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Perturbed Differential Equations with Singular Points

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Additional information is available at the end of the chapter

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Dedicated to academician of National Academy Sciences Kyrgyz Republic
and Corresponding member of RAS Imanaliev Murzabek

Abstract

Here, we generalize the boundary layer functions method (or composite asymptotic expansion) for bisingular perturbed differential equations (BPDE that is perturbed differential equations with singular point). We will construct a uniform valid asymptotic solution of the singularly perturbed first-order equation with a turning point, for BPDE of the Airy type and for BPDE of the second-order with a regularly singular point, and for the boundary value problem of Cole equation with a weak singularity. A uniform valid expansion of solution of Lighthill model equation by the method of uniformization and the explicit solution—this one by the generalization method of the boundary layer function—is constructed. Furthermore, we construct a uniformly convergent solution of the Lagerstrom model equation by the method of fictitious parameter.

Keywords: turning point, singularly perturbed, bisingularly perturbed, Cauchy problem, Dirichlet problem, Lagerstrom model equation, Lighthill model equation, Cole equation, generalization boundary layer functions

1. Preliminary

1.1. Symbols O , o , \sim . Asymptotic expansions of functions

Let a function $f(x)$ and $\varphi(x)$ be defined in a neighborhood of $x = 0$.

Definition 1. If $\lim_{x \rightarrow 0} \frac{f(x)}{\varphi(x)} = M$, then write $f(x) = O(\varphi(x))$, $x \rightarrow 0$, and M is constant.

If $\lim_{x \rightarrow 0} \frac{f(x)}{\varphi(x)} = 0$, then write $f(x) = o(\varphi(x))$, $x \rightarrow 0$.

If $\lim_{x \rightarrow 0} \frac{f(x)}{\varphi(x)} = 1$, then write $f(x) \sim \varphi(x)$, $x \rightarrow 0$.

Definition 2. The sequence $\{\delta_n(\varepsilon)\}$, where $\delta_n(\varepsilon)$ defined in some neighborhood of zero, is called the asymptotic sequence in $\varepsilon \rightarrow 0$, if

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_{n+1}(\varepsilon)}{\delta_n(\varepsilon)} = 0, \quad \forall n = 1, 2, \dots$$

For example.

$$\{\varepsilon^n\}, \quad \left\{ \left(1/\ln(1/\varepsilon) \right)^n \right\}, \quad \left\{ \left(\varepsilon \ln(1/\varepsilon) \right)^n \right\}.$$

Note 1. Everywhere below ε denotes a small parameter.

Definition 3. We say that $f(x)$ function can be expanded in an asymptotic series by the asymptotic sequence $\{\varphi_n(x)\}$, $x \rightarrow 0$, if there exists a sequence of numbers $\{f_n\}$ and has the relation

$$f(x) = \sum_{k=0}^n f_k \varphi_k(x) + O(\varphi_{n+1}(x)), \quad x \rightarrow 0,$$

and write

$$f(x) \sim \sum_{k=0}^{\infty} f_k \varphi_k(x), \quad x \rightarrow 0.$$

1.2. The asymptotic expansion of infinitely differentiable functions

Theorem (Taylor (1715) and Maclaurin (1742)). If the function $f(x) \in C^\infty$ in some neighborhood of $x = 0$, then it can be expanded in an asymptotic series for the asymptotic sequence $\{x^n\}$, i.e.,

$$f(x) \sim \sum_{n=1}^{\infty} f_n x^n, \quad \text{where } f_n = f^{(n)}(0)/n!.$$

Thus, the concept of an asymptotic expansion was given for the first time by Taylor and Maclaurin, although an explicit definition was given by Poincaré in 1886.

1.3. The asymptotic expansion of the solution of the ordinary differential equation

Consider the Cauchy problem for a normal ordinary differential equation

$$y'(x) = f(x, y, \varepsilon), \quad y(0) = 0. \quad (1)$$

The function $f(x, y, \varepsilon)$ is infinitely differentiable on the variables x, y, ε in some neighborhood $O(0, 0, 0)$. It is correct next.

Theorem 1. The solution $y = y(x, \varepsilon)$ of problem (1) exists and unique in some neighborhood point $O(0, 0, 0)$ and $y(x, \varepsilon) \in C^\infty$, for small x, ε .

Corollary. The solution of problem (1) can be expanded in an asymptotic series by the small parameter ε , i.e.,

$$y(x, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k y_k(x). \quad (2)$$

Here and below, the equality is understood in an asymptotic sense.

Note 2. Theorem 1 for the case when $f(x, y, \varepsilon)$ is analytical was given in [1] by Duboshin.

Note 3. This theorem 1 is not true if $f(x, y, \varepsilon)$ is not smooth at ε . For example, the solution of a singularly perturbed equation

$$\varepsilon y'(x) = -y(x), \quad y(0) = a$$

function $y(x) = ae^{-x/\varepsilon}$ and is not expanded in an asymptotic series in powers of ε , because here $f(x, y, \varepsilon) = -y(x)/\varepsilon$ and f have a pole of the first order with respect to ε .

Note 4. The series 2 is a uniform asymptotic expansion of the function $y(x)$ in a neighborhood of $x = 0$.

For example. Series

$$y(x, \varepsilon) = 1 + \varepsilon x^{-1} + (\varepsilon x^{-1})^2 + \dots + (\varepsilon x^{-1})^n + \dots$$

It is not uniform valid asymptotic series on the interval $[0, 1]$, but it is a uniform valid asymptotic expansion of the segment $[\varepsilon^\alpha, 1]$, where $0 < \alpha < 1$.

1.4. Singularly perturbed ordinary differential equations

We divide such equations into three types:

- (I) Singular perturbations of ordinary differential equations such as the Prandtl-Tikhonov [2–56], i.e., perturbed equations that contain a small parameter at the highest derivative, i.e., equations of the form

$$y'(x) = f(x, y, \varepsilon), \quad y(0) = 0, \quad \varepsilon z'(x) = g(x, y, \varepsilon), \quad z(0) = 0,$$

where f, g are infinitely differentiable in the variables x, y, ε in the neighborhood of $O(0, 0, 0)$. It is obvious that unperturbed equation ($\varepsilon = 0$)

$$y_0'(x) = f(x, y, 0), \quad 0 = g(x, y, 0)$$

is a first order.

Definition 4. Singularly perturbed equation will be called bisingularly perturbed if the corresponding unperturbed differential equation has a singular point, or this one is an unbounded solution in the considering domain.

For example

1. Equation $\varepsilon y'(x) = -y(x)$ is a singularly perturbed ordinary differential equation.
2. Equation Vander Pol

$$\varepsilon y''(x) + (1 - y^2(x))y'(x) + y(x) = 0.$$

It is a bisingularly perturbed ordinary differential equation with singular points, if $y(x) = \pm 1$.

3. $\varepsilon y'(x) - xy(x) = 1$, $x \in [0, 1]$ is a bisingularly perturbed equation, because the unperturbed equation has an unbounded solution $y_0(x) = -x^{-1}$.

4. $\varepsilon y''(x) - xy(x) = 1$, $x \in [0, 1]$ is a bisingularly perturbed equation also.

(II) Singularly perturbed differential equations such as the Lighthill's type [57–69], in which the order of the corresponding unperturbed equation is not reduced, but has a singular point in the considering domain.

For example, a Lighthill model equation

$$(x + \varepsilon y(x))y'(x) + p(x)y(x) = r(x), \quad y(1) = a$$

where $x \in [0, 1]$, $p(x)$, $r(x) \in C^\infty[0, 1]$. For unperturbed equation

$$xy'_0(x) + p(x)y_0(x) = r(x),$$

point $x = 0$ is a regular singular point.

(III) A singularly perturbed equation with a small parameter is considered on an infinite interval. For example, the Lagerstrom equation [70–81]

$$y''(x) + nx^{-1}y'(x) + y(x)y'(x) = \beta(y'(x))^2, \\ y(\varepsilon) = 0, y(\infty) = 1.$$

where $0 < \beta$ is a given number and n is the dimension space.

Remark. The division into such classes is conditional, because singularly perturbed equation of Van der Pol in the neighborhood of points $y = \pm 1$ leads to an equation of Lighthill type [2, 3].

1.5. Methods of construction of asymptotic expansions of solutions of singularly perturbed differential equations

1. The method of matching of outer and inner expansions [13, 19, 28, 29, 37, 49] is the most common method for constructing asymptotic expansions of solutions of singularly perturbed differential equations. Justification for this method is given by Il'in [22]. However, this method is relatively complex for applied scientists.
2. The boundary layer function method (or composite asymptotic expansion) dates back to the work of many mathematicians. For the first time, this method for a singularly perturbed differential equations in partial derivatives is developed by Vishik and Lyusternik [52] and for nonlinear integral-differential equations (thus for the ordinary differential equations) Imanaliev [24], O'Malley (1971) [38], and Hoppenstedt (1971) [42].

It should be noted that, for the first time, the uniform valid asymptotic expansion of the solution of Eq. (5) is constructed by Vasil'eva (1960) [50] after Wasow [69] and Sibuya in 1963 [68] by the method of matching.

This method is constructive and understandable for the applied scientists.

3. The method of Lomov or regularization method [33] is applied for the construction of uniformly valid solutions of a singularly perturbed equation and will apply Fredholm ideas.
4. The method WKB or Liouville-Green method is used for the second-order differential equations.
5. The method of multiple scales.
6. The averaging method is applicable to the construction of solutions of a singularly perturbed equation on a large but finite interval.

Here, we consider a bisingularly perturbed differential equations and types of equations of Lighthill and Lagerstrom.

Here, we generalize the boundary layer function method for bisingular perturbed equations. We will construct a uniform asymptotic solution of the Lighthill model equation by the method of uniformization and construct the explicit solution of this one by the generalized method of the boundary layer functions.

Furthermore, we construct a uniformly convergent solution of the Lagerstrom model equation by the method of fictitious parameter.

