbol{S}^{\circ}_{cij} + \boldsymbol{S}^{\circ}_{cji}}{2}\boldsymbol{p}^{\circ}(t) < 0, \tag{104}$$

which, using Eq. (87), leads to the following sets of inequalities

$$\boldsymbol{A}^{\circ}_i\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}^{\circ T}_i + \boldsymbol{B}^{\circ}_i\boldsymbol{K}^{\circ}_j\boldsymbol{C}^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{C}^{\circ T}\boldsymbol{K}^{\circ T}_j\boldsymbol{B}^{\circ T}_i + \boldsymbol{X}^{\circ}_{ij} < 0, \tag{105}$$

$$\begin{aligned}\frac{\left(\boldsymbol{A}^{\circ}_i + \boldsymbol{B}^{\circ}_i\boldsymbol{K}^{\circ}_j\boldsymbol{C}^{\circ}\right)\boldsymbol{S}^{\circ}}{2} + \frac{\left(\boldsymbol{A}^{\circ}_j + \boldsymbol{B}^{\circ}_j\boldsymbol{K}^{\circ}_i\boldsymbol{C}^{\circ}\right)\boldsymbol{S}^{\circ}}{2} + \frac{\boldsymbol{S}^{\circ}\left(\boldsymbol{A}^{\circ}_i + \boldsymbol{B}^{\circ}_i\boldsymbol{K}^{\circ}_j\boldsymbol{C}^{\circ}\right)^T}{2}\\ + \frac{\boldsymbol{S}^{\circ}\left(\boldsymbol{A}^{\circ}_j + \boldsymbol{B}^{\circ}_j\boldsymbol{K}^{\circ}_i\boldsymbol{C}^{\circ}\right)^T}{2} + \frac{\boldsymbol{X}^{\circ}_{ij} + \boldsymbol{X}^{\circ}_{ji}}{2} < 0,\end{aligned} \tag{106}$$

for $i = 1, 2, \ldots, s$, as well as $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$, and $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$.

Analyzing the product $\boldsymbol{B}_i^{\circ}\boldsymbol{K}_j^{\circ}\boldsymbol{C}^{\circ}\boldsymbol{S}^{\circ}$, it can set

$$\boldsymbol{B}_i^{\circ}\boldsymbol{K}_j^{\circ}\boldsymbol{C}^{\circ}\boldsymbol{S}^{\circ} = \boldsymbol{B}_i^{\circ}\boldsymbol{K}_j^{\circ}\boldsymbol{H}^{\circ}(\boldsymbol{H}^{\circ})^{-1}\boldsymbol{C}^{\circ}\boldsymbol{S}^{\circ} = \boldsymbol{B}_i^{\circ}\boldsymbol{Y}_j^{\circ}\boldsymbol{C}^{\circ}, \tag{107}$$

where

$$\boldsymbol{K}_j^{\circ}\boldsymbol{H}^{\circ} = \boldsymbol{Y}_j^{\circ}, \quad (\boldsymbol{H}^{\circ})^{-1}\boldsymbol{C}^{\circ} = \boldsymbol{C}^{\circ}(\boldsymbol{S}^{\circ})^{-1} \tag{108}$$

and $\boldsymbol{H}^{\circ} \in \mathbb{R}^{(m+n)\times(m+n)}$ is a regular square matrix. Thus, with Eq. (108), then Eqs. (105) and (106) implies Eqs. (89) and (90) and Eq. (108) gives Eq. (91). This concludes the proof.

This theorem provides the sufficient condition under LMIs and LME formulations for the synthesis of the dynamic output controller reflecting the membership function properties.

For the same reasons as in Theorem 1, the following theorem is proven.