2. Bisingularly perturbed ordinary differential equations

2.1. Singularly perturbed of the first-order equation with a turning point

Consider the Cauchy problem [5]

$$\varepsilon y'(x) + xy(x) = f(x), \quad 0 < x \leq 1, \quad y(0) = a, \quad (3)$$

where $f(x) \in C^\infty[0, 1]$, $f(x) = \sum_{k=0}^{\infty} f_k x^k$, $f_k = f^{(k)}(0)/k!$, $f_0 \neq 0$; a is the constant

Explicit solution of the problem (3) has the form: $y(x) = ae^{-x^2/2\varepsilon} + \frac{1}{\varepsilon} \int_0^x e^{(s^2-x^2)/2\varepsilon} f(s) ds$.

The corresponding unperturbed equation ($\varepsilon = 0$)

$$-x\tilde{y}(x) + f(x) = 0,$$

has a solution $\tilde{y}(x) = f(x)/x$, which is unbounded at $x = 0$.

If you seek a solution to problem (1) in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots, \quad (4)$$

then

$$\begin{aligned}
y_0(x) &= \frac{f(x)}{x} \sim f_0 x^{-1}, \quad x \rightarrow 0, \\
y_1(x) &= x^{-1} y'_0(x) \sim f_0 x^{-3}, \quad x \rightarrow 0, \\
y_2(x) &= x^{-1} y'_1(x) \sim 3f_0 x^{-5}, \quad x \rightarrow 0, \\
y_3(x) &= x^{-1} y'_2(x) \sim 3 \cdot 5 f_0 x^{-7}, \quad x \rightarrow 0, \\
y_n(x) &= x^{-1} y'_{n-1}(x) \sim 3 \cdot 5 \cdot \dots \cdot (2n-1) f_0 x^{-(2n+1)}, \quad x \rightarrow 0,
\end{aligned}$$

and a series of Eq. (4) is asymptotic in the segment $(\sqrt{\varepsilon}, 1]$, and the point $x_0 = \sqrt{\varepsilon} = \mu$ is singular point of the asymptotic series of Eq. (4). Therefore, the solution of problem (3) we will seek in the form

$$y(x) = \mu^{-1} \pi_{-1}(t) + Y_0(x) + \pi_0(t) + \mu(Y_1(x) + \pi_1(t)) + \mu^2(Y_2(x) + \pi_2(t)) + \dots, \quad \mu \rightarrow 0, \quad (5)$$

where $Y_k(x) \in C^{(\infty)}[0, 1]$, $\pi_k(t) \in C^{(\infty)}[0, \mu^{-1}]$, $x = \mu t$ and boundary layer functions $\pi_k(t)$ decreasing by power law as $t \rightarrow \infty$, that is, $\pi_k(t) = O(t^{-m})$, $t \rightarrow \infty, m \in N$.

Substituting Eq. (5) into Eq. (3), we obtain

$$\begin{aligned}
&\pi'_{-1}(t) + \mu^2 Y'_0(x) + \mu \pi'_0(t) + \mu^3 Y'_1(x) + \mu^2 \pi'_1(t) + \mu^4 Y'_2(x) + \mu^3 \pi'_2(t) + \mu^5 Y'_3(x) + \mu^4 \pi'_3(t) + \dots \\
&+ x Y_0(x) + \mu x Y_1(x) + \mu^2 x Y_2(x) + \mu^3 x Y_3(x) + \dots + t \pi_{-1}(t) + \mu t \pi_0(t) + \mu^2 t \pi_1(t) + \mu^3 t \pi_2(t) \\
&+ \mu^4 t \pi_3(t) + \dots = f(x).
\end{aligned} \quad (6)$$

The initial conditions for the functions $\pi_{k-1}(t)$, $k = 0, 1, \dots$ we take in the next form

$$\pi_{-1}(0) = 0, \quad \pi_0(0) = a - Y_0(0), \quad \pi_k(0) = -Y_k(0), \quad k = 1, 2, \dots$$

From Eq. (6), we have

$$\mu^0 : \quad \pi'_{-1}(t) + t \pi_{-1}(t) + x Y_0(x) = f(x), \quad (7.1)$$

$$\mu^1 : \quad \pi'_0(t) + t \pi_0(t) + x Y_1(x) = 0, \quad (7.0)$$

$$\mu^{k+1} : \quad \pi'_k(t) + t \pi_k(t) + x Y_{k+1}(x) + Y'_{k-1}(x) = 0, \quad k = 1, 2, \dots \quad (7.k)$$

To $Y_0(x)$ function has been smooth, and we define it from the equation

$$x Y_0(x) = f(x) - f_0 \Rightarrow Y_0(x) = (f(x) - f_0)/x,$$

and then from Eq. (7.1), we have obtained the equation

$$\pi'_{-1}(t) + t\pi_{-1}(t) = f_0.$$

Therefore

$$\pi_{-1}(t) = f_0 e^{-t^2/2} \int_0^t e^{s^2/2} ds \in C^\infty[0, \mu^{-1}],$$

Obviously, this function bounded and is infinitely differentiable on the segment $[0, \mu^{-1}]$, and

$$\pi_{-1}(t) = -\frac{f_0}{t} \left(1 + \frac{1}{t^2} + \frac{3}{t^4} + \dots\right), \quad t \rightarrow \infty.$$

This asymptotic expression can be obtained by integration by parts the integral expression for $\pi_{-1}(t)$.

Eq. (7.0) define $Y_1(x)$ and $\pi_0(t)$. Let $Y_1(x) \equiv 0$, then

$$\pi'_0(t) + t\pi_0(t) = 0, \quad \pi_0(0) = a - f_1.$$

Hence, we find

$$\pi_0(t) = (a - f_1)e^{-t^2/2}.$$

From Eq. (7c) for $k = 1$, we have

$$\pi'_1(t) + t\pi_1(t) + xY_2(x) + Y'_0(x) = 0.$$

Let $xY_2(x) = Y'_0(0) - Y'_0(x)$, then $\pi'_1(t) + t\pi_1(t) = -Y'_0(0)$.

From these, we get

$$Y_2(x) = (Y'_0(0) - Y'_0(x))/x, \quad \pi_1(t) = -f_2 e^{-t^2/2} \int_0^t e^{s^2/2} ds \in C^\infty[0, \mu^{-1}],$$

and

$$\pi_{-1}(t) = \frac{f_2}{t} \left(1 + \frac{1}{t^2} + \frac{3}{t^4} + \dots\right), \quad t \rightarrow \infty.$$

From Eq. (7c) for $k = 2$, we have

$$\pi'_2(t) + t\pi_2(t) + xY_3(x) + Y'_1(x) = 0 \text{ or } \pi'_2(t) + t\pi_2(t) + xY_3(x) = 0.$$

Let $Y_3(x) \equiv 0$, then

$$\pi_2'(t) + t\pi_2(t) = 0, \quad \pi_2(0) = -Y_2(0) = 2f_3.$$

From this, we get

$$\pi_2(t) = 2f_3 e^{-t^2/2}.$$

Analogously continuing this process, we determine the others of the functions $Y_k(x)$, $\pi_k(t)$.

In order to show that the constructed series of [Eq. (5)] is asymptotic series, we consider remainder term $R_m(x) = y(x) - y_m(x)$,

$$\text{where } y_m(x) = \frac{1}{\mu} \pi_{-1}(t) + Y_0(x) + \pi_0(t) + \mu(Y_1(x) + \pi_1(t)) + \dots + \mu^m(Y_m(x) + \pi_m(t)).$$

For the remainder term $R_m(x)$, we obtain a problem:

$$\varepsilon R_m'(x) + xR_m(x) = -\mu^{m+2}Y_m'(x), \quad 0 < x \leq 1, \quad R_m(0) = 0. \quad (8)$$

We note that if m is odd, then $Y_m'(x) \equiv 0$.

The problem (8) has a unique solution

$$R_m(x) = -\mu^m e^{-x^2/2\varepsilon} \int_0^x Y_m'(s) e^{s^2/2\varepsilon} ds,$$

and from this, we have $R_m(x) = O(\mu^m)$, $\mu \rightarrow 0$, $x \in [0, 1]$.

2.2. Bisingularly perturbed in a homogenous differential equation of the Airy type

Consider the boundary value problem for the second-order ordinary in a homogenous differential equation with a turning point

$$\varepsilon y''(x) - xy(x) = f(x), \quad x \in (0, 1), \quad (9)$$

$$y(0) = 0, \quad y(1) = 0. \quad (10)$$

where $f(x) = \sum_{k=0}^{\infty} f_k x^k$, $x \rightarrow 0$, $f_k = f^{(k)}(0)/k!$, $f_0 \neq 0$.

Note 5. It is the general case of this one was considered in Ref. [8, 45–47].

Without loss of generality, we consider the homogeneous boundary conditions, since $y(0) = a$, $y(1) = b$, $a^2 + b^2 \neq 0$, using transformation

$$y(x) = a + (b - a)x + z(x),$$

can lead to conditions (10).

If the asymptotic solution of the problems (9)–(10) we seek in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots, \quad (11)$$

then we have

$$\begin{aligned} y_0(x) &= -\frac{f(x)}{x} \sim f_0 x^{-1}, \quad x \rightarrow 0, \\ y_1(x) &= x^{-1} y_0''(x) \sim 1 \cdot 2 f_0 x^{-4}, \quad x \rightarrow 0, \\ y_2(x) &= x^{-1} y_1''(x) \sim 1 \cdot 2 \cdot 4 \cdot 5 f_0 x^{-7}, \quad x \rightarrow 0, \\ y_3(x) &= x^{-1} y_2''(x) \sim 1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 f_0 x^{-10}, \quad x \rightarrow 0, \\ y_n(x) &= x^{-1} y_{n-1}''(x) \sim 1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot \dots \cdot (3n-2) \cdot (3n-1) f_0 x^{-(3n+1)}, \quad 0 < n, \quad x \rightarrow 0, \end{aligned}$$

and the series (11) is asymptotic in the segment $(\sqrt[3]{\varepsilon}, 1]$. The point $x_0 = \sqrt[3]{\varepsilon} = \mu$ is singular point of asymptotic series (11).

The solution of problems (9) and (10) will be sought in the form

$$y(x) = \mu^{-1} \pi_{-1}(t) + \sum_{k=0}^{\infty} \mu^k \left(Y_k(x) + \pi_k(t) \right) + \sum_{k=0}^{\infty} \lambda^k w_k(\eta), \quad (12)$$

where $t = x/\mu$, $\mu = \sqrt[3]{\varepsilon}$, $\eta = (1-x)/\lambda$, $\lambda = \sqrt{\varepsilon}$. Here, $Y_k(x) \in C^\infty[0, 1]$, $\pi_k(t) \in C^\infty[0, 1/\mu]$ is boundary layer function in a neighborhood of $t = 0$ and decreases by the power law as $t \rightarrow \infty$, and the function $w_k(t) \in C^\infty[0, 1/\lambda]$ is boundary function in a neighborhood of $\eta = 0$ and decreases exponentially as $\eta \rightarrow \infty$.

Substituting Eq. (12) in Eq. (9), we get

$$\sum_{k=0}^{\infty} \mu^k (\pi_{k-1}''(t) - t \pi_{k-1}(t)) + \sum_{k=0}^{\infty} \mu^{k+3} Y_k''(x) - x \sum_{k=0}^{\infty} \mu^k Y_k(x) = f(x) \quad (13)$$

$$\sum_{k=0}^{\infty} \lambda^k (w_k''(\eta) - (1 - \lambda \eta) w_k(\eta)) = 0. \quad (14)$$

From Eq. (13), we have

$$\mu^0 : \quad \pi_{-1}''(t) - t \pi_{-1}(t) - x Y_0(x) = f(x), \quad (15.-1)$$

$$\mu^1 : \quad \pi_0''(t) - t \pi_0(t) - x Y_1(x) = 0, \quad (15.0)$$

$$\mu^2 : \quad \pi_1''(t) - t \pi_1(t) - x Y_2(x) = 0, \quad (15.1)$$

$$\mu^3 : \quad \pi_2''(t) - t \pi_2(t) + Y_0''(x) - x Y_3(x) = 0, \quad (15.2)$$

$$\mu^k : \quad \pi_{k-1}''(t) - t \pi_{k-1}(t) + Y_{k-3}''(x) - x Y_k(x) = 0, \quad k > 3, \quad (15.k)$$

Boundary conditions for functions $\pi_{k-1}(t)$, $k = 0, 1, \dots$ we take next form

$$\pi_{-1}(0) = 0, \quad \pi_k(0) = -Y_k(0), \quad \lim_{\mu \rightarrow 0} \pi_{k-1}(1/\mu) = 0, \quad k = 0, 1, 2, \dots$$

To $Y_0(x)$ function has been smooth; therefore, we define it from the equation

$$-xY_0(x) = f(x) - f_0 \Rightarrow Y_0(x) = -(f(x) - f_0)/x,$$

then from Eq. (15.1), we have the equation

$$\pi''_{-1}(t) - t\pi_{-1}(t) = f_0.$$

Let us prove an auxiliary lemma.

Lemma 1. Next boundary value problem

$$z''(t) - tz(t) = b, \quad 0 < t < 1/\mu, \quad \text{here } b \text{ is the constant,} \quad (16)$$

$$z(0) = z^0, \quad z(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0 \quad (17)$$

will have the unique solution and this one have next form

$$z(t) = z^0 \frac{Ai(t)}{Ai(0)} - \pi b \left(Ai(t) \int_0^t Bi(s) ds + Bi(t) \int_t^{1/\mu} Ai(s) ds - Ai(t) \sqrt{3} \int_0^{1/\mu} Ai(s) ds \right),$$

and $z(t) \in C^\infty[0, \mu^{-1}]$.

Proof. We verify the boundary conditions:

$$z(0) = z^0 - \pi b \left(Bi(0) \int_0^{1/\mu} Ai(s) ds - Ai(0) \sqrt{3} \int_0^{1/\mu} Ai(s) ds \right),$$

as $Bi(0) = Ai(0)\sqrt{3}$, so $z(0) = z^0$.

$$z(1/\mu) = z^0 \frac{Ai(1/\mu)}{Ai(0)} - \pi b (1 - \sqrt{3}) Ai(1/\mu) \int_0^{1/\mu} Bi(s) ds,$$

as $Ai(t) \sim t^{-1/4} e^{-\frac{2}{3}t^{3/2}}$, $Bi(t) \sim t^{-1/4} e^{\frac{2}{3}t^{3/2}}$, $t \rightarrow \infty$, so $z(1/\mu) = O(\mu)$, $\mu \rightarrow 0$.

Now we show that $z(t)$ satisfies Eq. (16). For this, we compute derivatives:

$$z'(t) = z^0 \frac{Ai'(t)}{Ai(0)} - \pi b \left(Ai'(t) \int_0^t Bi(s) ds + Bi'(t) \int_t^{1/\mu} Ai(s) ds - Ai'(t) \sqrt{3} \int_0^{1/\mu} Ai(s) ds \right)$$

$$z''(t) = z^0 \frac{Ai''(t)}{Ai(0)} - \pi b \left(Ai''(t) \int_0^t Bi(s) ds + Bi''(t) \int_t^{1/\mu} Ai(s) ds - \frac{1}{\pi} - Ai''(t) \sqrt{3} \int_0^{1/\mu} Ai(s) ds \right)$$

Substituting the expressions for $z''(t)$ and $z(t)$ in Eq. (17), and given that $Ai''(t) - tAi(t) \equiv 0$ and $Bi''(t) - tBi(t) \equiv 0$, we get: $b \equiv b$.

The uniqueness of $z(t)$ the solution is proved by contradiction. Let $u(t)$ also be a solution of problems (16) and (17), $z(t) \neq u(t)$. Considering the function $r(t) = z(t) - u(t)$, for the function $r(t)$, we obtain the problem

$$r''(t) - tr(t) = 0, \quad 0 < t < 1/\mu, \quad r(0) = 0, \quad r(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0.$$

The general solution of the homogeneous equation is

$$r(t) = c_1 Ai(t) + c_2 Bi(t); \quad c_{1,2} \text{ is the constant.}$$

Considering the boundary condition $r(1/\mu) \rightarrow 0, \mu \rightarrow 0$, we have $c_2 = 0$; $r(t) = c_1 Ai(t)$. And the second condition $r(0) = 0, c_1 = 0$ follows. This implies that $r(t) \equiv 0$.

Therefore, $z(t) \equiv u(t)$. It is obvious that $z(t) \in C^\infty[0, \mu^{-1}]$. Lemma 1 is proved.

This Lemma 1 implies the existence and uniqueness of $\pi_{-1}(t) \in C^\infty[0, \mu^{-1}]$ solution of the problem:

$$\pi_{-1}''(t) - t\pi_{-1}(t) = f_0, \quad 0 < t < 1/\mu, \quad \pi_{-1}(0) = 0, \quad \pi_{-1}(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0.$$

This function bounded and is infinitely differentiable on the segment $[0, \mu^{-1}]$, and as $t \rightarrow \infty$:

$$\pi_{-1}(t) = -\frac{f_0}{t} \left(1 + \frac{1 \cdot 2}{t^3} + \frac{1 \cdot 2 \cdot 4 \cdot 5}{t^6} + \dots \right).$$

This asymptotic expression can be obtained by integration by parts the integral expression for $\pi_{-1}(t)$.

From Eq. (15.0), we define $Y_1(x)$ and $\pi_0(t)$. Let $Y_1(x) \equiv 0$, then

$$\pi_0''(t) - t\pi_0(t) = 0, \quad \pi_0(0) = f_1, \quad \pi_0(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0,$$

And by Lemma 1, we have

$$\pi_0(t) = f_1 Ai(t)/Ai(0).$$

Analogously, from Eq. (15.1), we define $Y_2(x)$ and $\pi_1(t)$. Let $Y_2(x) \equiv 0$, then

$$\pi_1''(t) - t\pi_1(t) = 0, \quad \pi_1(0) = 0, \quad \pi_1(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0.$$

In view of Lemma 1, we have $\pi_1(t) \equiv 0$.

To $Y_3(x)$ function has been smooth; as above, we define it from the equation

$$xY_3(x) = Y_0''(x) - Y_0''(0) \Rightarrow Y_3(x) = (Y_0''(x) - Y_0''(0))/x, \quad (Y_0''(0) = -2f_3),$$

then Eq. (15.2) to $\pi_2(t)$ has the problem

$$\pi_2''(t) - t\pi_2(t) = 2f_3, \quad \pi_2(0) = 0, \quad \pi_2(1/\mu) \rightarrow 0, \quad \mu \rightarrow 0.$$

By Lemma 1, we can write an explicit solution to this problem, and this solution bounded and is infinitely differentiable on the segment $[0, \mu^{-1}]$, and as $t \rightarrow \infty$:

$$\pi_2(t) = -\frac{2f_3}{t} \left(1 + \frac{1 \cdot 2}{t^3} + \frac{1 \cdot 2 \cdot 4 \cdot 5}{t^6} + \dots \right).$$

Analogously continuing this process, we determine the rest of the functions $Y_k(x)$, $\pi_k(t)$.

Now we will define functions $w_k(\eta)$ from the equality (14) by using the boundary conditions $y(1) = 0$. We state problems

$$Lw_0 \equiv w_0''(\eta) - w_0(\eta) = 0, \quad w_0(0) = Y_0(1), \quad \lim_{\eta \rightarrow \infty} w_0(\eta) = 0 \quad (18.0)$$

$$Lw_k = -\eta w_{k-1}(\eta), \quad w_{2i}(0) = Y_{3i}(1), \quad w_{2i-1}(0) = 0, \quad \lim_{\eta \rightarrow \infty} w_k(\eta) = 0, \quad k, i \in \mathbb{N}. \quad (18.k)$$

One can easily make sure that all these problems (18.0) and (18.k) have unique solutions such that $w_k(\eta) \in C^\infty[0, \infty)$, $w_k(\eta) = O(e^{-\eta})$ with $\eta \rightarrow \infty$.