Theorem 4 (*enhanced relaxed design conditions*). *The equilibrium of the fuzzy system Eqs.* (2) *and* (3) *controlled by the fuzzy dynamic output controller Eqs.* (79) *and* (80) *is global asymptotically stable if for given a positive* $\delta \in \mathbb{R}$ *there exist positive definite symmetric matrices* $\boldsymbol{V}^{\circ}$, $\boldsymbol{S}^{\circ} \in \mathbb{R}^{n\times n}$, *and matrices* $\boldsymbol{Y}_j^{\circ} \in \mathbb{R}^{r\times n}$, $\boldsymbol{H}^{\circ} \in \mathbb{R}^{m\times m}$ *such that*

$$\boldsymbol{S}^{\circ} = \boldsymbol{S}^{\circ T} > 0, \quad \boldsymbol{V}^{\circ} = \boldsymbol{V}^{\circ T} > 0, \quad \begin{bmatrix} \boldsymbol{X}_{11}^{\circ} & \boldsymbol{X}_{12}^{\circ} & \cdots & \boldsymbol{X}_{1s}^{\circ} \\ \boldsymbol{X}_{21}^{\circ} & \boldsymbol{X}_{22}^{\circ} & \cdots & \boldsymbol{X}_{2s}^{\circ} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{s1}^{\circ} & \boldsymbol{X}_{s2}^{\circ} & \cdots & \boldsymbol{X}_{ss}^{\circ} \end{bmatrix} > 0, \tag{109}$$

$$\begin{bmatrix} \boldsymbol{A}_i^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}_i^{\circ T} + \boldsymbol{B}_i^{\circ}\boldsymbol{Y}_i^{\circ}\boldsymbol{C}^{\circ} + \boldsymbol{C}^{\circ T}\boldsymbol{Y}_i^{\circ T}\boldsymbol{B}_i^{\circ T} & * \\ \boldsymbol{V}^{\circ} - \boldsymbol{S}^{\circ} + \delta\boldsymbol{A}_i^{\circ}\boldsymbol{S}^{\circ} + \delta\boldsymbol{B}_i^{\circ}\boldsymbol{Y}_i^{\circ}\boldsymbol{C}^{\circ} & -2\delta\boldsymbol{S}^{\circ} \end{bmatrix} < 0, \tag{110}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{ij}^{\circ} & * \\ \boldsymbol{V}^{\circ} - \boldsymbol{S}^{\circ} + \delta\dfrac{\boldsymbol{A}_i^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{A}_j^{\circ}\boldsymbol{S}^{\circ}}{2} + \delta\dfrac{\boldsymbol{B}_i^{\circ}\boldsymbol{Y}_j^{\circ} + \boldsymbol{B}_j^{\circ}\boldsymbol{Y}_i^{\circ}}{2}\boldsymbol{C}^{\circ} & -2\delta\boldsymbol{S}^{\circ} \end{bmatrix} < 0, \tag{111}$$

$$\boldsymbol{C}^{\circ}\boldsymbol{S}^{\circ} = \boldsymbol{H}^{\circ}\boldsymbol{C}^{\circ}, \tag{112}$$

for $i = 1, 2, \ldots, s$, *as well as* $i = 1, 2, \ldots, s-1$, $j = 1+1, i+2, \ldots, s$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *and*

$$\boldsymbol{\Phi}_{ij}^{\circ} = \frac{\boldsymbol{A}_i^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}_i^{\circ T}}{2} + \frac{\boldsymbol{A}_j^{\circ}\boldsymbol{S}^{\circ} + \boldsymbol{S}^{\circ}\boldsymbol{A}_j^{\circ T}}{2} + \frac{\boldsymbol{B}_i^{\circ}\boldsymbol{Y}_j^{\circ}\boldsymbol{C}^{\circ} + \boldsymbol{C}^{\circ T}\boldsymbol{Y}_j^{\circ T}\boldsymbol{B}_i^{\circ T}}{2} + \frac{\boldsymbol{B}_j^{\circ}\boldsymbol{Y}_i^{\circ}\boldsymbol{C}^{\circ} + \boldsymbol{C}^{\circ T}\boldsymbol{Y}_i^{\circ T}\boldsymbol{B}_j^{\circ T}}{2}. \tag{113}$$

When the above conditions hold, the control law gain matrices are given as

$$\boldsymbol{K}_i^{\circ} = \boldsymbol{Y}_i^{\circ}(\boldsymbol{H}^{\circ})^{-1}. \tag{114}$$

Proof. Since Eq. (82), Eq. (87) takes formally the same structure as Eqs. (6) and (7), following the same way as in the proof of Theorem 1, the conditions given in Theorem 4 can be obtained. From this reason, the proof is omitted. Compare, for example, Ref. [17].