Thus, all functions $Y_k(x)$, $w_k(\eta)$, and $\pi_k(t)$ in equality (12) are defined, i.e., a formally asymptotic expansion is constructed. Let us justify the constructed expansion. Let

$$y_m(x) = \mu^{-1}\pi_{-1}(t) + \sum_{k=0}^{3m} \mu^k (Y_k(x) + \pi_k(t)) + \sum_{k=0}^{2m} \lambda^k w_k(\eta), \quad r_m(x) = y(x) - y_m(x).$$

Then for the remainder term, we state the following problem:

$$\varepsilon r_m''(x) - x r_m(x) = O(\varepsilon^{m+1/2}), \quad \varepsilon \rightarrow 0, \quad x \in (0, 1). \quad (19)$$

$$r_m(0) = O(e^{-1/\sqrt{\varepsilon}}), \quad r_m(1) = O(\varepsilon^{m+1}), \quad \varepsilon \rightarrow 0. \quad (20)$$

Let $r_m(x) = (2 - x^2)R_m(x)/2$, and then problems (19) and (20) take the form

$$\varepsilon R_m''(x) - \frac{4x\varepsilon}{2 - x^2} R_m'(x) - \left(\frac{2\varepsilon}{2 - x^2} + x \right) R_m(x) = O(\varepsilon^{m+1/2}), \quad \varepsilon \rightarrow 0,$$

$$R_m(0) = O(e^{-1/\sqrt{\varepsilon}}), \quad R_m(1) = O(\varepsilon^{m+1}), \quad \varepsilon \rightarrow 0.$$

According to the maximum principle [23, p. 117, 82], we have $R_m(x) = O(\varepsilon^{m-1/2})$, $\varepsilon \rightarrow 0$, $x \in [0, 1]$.

Hence, we get $r_m(x) = O(\varepsilon^{m-1/2})$, $\varepsilon \rightarrow 0$, $x \in [0, 1]$.

Thus, we have proved.

Theorem 2. Let $f(0) \neq 0$, then the solution to problem (9) and (10) will have next form

$$y(x) = \frac{1}{\sqrt[3]{\varepsilon}} \pi_{-1} \left(\frac{x}{\sqrt[3]{\varepsilon}} \right) + \sum_{k=0}^{\infty} \sqrt[3]{\varepsilon^k} \left(y_k(x) + \pi_k \left(\frac{x}{\sqrt[3]{\varepsilon}} \right) \right) + \sum_{k=0}^{\infty} \sqrt{\varepsilon^k} w_k \left(\frac{1-x}{\sqrt{\varepsilon}} \right).$$

Example. Consider the problem

$$\varepsilon y''(x) - xy(x) = 1 + x, \quad x \in (0, 1), \quad y(0) = 0, \quad y(1) = 0.$$

The asymptotic solution this problem we can represent in the form $y(x) = \mu^{-1} \pi_{-1}(t) +$

$$\sum_{k=0}^3 \mu^k (Y_k(x) + \pi_k(t)) + w_0(\eta) + \lambda w_1(\eta) + \lambda^2 w_2(\eta) + R(x).$$

We have got $Y_0(x) = -(1+x-1)/x = -1$, $Y_{1,2,3}(x) \equiv 0$,

$$\pi_{-1}(t) = -\pi \left(Ai(t) \int_0^t Bi(s) ds + Bi(t) \int_t^{1/\mu} Ai(s) ds - Ai(t) \sqrt{3} \int_0^{1/\mu} Ai(s) ds \right),$$

$$\pi_0(t) = Ai(t)/Ai(0), \quad \pi_{1,2,3}(t) \equiv 0, \quad w_0(\eta) = 2e^{-\eta}, \quad w_k(\eta) = O(e^{-\eta}), \quad k = 1, 2.$$

$$\varepsilon R''(x) - xR(x) = O(\varepsilon^{3/2}), \quad 0 < x < 1, \quad R(0) = O(e^{-1/\sqrt{\varepsilon}}), \quad R(1) = O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

We have

$$y(x) = \varepsilon^{-1/3} \pi_{-1}(t) - 1 + 2e^{-(1-x)/\sqrt{\varepsilon}} + \pi_0(t) + \sqrt{\varepsilon} w_1(\eta) + \varepsilon w_2(\eta) + O(\sqrt{\varepsilon}), \quad \varepsilon \rightarrow 0.$$

2.3. Bisingularly perturbed equation of the second order with a regularly singular point

Consider the boundary value problem [6, 7]

$$L_\varepsilon y \equiv \varepsilon y'' + xy' - q(x)y = f(x), \quad x \in [0, 1], \quad (21)$$

$$y(0) = 0, \quad y(1) = 0, \quad (22)$$

where $q(x), f(x) \in C^\infty[0, 1]$.

Here, for simplicity, we consider the case $q(0) = 1$, $q(x) \geq 1$.

The solution of the unperturbed problem

$$My \equiv xy' - q(x)y = f(x),$$

represented as

$$y_0(x) \equiv xp(x) \int_1^x r(s)s^{-2}ds, \quad (23)$$

where

$$r(x) = p^{-1}(x)f(x), \quad p(x) = \exp \left\{ \int_1^x (q(s) - 1)s^{-1}ds \right\}.$$

Extracting in Eq. (23), the main part of the integral in the sense of Hadamard [34], it can be represented as

$$y_0(x) = a(x) + r_1xp(x)\ln x, \quad (24)$$

where

$$\begin{aligned} a(x) &= xp(x) \int_1^x (r(s) - r_0 - r_1s)s^{-2}ds + r_0p(x)[x - 1], \\ r_0 &= r(0), \quad r_1 = r'(0) = p(0)^{-1}[f'(0) - q'(0)f(0)]. \end{aligned} \quad (25)$$

Function $a(x) \in C^\infty[0, 1]$.

Theorem 3. Suppose that the conditions referred to the above with respect to $q(x)$ and $f(x)$. Then the asymptotic behavior of the solution of the problems (21) and (22) can be written as:

$$\sum_{k=0}^{\infty} \mu^k (z_k(x) + \pi_k(t)), \quad \varepsilon = \mu^2, \quad x = \mu t, \quad (26)$$

where $z_k(x) \in C^\infty[0, 1]$, $\pi_k(t) \in C^\infty[0, \mu^{-1}]$.

Function $z_0(x)$ is a solution of equation

$$Mz_0 = f(x) - c_0xp(x),$$

where $c_0 = p(0)^{-1}[f'(0) - q'(0)f(0)]$.

The coefficients $z_k(x)$ of the series (26) will be determined as the solution of equations

$$Mz_k = -z''_{k-1}(x) - c_kxp(x),$$

where $c_k = p(0)^{-1}[-z'''_{k-1}(0) + z''_{k-1}(0)q'(0)]$, with boundary conditions $z_k(1) = 0$, $k \geq 1$.

Functions $\pi_k(t)$ is the solution of the equations

$$L\pi_k \equiv \pi_k''(t) + t\pi_k'(t) - q(\mu t)\pi_k(t) - c_k\mu t p(\mu t)$$

with boundary conditions $\pi_k(0) = -z_k(0)$, $\pi_k(\mu^{-1}) = 0$.

Next, we use the following lemma.

Lemma 2. The problem

$$My = f(x) - r_1xp(x)$$

It has a unique solution $y(x) \in C^\infty[0, 1]$.

The proof of Lemma 2 follows from Eqs. (24) and (25).

Lemma 3. A boundary value problem

$$L_0v \equiv v'' + tv' - v(t) = 0, \quad v(0) = a, \quad v(1/\mu) = 0,$$

has solution $v(t) = aX(t)$, where

$$X(t) = t \int_t^{\mu^{-1}} s^{-2} \exp\left(\frac{-s^2}{2}\right) ds, \quad 0 \leq X(t) \leq 1, \quad X(0) = 1.$$

The proof of Lemma 3 is obvious.

Lemma 4. In order to solve the boundary value problem

$$L_0W = -\mu t, \quad W(0) = W(\mu^{-1}) = 0,$$

we have the estimate

$$0 \leq W(\mu, t) \leq e^{-1} \ln \mu^{-1}.$$

Proof. This follows from the fact that the solution of this problem exists uniquely by the maximum principle [23, 82] and will be represented in the form

$$W(\mu, t) = \mu t \int_t^{\mu^{-1}} y^{-2} \exp\left(-\frac{y^2}{2}\right) \int_0^y s^2 \exp\left(\frac{s^2}{2}\right) ds dy.$$

Lemma 5. The estimate

$$|\pi_k(\mu, t)| < B_k,$$

where $0 < B_k$ is constant.

Proof. Consider the function

$$V_{\pm}(\mu, t) = \gamma_1 W(\mu, t) + \gamma_2 X(t) \pm \pi_k(\mu, t),$$

where γ_1 and γ_2 are positive constants such that

$$\gamma_1 > \max_{[0,1]} |p(x)|, \gamma_2 > |z_k(0)|.$$

It is obvious that

$$V_{\pm}(\mu, 0) > 0, \quad V_{\pm}(\mu, \mu^{-1}) > 0, \quad L_0 V_{\pm} \equiv V''_{\pm}(t) + tV'_{\pm}(t) - V_{\pm}(t) < 0$$

From the maximum principle, it follows that $|\pi_k(\mu, t)| < \gamma_1 W(\mu, t) + \gamma_2 X(t)$.

Now the proof of the lemma 5 follows from estimates of $W(\mu, t)$ and $X(t)$.

If we introduce the notation

$$Y_n(x, \varepsilon) = \sum_{k=0}^n \varepsilon^k (z_k(x) + \pi_k(\mu, t)),$$

where $z_k(x)$, $\pi_k(\mu, t)$ are constructed above functions, then

$$L_{\varepsilon} Y_n(x, \varepsilon) = f(x) + \varepsilon^{n+1} z''_n.$$

Let $y(x, \varepsilon)$ be the solution of the problems (21) and (22). Then

$$|L_{\varepsilon} (Y_n(x, \varepsilon) - y(x, \varepsilon))| < B_n \varepsilon^{n+1}, \quad Y_n(0, \varepsilon) - y(0, \varepsilon) = Y_n(1, \varepsilon) - y(1, \varepsilon) = 0.$$

Therefore, $|Y_n(x, \varepsilon) - y(x, \varepsilon)| < B_n \varepsilon^{n+1}$.

2.4. The bisingular problem of Cole equation with a weak singularity

The following problem is considered [9, 13, 28, 29],

$$\varepsilon y''(x) + \sqrt{x} y'(x) - y(x) = 0, \quad 0 < x < 1, \quad (27)$$

$$y(0) = a, \quad y(1) = b \quad (28)$$

where $x \in [0, 1]$; a, b are the given constants.

The unperturbed equation $\sqrt{x} y'(x) - y(x) = 0$, $0 < x < 1$,

has the general solution

$$y_0(x) = c e^{2\sqrt{x}}, \quad c - \text{const.}$$

This is a nonsmooth function in $[0, 1]$.

We seek asymptotic representation of the solution of the problems (27) and (28) in the form:

$$y(x) = \sum_{k=0}^n \varepsilon^k y_k(x) + \sum_{k=0}^{3(n+1)} \mu^k \pi_k(t) + R(x, \varepsilon), \quad (29)$$

where $t = x/\mu^2$, $\varepsilon = \mu^3$, $y_k(x) \in C[0, 1]$, $\pi_k(t) \in C[0, 1/\mu^2]$, $R(x, \varepsilon)$ is the reminder term.

Substituting Eq. (29) into Eq. (27), we have

$$\begin{aligned} & \sum_{k=0}^n \varepsilon^k (\varepsilon y_k''(x) + \sqrt{x} y_k'(x) - y_k(x)) + \frac{1}{\mu} (\pi_0''(t) + \sqrt{t} \pi_0'(t)) \\ & + \sum_{k=1}^{3(n+1)} \mu^{k-1} (\pi_k''(t) + \sqrt{t} \pi_k'(t) - \pi_{k-1}(t)) - \mu^{3(n+1)} \pi_{3(n+1)}(t) + \varepsilon R''(x, \varepsilon) + \sqrt{x} R'(x, \varepsilon) \\ & - R(x, \varepsilon) - h(x, \varepsilon) + h(x, \varepsilon) = 0 \end{aligned} \quad (30)$$

By the method of generalized boundary layer function, we put the term $h(x, \varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k h_k(x)$ into the equation. We choose functions $h_k(x)$ so that $y_k(x) \in C[0, 1]$.

Taking into account the boundary condition (28), from Eq. (30), we obtain

$$\sqrt{x} y_0'(x) - y_0(x) = 0, \quad 0 < x < 1, \quad y_0(1) = b. \quad (31)$$

$$\sqrt{x} y_k'(x) - y_k(x) = h_{k-1}(x) - y_{k-1}''(x), \quad 0 < x < 1, \quad k \in N, \quad y_k(1) = 0. \quad (32)$$

The solution of the problems (31) and (32) exists. It is unique and has the form

$$y_0(x) = b e^{2(\sqrt{x}-1)}, y_k(x) = e^{2\sqrt{x}} \int_1^x \frac{h_{k-1}(s) - y_{k-1}''(s)}{\sqrt{s}} e^{-2\sqrt{s}} ds, \quad k \in N.$$

We choose indefinite functions $h_k(x)$ as follows: $y_{k-1}''(x) - h_{k-1}(x) \in C[0, 1]$. We can represent

$$y_0(x) = b e^{-2} \left(1 + 2\sqrt{x} + \frac{(2\sqrt{x})^2}{2!} + \frac{(2\sqrt{x})^3}{3!} + \frac{(2\sqrt{x})^4}{4!} + \dots + \frac{(2\sqrt{x})^n}{n!} + \dots \right).$$

$$\text{Let } h_1(x) = b e^{-2} \left(2\sqrt{x} + \frac{(2\sqrt{x})^3}{3!} \right)'' = -b e^{-2} \left(\frac{1}{2\sqrt{x^3}} - \frac{1}{\sqrt{x}} \right).$$

Then

$$y_0''(x) - h_0(x) \in C[0, 1], \mu^3 h_1(t\mu^2) = -c_1 \left(\frac{1}{2\sqrt{t^3}} - \frac{\mu^2}{\sqrt{t}} \right), \quad c_1 = b e^{-2},$$

$$y_1(x) = c_1 e^{2\sqrt{x}} \int_1^x \left(-\frac{1}{2s^2} + \frac{1}{s} + \frac{1}{2s^2} e^{2\sqrt{s}} - \frac{1}{\sqrt{s^3}} e^{2\sqrt{s}} \right) e^{-2\sqrt{s}} ds.$$

We can rewrite $y_1(x)$ in the form:

$$y_1(x) = y_{1,0} + y_{1,1}(2\sqrt{x}) + y_{1,2}(2\sqrt{x})^2 + y_{1,3}(2\sqrt{x})^3 + \dots,$$

$$\text{where } y_{1,0} = \left(\frac{3}{2} + \frac{1}{2e^2}\right)c_1, \quad y_{1,1} = \left(\frac{1}{6} + \frac{1}{2e^2}\right)c_1, \quad y_{1,2} = \left(\frac{-1}{6} + \frac{1}{4e^2}\right)c_1, \quad y_{1,3} = \left(\frac{-1}{10} + \frac{1}{12e^2}\right)c_1.$$

Analogously, we have obtained

$$h_1(x) = \left(y_{1,1}(2\sqrt{x}) + y_{1,3}(2\sqrt{x})^3\right)' = -\frac{y_{1,1}}{2\sqrt{x^3}} + \frac{6y_{1,3}}{\sqrt{x}}.$$

Then

$$y_2''(x) - h_2(x) \in C[0, 1], \quad \mu^6 h_2(t\mu^2) = -\frac{\mu^3 y_{1,1}}{2\sqrt{t^3}} + \frac{\mu^5 y_{1,3}}{\sqrt{t}}.$$

Continuing this process, we have

$$h_{k-1}(x) = -\frac{y_{k-1,1}}{2\sqrt{x^3}} + \frac{6y_{k-1,3}}{\sqrt{x}}, \quad k = 4, \dots, n,$$

where $y_{k-1,1}$, $y_{k-1,3}$ are corresponding coefficients of the expansion of $y_{k-1,1}(x)$ in powers of $(2\sqrt{x})$.

From Eq. (30), we have the following equations for the boundary functions $\pi_k(t)$:

$$L\pi_0 \equiv \pi_0''(t) + \sqrt{t}\pi_0'(t) = 0, \quad 0 < t < \tilde{\mu}, \quad \pi_0(0) = a - y_0(0), \quad \pi_0(\tilde{\mu}) = 0, \quad \tilde{\mu} = 1/\mu^2, \quad (33)$$

$$L\pi_{3k+1}(t) = \pi_{3k}(t) + \frac{y_{k,1}}{2\sqrt{t^3}}, \quad 0 < t < \tilde{\mu}, \quad \pi_{3k+1}(0) = 0, \quad \pi_{3k+1}(\tilde{\mu}) = 0, \quad k = 0, 1, \dots, n \quad (34)$$

$$L\pi_{3k+2}(t) = \pi_{3k+1}(t), \quad 0 < t < \tilde{\mu}, \quad \pi_{3k+2}(0) = 0, \quad \pi_{3k+2}(\tilde{\mu}) = 0, \quad k = 0, 1, \dots, n \quad (35)$$

$$L\pi_{3k+3} = \pi_{3k+2}(t) - \frac{y_{k,3}}{\sqrt{t}}, \quad 0 < t < \tilde{\mu}, \quad \pi_{3k}(0) = -y_k(0), \quad \pi_{3k}(\tilde{\mu}) = 0, \quad k = 0, 1, \dots, n-1 \quad (36)$$

$$L\pi_{3(n+1)}(t) = \pi_{3n+2}(t) - \frac{y_{n,3}}{\sqrt{t}}, \quad 0 < t < \tilde{\mu}, \quad \pi_{3n}(0) = 0, \quad \pi_{3n}(\tilde{\mu}) = 0 \quad (37)$$

The solution of problem (33) is represented in the form

$$\pi_0(t) = (a - be^{-2})A \int_t^{\tilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds, \quad A = \left(\int_0^{\tilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds \right)^{-1}.$$

We note that $\pi_0(t)$ will exponentially decrease as $t \rightarrow \tilde{\mu}$.

Lemma 6. The general solution of this equation $Lz(t) = 0$ will have $z(t) = c_1 Y(t) + c_2 X(t)$; here c_1 , c_2 are constants, and

$$Y(t) = 1 - X(t), \quad X(t) = \alpha \int_t^{\tilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds \quad \left(\alpha \int_0^{\tilde{\mu}} e^{-\frac{2}{3}s^{3/2}} ds = 1 \right).$$

Two linearly independent solutions and $Y(t) = O(t)$, $t \rightarrow 0$, $0 < X(t) \leq 1$,

$$X(t) = t^{-\frac{1}{2}} e^{-\frac{2}{3}t^{3/2}} \left(1 - \frac{1}{2}t^{-\frac{3}{2}} + \dots + \frac{(-1)^n}{2^n} \prod_{k=1}^n 1 \cdot 4 \cdot \dots \cdot (3k-2) t^{-\frac{3n}{2}} + \dots \right), \quad t \rightarrow \tilde{\mu} \quad (38)$$

Lemma 7. The boundary problem $Lz(t) = 0$, $z(0) = z(\tilde{\mu}) = 0$ will have only trivial solution.