Following the presented results, it is evident that the standard as well as the enhanced conditions for biproper dynamic output controller design can be derived from Theorem 3 and Theorem 4 in a simple way.

6. Illustrative example

The nonlinear dynamics of the system is represented by TS model with s = 3 and the system model parameters [20]

$$\boldsymbol{A}_1 = \begin{bmatrix} -1.0522 & -1.8666 & 0.5102 \\ -0.4380 & -5.4335 & 0.9205 \\ -0.5522 & 0.1334 & -0.4898 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} -1.0565 & -1.8661 & 0.5116 \\ -0.4380 & -5.4359 & 0.9214 \\ -0.5565 & 0.1339 & -0.4884 \end{bmatrix},$$
$$\boldsymbol{A}_3 = \begin{bmatrix} -1.0602 & -1.8657 & 0.5133 \\ -0.4381 & -5.4353 & 0.9216 \\ -0.5602 & 0.1343 & -0.4867 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -0.1765 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.1176 & 0.4721 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

To the state vector $\boldsymbol{q}(t)$ are associated the premise variables and the membership functions as follows

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix}, \theta_i(t) = \begin{cases} \theta_1(t) \text{ if } q_1(t) \text{ is about } 2.5, & h_1(\theta_2(t)) = 1 - \dfrac{1}{2.5}|\theta_2(t) - 2.5| \\ \theta_2(t) \text{ if } q_1(t) \text{ is about } 0, & h_2(\theta_1(t)) = 1 - \dfrac{1}{2.5}|\theta_1(t)| \\ \theta_3(t) \text{ if } q_1(t) \text{ is about } -2.5, & h_3(\theta_3(t)) = 1 - \dfrac{1}{2.5}|\theta_3(t) + 2.5| \end{cases}$$

while the generalized premise variable is $\theta(t) = q_1(t)$.

Thus, solving Eqs. (46)–(49) for prescribed $\delta = 1.2$ with respect to the LMI matrix variables $\boldsymbol{S}$, $\boldsymbol{V}$ $\boldsymbol{H}$, $\boldsymbol{Y}_i$, $j = 1, 2, 3$, and $\boldsymbol{X}_{ij}$, $i, j = 1, 2, 3$ using Self–Dual–Minimization (SeDuMi) package for Matlab [26], then the feedback gain matrix design problem was feasible with the results

$$\boldsymbol{S} = \begin{bmatrix} 0.3899 & -0.0102 & -0.0000 \\ -0.0102 & 0.1596 & -0.0000 \\ -0.0000 & -0.0000 & 0.4099 \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} 0.9280 & 0.1235 & -0.1525 \\ 0.1235 & 1.1533 & -0.3979 \\ -0.1525 & -0.3979 & 0.7574 \end{bmatrix},$$
$$\boldsymbol{H} = \begin{bmatrix} 0.3899 & -0.0102 \\ -0.0102 & 0.1596 \end{bmatrix},$$
$$\boldsymbol{X} = \begin{bmatrix} 0.4567 & 0.0983 & -0.0517 & 0.0694 & 0.0463 & -0.0174 & 0.0694 & 0.0463 & -0.0174 \\ 0.0983 & 0.7153 & -0.1118 & 0.0463 & 0.1906 & -0.0441 & 0.0463 & 0.1905 & -0.0440 \\ -0.0517 & -0.1118 & 0.1883 & -0.0175 & -0.0442 & 0.0143 & -0.0175 & -0.0442 & 0.0142 \\ 0.0694 & 0.0463 & -0.0175 & 0.4573 & 0.0981 & -0.0515 & 0.0695 & 0.0463 & -0.0174 \\ 0.0463 & 0.1906 & -0.0442 & 0.0981 & 0.7154 & -0.1115 & 0.0463 & 0.1905 & -0.0440 \\ -0.0174 & -0.0441 & 0.0143 & -0.0515 & -0.1115 & 0.1876 & -0.0175 & -0.0441 & 0.0142 \\ 0.0694 & 0.0463 & -0.0175 & 0.0695 & 0.0463 & -0.0175 & 0.4578 & 0.0978 & -0.0514 \\ 0.0463 & 0.1905 & -0.0442 & 0.0463 & 0.1905 & -0.0441 & 0.0978 & 0.7152 & -0.1111 \\ -0.0174 & -0.0440 & 0.0142 & -0.0174 & -0.0440 & 0.0142 & -0.0514 & -0.1111 & 0.1868 \end{bmatrix},$$
$$\boldsymbol{Y}_1 = \begin{bmatrix} 0.5607 & -0.4590 \\ 0.1544 & -0.1191 \end{bmatrix}, \boldsymbol{Y}_2 = \begin{bmatrix} 0.5558 & -0.4577 \\ 0.1579 & -0.1207 \end{bmatrix}, \boldsymbol{Y}_3 = \begin{bmatrix} 0.5518 & -0.4566 \\ 0.1606 & -0.1222 \end{bmatrix},$$