The proofs of Lemmas 6 and 7 are evident.

Theorem 4. The problem

$$Lz(t) = f(t), \quad z(0) = 0, \quad z(\tilde{\mu}) = 0,$$

will have the unique solution and this one has the next form

$$z(t) = \int_0^{\tilde{\mu}} G(t, s) e^{\frac{2}{3}s^{3/2}} f(s) ds,$$

$$\text{and } G(t, s) = \begin{cases} -Y(t)X(s), & 0 \leq t \leq s, \\ -Y(s)X(t), & s \leq t \leq \tilde{\mu}, \end{cases}$$

is the function of Green and $f(t) \in C(0, \tilde{\mu}]$.

Theorem 4 implies the existence and uniqueness of the solution of problem (34)–(37): $|\pi_k(t)| < l = \text{const}$, $t \in [0, \tilde{\mu}]$.

Lemma 8. Asymptotical expansions of functions $\pi_k(t)$, $t \rightarrow \tilde{\mu}$ ($k = 1, 2, \dots$) will have the next forms

$$\begin{aligned} \pi_1(t) &= -\frac{y_{0,1}}{2t} \left(1 + \frac{4}{5\sqrt{t^3}} + \frac{7}{4t^3} + \frac{42}{11\sqrt{t^9}} + \frac{39}{2t^7} + \dots \right), \\ \pi_2(t) &= \frac{y_{0,1}}{\sqrt{t}} \left(1 + \frac{23}{40\sqrt{t^3}} + \frac{173}{2t^3} + \dots \right), \quad \pi_3(t) = -\frac{23y_{0,1}}{60\sqrt{t^3}} + O\left(\frac{1}{t^3}\right), \\ \pi_{3k+1}(t) &= t^{-1} \sum_{j=0}^{\infty} l_{3k+1,j} t^{-\frac{3}{2}j}, \quad \pi_{3k+2}(t) = t^{-1/2} \sum_{j=0}^{\infty} l_{3k+2,j} t^{-\frac{3}{2}j}, \quad \pi_{3k}(t) = \sum_{j=1}^{\infty} l_{3k,j} t^{-\frac{3}{2}j}. \end{aligned}$$

Proof for Lemma 8.

Firs proof. We can prove this lemma by applying formulas (38) and Theorem 4.

Second proof. We can receive these representations from Eqs. (34)–(37) directly.

Now we will prove the boundedness of the reminder function $R(x, \varepsilon)$. This function will satisfy the next equation:

$$\varepsilon R''(x, \varepsilon) + \sqrt{x} R'(x, \varepsilon) - R(x, \varepsilon) = \mu^{3(n+1)} \pi_{3(n+1)}(t) + \varepsilon^{n+1} (h_n(x) - y_n''(x)),$$

$$R(0, \varepsilon) = 0, \quad R(1, \varepsilon) = 0.$$

Applying to this problem theorem [23, p.117, 82], we obtained

$$|R(x, \varepsilon)| \leq \varepsilon^{n+1} C \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq \tilde{\mu}}} |\pi_{3(n+1)}(t) + h_n(x) - y_n''(x)|.$$

Therefore, we have $R(x, \varepsilon) = O(\varepsilon^{n+1})$, $\varepsilon \rightarrow 0$, $x \in [0, 1]$.

We prove next.

Theorem 5. The asymptotical expansion of the solution of the problems (27) and (28) and will have the next form

$$y(x) = \sum_{k=0}^n \varepsilon^k y_k(x) + \sum_{k=0}^{3(n+1)} \mu^k \pi_k(t) + O(\varepsilon^{n+1}), \quad \varepsilon \rightarrow 0.$$

3. Singularly perturbed differential equations Lighthill type

3.1. The idea of the method of Poincare

Consider the equation

$$My(x) := y''(x) + y(x) - \varepsilon y^3(x) = 0. \quad (39)$$

Unperturbed equation has solutions $y_0(x) = a_1 \cos x + b_1 \sin x$ (where a_1, b_1 are arbitrary constants) with period 2π . We are looking for the periodic solution of the equation $y(x, \varepsilon)$ with a period of $\omega(\varepsilon) = \omega(0) = 2\pi$.

Note that the operator M transforms Fourier series $\sum_{k=1}^{\infty} a_k \cos kx$ and $\sum_{k=1}^{\infty} a_k \sin kx$ in itself.

Poincare's method reduces the existence of periodic solutions of differential equations to the existence of the solution of an algebraic equation.

We will seek a periodic solution of Eq. (39) with the initial condition

$$y(0) = 1, \quad y'(0) = 0.$$

If we seek the solution in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

with the initial conditions

$$y_0(0) = 1, \quad y'_0(1) = 0, \quad y_k(0) = y'_k(1) = 0, \quad k = 1, 2, \dots$$

then for $y_s(x)$, $s = 0, 1, \dots$ we have next equations

$$Ly_0 := y''_0(x) + y_0(x) = 0 \Rightarrow y_0(x) = \cos x$$

$$Ly_1 = \cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x \Rightarrow y_1(x) = \frac{3}{8} x \sin x - \frac{1}{32} \cos 3x + \frac{1}{32} \cos x,$$

Thus, $y(x) = \cos x + \frac{\varepsilon}{8} (3x \sin x - \frac{1}{4} \cos 3x + \frac{1}{4} \cos x) + \dots$ it is not a uniform expansion of the $y(x)$ on the segment $[-\infty, \infty]$, since the term $\varepsilon x \sin x$ is present here.

If these secular terms do not appear in Eq. (39), it is necessary to make the substitution

$$x = t(1 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \dots)$$

where the constant α_k should be selected so as not to have secular terms in t .

Thus, the solution of Eq. (39) must be sought in the form

$$\begin{aligned} y(t) &= y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots \\ x &= t(1 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \dots) \end{aligned} \quad (40)$$

Then Eq. (39) has the form

$$z''(t) + (1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \dots)z(t) = \varepsilon(1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \dots)z^3(t)$$

where $y(w(\varepsilon)t) = z(t)$.

We will seek the 2π periodic solution of this equation in the form

$$z(t) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + \dots$$

Then

$$Lz_0 := z''_0(t) + z_0(t) = 0 \Rightarrow z_0(t) = \cos t.$$

$$Lz_1(t) = \alpha_1 \cos t + \frac{3}{4} \cos t + \frac{1}{4} \cos 3t.$$

The function $Z_1(t)$ will have the periodical solution we take $\alpha_1 = -3/4$. Then $z_1(t) = -\frac{1}{32} \cos 3t$.

Similarly, from equations

$$\alpha z_n(t) = -\alpha_n \cos t + g(\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \cos t + \sum_{m=1}^{2n+1} \beta_n \cos mt$$

α_n and etc. are uniquely determined.

Theorem 6. Equation (39) has a unique $2\pi/\omega$ periodic solution, and it can be represented in the form (40).

3.2. The idea of the Lighthill method

Lighthill in 1949 [67] reported an important generalization of the method of Poincare.

He considered the model equation [67, 82]:

$$(x + \varepsilon y(x))y'(x) + q(x)y(x) = r(x), \quad y(1) = a \quad (41)$$

where $x \in [0, 1]$, $q(x), r(x) \in C^\infty[0, 1]$.

Lighthill proposed to seek the solution of Eq. (41) in the form

$$\begin{aligned} y(\xi) &= y_0(\xi) + \varepsilon y_1(\xi) + \varepsilon^2 y_2(\xi) + \dots \\ x &= \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \dots \end{aligned} \quad (42)$$

It is obvious that Eq. (42) has generalized the Poincare ideas (see, the transformation Eq. (40)).

At first, we consider the example

$$(x + \varepsilon y(x))y'(x) + y(x) = 0, \quad y(1) = b. \quad (43)$$

It has exact solution

$$y(x) = (\sqrt{x^2 + 2b\varepsilon + \varepsilon^2 b^2} - x)/\varepsilon. \quad (44)$$

It is obvious that for $b > 0$, the solution (43) exists on the interval $[0, 1]$ and

$$y(0) = \sqrt{2b + \varepsilon b^2}/\sqrt{\varepsilon}.$$

The solution of Eq. (43) is obtained by the method of small parameter that can be obtained from Eq. (44). For this purpose, we write Eq. (44) in the form

$$y(x) = \frac{x}{\varepsilon} \left(-1 + \sqrt{1 + 2b\frac{\varepsilon}{x} + b^2\left(\frac{\varepsilon}{x}\right)^2} \right)$$

and considering $x^2 > 2\varepsilon b$, this expression can be expanded in powers of ε , and then we have

$$y(x) = \frac{b}{x} + \frac{b^2}{2x} \frac{\varepsilon}{x^2} (x^2 - 1) + \dots + O\left(\frac{1}{x} \left(\frac{\varepsilon}{x^2}\right)^n\right) + \dots \quad (45)$$

The series (45) is uniformly convergent asymptotic series only on the segment $[\varepsilon^\alpha, 1]$, $0 < \alpha < 1/2$.

First, we write Eq. (43) in the form

$$(x + \varepsilon y(\xi))y'(\xi) + y(\xi)x'(\xi) = 0 \quad (46)$$

Substituting Eq. (42) into Eq. (46):

$$(\xi + \varepsilon(y_0(\xi) + x_1(\xi)) + \dots + \varepsilon^n(y_{n-1}(\xi) + x_n(\xi)) + \dots)(y'_0(\xi) + \varepsilon y'_1(\xi) + \dots + \varepsilon^n y'_n(\xi) + \dots) + (y_0(\xi) + \varepsilon y'_1(\xi) + \dots \varepsilon^n y'_n(\xi) + \dots)(1 + \varepsilon x'_1(\xi) + \dots + \varepsilon^n x'_n(\xi) + \dots) = 0$$

and equating coefficients of the same powers ε , we have

$$\xi y'_0(\xi) + y_0(\xi) = 0 \quad (47)$$

$$\xi y'_n(\xi) + y_n(\xi) + \sum_{i=0}^{n-1} \left((y_i(\xi) + x_{i+1}(\xi))y'_{n-1-i}(\xi) + y_i(\xi)x'_{n-i}(\xi) \right) = 0, \quad y_n(1) = 0, \quad n = 1, 2, \dots \quad (48)$$

From Eq. (47), we have

$$y_0(\xi) = b\xi^{-1}.$$

Using Eq. (47), Eq. (48) for $n = 1$ can be written as

$$\xi y'_1(\xi) + y_1(\xi) = (\xi x'_1(\xi) - x_1(\xi) + y_0(\xi))y'_0(\xi) = 0, \quad y_1(1) = 0. \quad (49)$$

If we put $x_1(\xi) = 0$ in Eq. (49), we obtain

$$\xi y'_1(\xi) + y_1(\xi) = -b^2 \xi^{-3}, \quad y_1(1) = 0.$$

Hence, solving this equation, we have

$$y_1(\xi) = b^2(2\xi)^{-1} - b^2(2\xi^3)^{-1}.$$

Since differentiation increased singularity of nonsmooth function, we select $x_1(\xi)$ so that the expression in the right side of Eq. (49) is equal to zero, i.e.,

$$\xi x'_1(\xi) - x_1(\xi) + y_0(\xi) = 0, \quad x_1(1) = 0.$$

Hence, we have

$$x_1(\xi) = 2^{-1}b\xi - (2\xi)^{-1}b.$$

Then Eq. (49) takes the form

$$\xi y'_1(\xi) + y_1(\xi) = 0, \quad y_1(1) = 0.$$

Hence, we obtain $y_1(\xi) = 0$.

Now Eq. (48) for $n = 2$ takes the form

$$\xi y_2'(\xi) + y_2(\xi) = (\xi x_2'(\xi) - x_2(\xi))y_0'(\xi) = 0, \quad y_2(1) = 0.$$

Let $x_2(\xi) = 0$, and then $y_1(\xi) = 0$. Further also choose $x_i(\xi) = y_i(\xi) = 0$ ($i = 3, 4, \dots$), as they also satisfy the initial conditions. Thus, we have found that

$$y(\xi) = b\xi^{-1} \quad (50)$$

$$x(\xi) = \xi + \frac{b}{2} \left(\xi - \frac{1}{\xi} \right) \varepsilon. \quad (51)$$

Putting in Eq. (51) $x = 0$, we have

$$\eta = \sqrt{b\varepsilon/(2 + b\varepsilon)}. \quad (52)$$

For $b > 0$, the point $x = 0$ is achieved. Moreover, the except in variable ξ from Eq. (50) and to Eq. (51) setting ξ , we obtain the exact solution (44).

Now we will present the main idea of the Lighthill method to Eq. (41) under conditions: $q(x), r(x) \in C^\infty[0, 1]$ and $q_0 = q(0) > 0$. We will write it in the form of

$$(x(\xi) + \varepsilon y(\xi))y'(\xi) = [r(x(\xi)) - q(x(\xi))y(\xi)]x'(\xi), \quad y(1) = y^0. \quad (53)$$

It is obvious that we have one equation for two unknown functions, $y(\xi), x(\xi)$. Now we substitute the series (42) to Eq. (53):

$$\begin{aligned} & \left(\xi + \sum_{k=0}^{\infty} \varepsilon^k (y_k(\xi) + x_k(\xi)) \right) \sum_{k=0}^{\infty} \varepsilon^k y_k'(\xi) = \\ & = \left(\sum_{j=0}^{\infty} r_j(\xi) \left(\sum_{k=0}^{\infty} x_k(\xi) \varepsilon^k \right)^j - \sum_{j=0}^{\infty} q_j(\xi) \left(\sum_{k=0}^{\infty} x_k(\xi) \varepsilon^k \right)^j \right) \left(1 + \sum_{k=0}^{\infty} x_k'(\xi) \varepsilon^k \right), \end{aligned}$$

where $q_j = q_j(\xi) = \frac{1}{j!} q^{(j)}(\xi)$, $r_j = r_j(\xi) = \frac{1}{j!} r^{(j)}(\xi)$.

Hence, equating the coefficients of equal powers has ε

$$Lu_0 \equiv \xi y_0'(\xi) + q(\xi) y_0(\xi) = r(\xi), \quad y_0(1) = y^0, \quad (54)$$

$$Ly_1 = [\xi y_0' x_1' - y_0' x_1 - y_0 y_0'] + (r_1 - q_1 y_0) x_1, \quad y_1(1) = 0, \quad (55)$$

$$\begin{aligned} Ly_2 = & [\xi y_0' x_2' - (y_0 + x_1) y_1' - (y_1 + x_2) y_0' + ((r_1 - q_1 y_0) x_1 - q y_1) x_1'] + \\ & + \{r_1 x_2 + r_2 x_1^2 - q_1 x_1 y_1 - (q_1 x_2 + q_2 x_1^2) y_0\}, \quad y_2(1) = 0, \end{aligned} \quad (56)$$

...

$$\begin{aligned} Ly_n = & [y_0' x_n' - y_0' x_n + f_n(y_0, \dots, y_{n-1}, x_1, \dots, x_{n-1}, y_0', \dots, y_{n-1}', x_1', \dots, x_{n-1}')] + \\ & + \{g_n(y_0, \dots, y_{n-1}, x_1, \dots, x_{n-1})\}, \quad y_n(1) = 0; \dots \end{aligned} \quad (57)$$

where $q = q_0$, $r = r_0$,

$$\begin{aligned} f_n = & -(y_0 + x_1)y'_{n-1} - (y_1 + x_2)y'_{n-2} - \dots - (y_{n-2} + x_{n-1})y'_1 - y_{n-1}y'_0 + \\ & + (r_1x_1 - qy_1 - q_1x_1y_0)x'_{n-1} + (r_1x_2 + r_2x_1^2 - qy_2 - q_1x_1y_1 - (q_1x_2 + q_2x_1^2)y_0)x'_{n-2} + \dots \\ & + (r_1x_{n-1} + 2r_2x_1x_{n-2} + 2r_2x_2x_{n-3} + \dots + r_{n-1}x_1^{n-1} - q_1y_1x_{n-2} - (q_1x_{n-1} + 2q_2x_1x_{n-2} + \dots \\ & + q_{n-1}x_1^{n-1})y'_0)x'_1, \end{aligned}$$

$$\begin{aligned} g_n = & r_1x_n + 2r_2x_1x_{n-1} + \dots + r_nx_1^n - q_1x_1y_{n-1} - (q_1x_2 + q_2x_1^2)y_{n-2} - \dots \\ & - (q_1x_n + 2q_2x_1x_{n-1} + \dots + q_nx_1^n)y_0. \end{aligned}$$

In these equations, the coefficient $r(\xi) - q(\xi)y_0(\xi)$ of the derivative $x'_n(\xi)$ ($n = 1, 2, \dots$) was replaced by Eq. (54) on $\xi y'_0(\xi)$.

From Eq. (57) for $n = 1, 2, \dots$, it follows that if we want to define functions $x_n(\xi)$ ($n = 1, 2, \dots$) from this differential equations, then we must assume that

$$\xi y'_0(\xi) = r(\xi) - q(\xi)y_0(\xi) \neq 0, \quad \xi \in (0, 1]. \quad (58)$$

And this condition cannot be avoided by applying the Lighthill method to Eq. (41). Condition (58) first appeared in [69], justifying Lighthill method, then in the works Habets [66] and Sibuya, Takahashi [68]. Comstock [65] on the example shows that the condition (58) is not necessary for the existence of solutions on the interval $[0, 1]$. Further assume that the condition (58) holds. Note that the right-handside of Eq. (57) is linear with respect to $x_n(\xi)$, and f_n function depends from $y'_0, \dots, y'_{n-1}, x'_1, \dots, x'_{n-1}$ only.

The solution of Eq. (54) can be written as

$$y_0(\xi) = \xi^{-q_0} g(\xi) (y^0 + \int_1^\xi s^{q_0-1} r(s) g^{-1}(s) ds) := \xi^{-q_0} w(\xi), \quad (59)$$

where $g(\xi) = \exp \left(\int_1^\xi (q_0 - q(s)) s^{-1} ds \right)$.

Let

$$w_0 = y^0 - \int_0^1 s^{q_0-1} r(s) g^{-1}(s) ds \neq 0 \Leftrightarrow w_0 = w(0) \neq 0.$$

Hence, we have

$$y_0(\xi) \sim \xi^{-q_0} w_0, \quad \xi \rightarrow 0. \quad (60)$$

Since the differentiation of $y_0(\xi)$ increased of its singularity at the point $\xi = 0$, it is better to choose such that the first brace in Eq. (55) is equal to zero, i.e.,

$$\xi x'_1 = x_1 + y_0, \quad x(1) = 0.$$

Hence, using Eq. (60), we obtain

$$x_1(\xi) = \xi + \xi \int_1^\xi s^{-2} y_0(s) ds \sim -\frac{w_0}{1+q_0} \xi^{-q_0}. \quad (61)$$

Then Eq. (55) takes the form

$$Ly_1 = (r_1 - q_1 y_0) x_1 \sim \tilde{a}_1 \xi^{-2q_0},$$

where $\tilde{a}_1 = \text{const}$. Hence, we have

$$y_1(\xi) \sim a_1 \xi^{-2q_0} (a_1 = \text{const}), \xi \rightarrow 0. \quad (62)$$

Now equating to zero the expression in the first brace in the right-hand side of Eq. (56), we have

$$\xi x'_2 - x_2 = y_1 + ((y_0 + x_1)y'_1 - ((r_1 - q_1 y_0)x_1 - qy_1)x'_1)(y'_0)^{-1} \sim \tilde{b}_2 \xi^{-2q_0}, \tilde{b}_2 = \text{const}.$$