Substituting the above parameters into Eq. (51) to solve the controller parameters, the following gain matrices are obtained

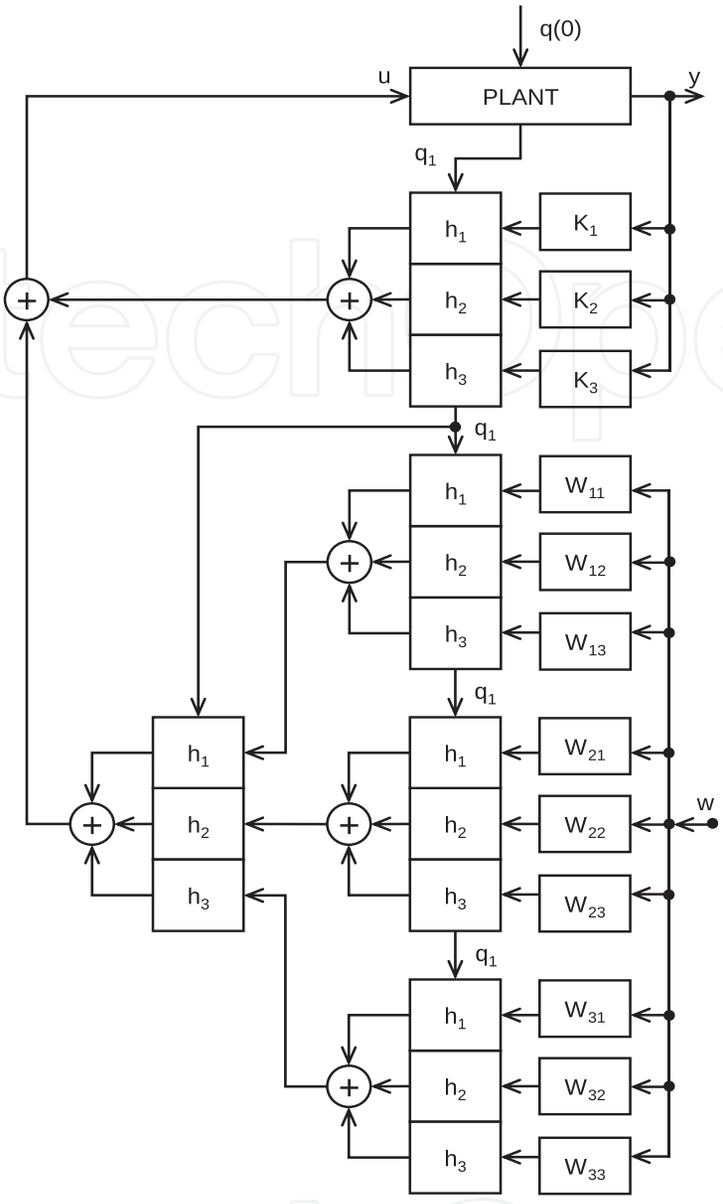


Figure 1. TS fuzzy static output control structure in a forced mode.

$$\boldsymbol{K}_1 = \begin{bmatrix} 1.3653 & -2.7895 \\ 0.3772 & -0.7224 \end{bmatrix}, \boldsymbol{K}_2 = \begin{bmatrix} 1.3530 & -2.7823 \\ 0.3860 & -0.7318 \end{bmatrix}, \boldsymbol{K}_3 = \begin{bmatrix} 1.3428 & -2.7761 \\ 0.3925 & -0.7406 \end{bmatrix},$$

$$\boldsymbol{A}_{c22} = \begin{bmatrix} -1.2953 & -1.3750 & 0.5116 \\ -0.4380 & -5.4359 & 0.9214 \\ -0.2152 & -0.5388 & -0.4884 \end{bmatrix}, \boldsymbol{A}_{c31} = \begin{bmatrix} -1.3012 & -1.3734 & 0.5133 \\ -0.4381 & -5.4353 & 0.9216 \\ -0.2216 & -0.5348 & -0.4867 \end{bmatrix},$$

For simplicity, other closed-loop matrices of subsystem dynamics are not listed here.

Since the diagonal elements of $\boldsymbol{A}_{cij}$, i, j = 1, 2, 3, are dominant, in terms of Gerschgorin theorem [27, 28], all eigenvalues of $\boldsymbol{A}_{cij}$ are real, resulting in the aperiodic dynamics, that is,

$$\tilde{n}(\mathbf{A}_{c11}) = \{-0.6751, \quad -1.0816, \quad -5.4598\}, \quad \tilde{n}(\mathbf{A}_{c21}) = \{-0.6756, \quad -1.0842, \quad -5.4620\},$$
$$\tilde{n}(\mathbf{A}_{c31}) = \{-0.6757, \quad -1.0861, \quad -5.4613\}, \quad \tilde{n}(\mathbf{A}_{c12}) = \{-0.6745, \quad -1.0805, \quad -5.4593\},$$
$$\tilde{n}(\mathbf{A}_{c22}) = \{-0.6750, \quad -1.0831, \quad -5.4615\}, \quad \tilde{n}(\mathbf{A}_{c32}) = \{-0.6751, \quad -1.0851, \quad -5.4609\},$$
$$\tilde{n}(\mathbf{A}_{c13}) = \{-0.6742, \quad -1.0795, \quad -5.4588\}, \quad \tilde{n}(\mathbf{A}_{c23}) = \{-0.6748, \quad -1.0820, \quad -5.4610\},$$
$$\tilde{n}(\mathbf{A}_{c33}) = \{-0.6748, \quad -1.0840, \quad -5.4604\}.$$

Figure 1 gives the associated TS fuzzy static output control structure in a forced mode.

For Eqs. (88)–(91), it can find the following feasible solutions by using the given design procedure

$$S^{\circ} = \begin{bmatrix} 0.6194 & -0.0614 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0614 & 0.1305 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.8724 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.7066 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.7066 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.7066 \end{bmatrix},$$

$$H^{\circ} = \begin{bmatrix} 0.7066 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7066 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.7066 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.6808 & -0.0614 \\ 0.0000 & 0.0000 & 0.0000 & 0.4889 & 0.0691 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -0.5668 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.5668 & 0.0000 & -0.0001 & 0.0000 \\ 0.0000 & 0.0000 & -0.5667 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.3612 & -0.2783 \\ 0.0000 & 0.0000 & 0.0000 & 0.7396 & -1.1397 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.5668 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.5668 & 0.0000 & -0.0001 & 0.0000 \\ 0.0000 & 0.0000 & -0.5667 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.3615 & -0.2784 \\ 0.0000 & 0.0000 & 0.0000 & 0.7397 & -1.1397 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} -0.5667 & -0.0001 & 0.0000 & 0.0000 & 0.0000 \\ -0.0001 & -0.5667 & 0.0000 & -0.0001 & 0.0000 \\ 0.0000 & 0.0000 & -0.5668 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.3519 & -0.2859 \\ -0.0001 & 0.0000 & -0.0001 & 0.7486 & -1.1421 \end{bmatrix}$$

and, computing the biproper dynamic output controller parameters, then

$$\boldsymbol{J}_1 = \begin{bmatrix} -0.8022 & 0.0000 & 0.0000 \\ 0.0000 & -0.8021 & 0.0000 \\ 0.0000 & 0.0000 & -0.8021 \end{bmatrix}, \quad \boldsymbol{L}_1 = 10^{-3} \begin{bmatrix} 0.0394 & 0.0048 \\ -0.3143 & 0.3318 \\ 0.0221 & -0.0508 \end{bmatrix},$$

$$\boldsymbol{M}_1 = 10^{-4} \begin{bmatrix} -0.2041 & 0.1504 & -0.0600 \\ -0.5318 & 0.0915 & -0.3275 \end{bmatrix}, \quad \boldsymbol{N}_1 = \begin{bmatrix} 2.0889 & -2.1701 \\ 7.8914 & -9.4765 \end{bmatrix},$$

$$\boldsymbol{J}_2 = \begin{bmatrix} -0.8022 & 0.0001 & 0.0000 \\ 0.0001 & -0.8022 & 0.0000 \\ 0.0000 & 0.0000 & -0.8021 \end{bmatrix}, \quad \boldsymbol{L}_2 = 10^{-3} \begin{bmatrix} -0.2009 & 0.3531 \\ 0.0453 & -0.2017 \\ -0.1903 & 0.1765 \end{bmatrix},$$

$$\boldsymbol{M}_2 = 10^{-4} \begin{bmatrix} -0.2022 & 0.1779 & 0.1796 \\ -0.4575 & 0.0413 & 0.2985 \end{bmatrix}, \quad \boldsymbol{N}_2 = \begin{bmatrix} 2.0897 & -2.1707 \\ 7.8915 & -9.4766 \end{bmatrix},$$

$$\boldsymbol{J}_3 = \begin{bmatrix} -0.8020 & -0.0001 & 0.0000 \\ -0.0001 & -0.8021 & 0.0000 \\ 0.0000 & 0.0000 & -0.8022 \end{bmatrix}, \quad \boldsymbol{L}_3 = 10^{-3} \begin{bmatrix} -0.0641 & 0.0139 \\ -0.1516 & 0.2382 \\ -0.2116 & 0.2630 \end{bmatrix},$$

$$\boldsymbol{M}_3 = 10^{-4} \begin{bmatrix} -0.0135 & 0.0020 & -0.0238 \\ -0.0917 & 0.0218 & -0.1102 \end{bmatrix}, \quad \boldsymbol{N}_3 = \begin{bmatrix} 2.1286 & -2.2445 \\ 7.9148 & -9.4907 \end{bmatrix}.$$

It is evident that all matrices $\boldsymbol{J}_i$, $i = 1.2.3$ are Hurwitz, which rise up a TS fuzzy stable dynamic output controller, and based on the solutions obtained, the TS fuzzy dynamic controller can be designed via the concept of PDC.

Verifying the closed-loop stability, it can compute the eigenvalue spectra as follows

$$\begin{aligned}
\tilde{\mathrm{n}}(\boldsymbol{A}_{c11}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3774, \quad -1.2919 \pm 0.2804i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c21}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3774, \quad -1.2919 \pm 0.2804i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c21}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3774, \quad -1.2919 \pm 0.2804i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c12}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3774, \quad -1.2919 \pm 0.2805i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c22}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3774, \quad -1.2919 \pm 0.2805i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c32}) &= \{-0.8022, \quad -0.8021, \quad -0.8021, \quad -4.3788, \quad -1.2946 \pm 0.2963i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c13}) &= \{-0.8020, \quad -0.8023, \quad -0.8023, \quad -4.3713, \quad -1.2919 \pm 0.2797i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c23}) &= \{-0.8020, \quad -0.8023, \quad -0.8023, \quad -4.3713, \quad -1.2919 \pm 0.2797i\}, \\
\tilde{\mathrm{n}}(\boldsymbol{A}_{c33}) &= \{-0.8020, \quad -0.8023, \quad -0.8023, \quad -4.3728, \quad -1.2945 \pm 0.2958i\}.
\end{aligned}$$

7. Concluding remarks

New approach for static and dynamic output feedback control design, taking into account the affine properties of the TS fuzzy model structure, is presented in the chapter. Applying the fuzzy output control schemes relating to the parallel-distributed output compensators, the method presented methods that significantly reduces the conservativeness in the control

design conditions. Sufficient existence conditions of the both output controller realization, manipulating the global stability of the system, implies the parallel decentralized control framework which stabilizes the nonlinear system in the sense of Lyapunov, and the design of controller parameters, resulting directly from these conditions, is a feasible numerical problem. An additional benefit of the method is that controllers use minimum feedback information with respect to desired system output and the approach is flexible enough to allow the inclusion of additional design conditions. The validity and applicability of the approach is demonstrated through numerical design examples.

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