From this, we get

$$x_2(\xi) \sim b_2 \xi^{-2q_0}, \quad b_2 = \text{const}, \quad \xi \rightarrow 0. \quad (63)$$

Now Eq. (56) takes the form

$$Ly_2 = g_2(y_0, y_1, x_1, x_2) \sim \tilde{a}_2 \xi^{-3q_0}, \quad \tilde{a}_2 = \text{const}, \quad \xi \rightarrow 0$$

Solving this equation, we have

$$y_2(\xi) \sim a_2 \xi^{-3q_0}, \quad a_2 = \text{const}, \quad \xi \rightarrow 0 \quad (64)$$

Next, the method of induction, it is easy to show that

$$x_j(\xi) \sim b_j \xi^{-jq_0}, \quad y_j(\xi) \sim a_j \xi^{-(j+1)q_0}, \quad j = 1, 2, \dots \quad (65)$$

Thus, the series (42) has the asymptotic

$$y(\xi) \sim \xi^{-q_0} (w_0 + a_1 \varepsilon \xi^{-q_0} + \dots + a_n (\varepsilon \xi^{-q_0})^n + \dots), \quad \xi \rightarrow 0, \quad (66)$$

$$x \sim \xi - \frac{w_0}{1+q_0} \xi^{-q_0} \varepsilon + b_2 (\varepsilon \xi^{-q_0})^2 + \dots + b_n (\varepsilon \xi^{-q_0})^n + \dots \quad (67)$$

From Eq. (67), it follows that the point $x = 0$ corresponds to the root of the equation

$$\eta + \varepsilon x_1(\eta) + \varepsilon^2 x_2(\eta) + \dots = 0 \quad (68)$$

Moreover, this equation should have a positive root and if the solution of Eq. (41) exists on the interval $(0, 1]$. Solving Eq. (68), we obtain

$$\eta \sim (w_0 \varepsilon / 1 + q_0)^{1/(1+q_0)}, \quad \varepsilon \rightarrow 0. \quad (69)$$

And, under the condition $w_0 > 0$, η_0 will be positive. It is obvious that on the interval $[\xi_0, 1]$ series (42) or (66) and (67) remains asymptotic. Substituting Eq. (69) into Eq. (66), we have

$$y(0) \sim w_0 \left(\frac{w_0 \varepsilon}{1 + q_0} \right)^{-q_0/(1+q_0)}, \quad \varepsilon \rightarrow 0.$$

If $w_0 < 0$ the point $x = 0$ does not have the positive root of Eq. (68), so that the solution of Eq. (41) goes to infinity, before reaching the point $x = 0$.

We have the

Theorem 7. Suppose that the conditions (1) $q(x), r(x) \in C^\infty[0, 1]$; (2) $q_0 > 0$; (3) $w_0 > 0$; (4) $\xi y'_0 \neq 0, \xi \in [0, 1]$. Then the solution of problem (41) exists on the interval $[0, 1]$, and it can be represented in the asymptotic series (42), (66) and (67).

Theorem 7 proved by Wasow [69], Sibuya and Takahashi [68] in the case where $q(x), r(x)$ are analytic functions on $[0, 1]$; proved by Habets [66] in the case $q(x), r(x) \in C^2[0, 1]$. Moreover, instead of the condition (3) Wasow impose a stronger condition: $a \gg 1$.

In the proof of Theorem 7, we will not stop because it is held by Majorant method.

From the foregoing, it follows that Wasow condition $y'_0(\xi) \neq 0, \xi \in (0, 1]$ is essential in the Lighthill method.

Comment 2. Prytula and later Martin [65] proposed the following variant of the Lighthill method. At first direct expansion determined using by the method of small parameter

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots \quad (70)$$

and further at second they will make transformation

$$x = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \dots \quad (71)$$

Here unknowns $x_j(\xi)$ are determined from the condition that function $y_j(\xi)$ was less singular function $y_{j-1}(\xi)$. We show that using the method Prytula or Martin, also cannot avoid Wasow conditions. Really, substituting Eq. (71) into Eq. (70) and expanding in a Taylor series in powers of ε , we have

$$y(\xi) = y_0(\xi) + \varepsilon \{y_1(\xi) + y'_0(\xi)x_1(\xi)\} + O(\varepsilon^2).$$

Hence, to obtain a uniform representation of the solution to the second order by ε , we must to put to zero the expression in the curly brackets, i.e., $x_1(\xi) = -y_1(\xi)/y'_0(\xi)$. Therefore, $y(\xi) = y_0(\xi) + O(\varepsilon^2)$. Hence, it is clear that we must make the condition of Wasow: $y'_0(\xi) \neq 0$ in the method of Prytula or Martin also.

3.3. Uniformization method for a Lighthill model equation

We will consider the problem (41) again [3, 58–60], i.e.,

$$(x + \varepsilon y(x))y'(x) = r(x) - q(x)y(x), \quad y(1) = a, \quad (72)$$

Theorem 8. Suppose that the problem (72) has a parametric representation of the solution $y = y(\xi)$, $x = x(\xi)$, where $\xi \in [\eta, 1]$, $\eta = \eta(\varepsilon) > 0$, then the problem (72) is equivalent to the problem

$$\begin{cases} \xi y'(\xi) = r(x(\xi)) - q(x(\xi))y(\xi), & y(1) = y^0, \\ \xi x'(\xi) = x(\xi) + \varepsilon y(\xi), & x(1) = 1, \quad \xi \in [\eta, 1], \end{cases} \quad (73)$$

where $\eta = \eta(\varepsilon)$ is the root equation $x(\eta) = 0$ and if the root $\eta = \eta(\varepsilon) > 0$ and $x(\xi) + \varepsilon y(\xi) \neq 0$ on the interval $[\eta, 1]$.

Proof. Sufficiency. Let the solution of the problem (72) exists and $x(\xi)$, $y(\xi)$ are a parametric representation of the solution of the problem (72). Then introducing the variable-parameter ξ , we obtain the problem (73).

Necessity. Let it fulfill the conditions of Theorem 8. Then dividing the first equation by second one, we get Eq. (72). Theorem 8 is proved.

Equation (73) on the proposal of the Temple [43], we will call uniformizing equation for the problem (72).

We have the following

Theorem 9. Suppose that the first three conditions of Theorem 8. i.e., (1) $q(x)$, $r(x) \in C^\infty[0, 1]$; (2) $q_0 > 0$; (3) $w_0 > 0$. Then the solution of problem (72) is represented in the form of an asymptotic series (42) and its solution can be obtained from uniformizing equation (73).

The proof of this theorem is completely analogous to the proof of Theorem 8, even more easily.

Only it remains to show that under the conditions of Theorem 9 we can get an explicit solution $y = y(x, \varepsilon)$. Really, since

$$\tilde{x} \xi - \frac{w_0}{1 + q_0} \xi^{-q_0} \varepsilon, \quad \xi \rightarrow 0.$$

Let

$$F(x, \xi, \varepsilon) = x - \xi + \frac{w_0}{1 + q_0} \xi^{-q_0} \varepsilon + O\left((\varepsilon \xi^{-q_0})^2\right), \quad \xi \rightarrow 0, \quad \eta = \sqrt[q_0+1]{\frac{w_0}{1 + q_0}} \varepsilon, \quad \varepsilon \rightarrow 0.$$

then

$$\frac{\partial F(x, \xi, \varepsilon)}{\partial \xi} \Big|_{\xi=\eta(\varepsilon)} = -1 - q_0 + O\left(\varepsilon^{1/(1+q_0)}\right) \neq 0, \quad \xi \in [\eta, 1].$$

Therefore, by the implicit function theorem, we can express ξ : $\xi = \varphi(x, \varepsilon)$.

Then when we put it in first equality (42), we obtain an explicit solution $y = y(x, \varepsilon)$.

Comment 3. Explicit asymptotic solution that this problem obtained in Section 3.4.

Example 43. Uniformized equation is

$$\begin{cases} \xi y'(\xi) = -y(\xi), & y(1) = b, \\ \xi x'(\xi) = x(\xi) + \varepsilon y(\xi), & x(1) = 1, \end{cases} \quad \xi \in [\eta, 1],$$

It is easy to integrate this system, and we obtain

$$y(\xi) = b\xi^{-1}, \quad x(\xi) = (1 + 2^{-1}b\varepsilon)\xi - (2\xi)^{-1}b\varepsilon,$$

Hence, excluding variable ξ , we have an exact solution (44).

Example 2 [37, 43])

$$(x + \varepsilon y(x))y'(x) + (2 + x)y(x) = 0, \quad y(1) = e^{-1}.$$

Uniformized equation is

$$\begin{cases} \xi x'(\xi) = x + \varepsilon y(\xi), & x(1) = 1, \\ \xi y'(\xi) = -(2 + x(\xi))y(\xi), & y(1) = e^{-1}, \end{cases} \quad \xi \in [\eta, 1], \quad (74)$$

Let

$$\begin{cases} x(\xi) = x_0(\xi) + \varepsilon x_1(\xi) + O(\varepsilon^2), \\ y(\xi) = y_0(\xi) + \varepsilon y_1(\xi) + O(\varepsilon^2), \end{cases} \quad (75)$$

Substituting Eq. (75) into Eq. (74), we have

$$x_0(\xi) = \xi, \quad x_1(\xi) = \xi \int_1^\xi e^{-s} s^{-4} ds, \quad y_0(\xi) = e^{-\xi} \xi^{-2}, \quad y_1(\xi) = -e^{-\xi} \xi^{-2} \int_1^\xi e^{-s} s^{-4} ds,$$

Hence if $\xi \rightarrow 0$, we obtain

$$x_0(\xi) = \xi, \quad x_1(\xi) = -\frac{1}{3}\xi^{-2} + \dots, \quad y_0(\xi) = \xi^{-2} + \dots, \quad y_1(\xi) = -\frac{1}{6}\xi^{-4} + \dots$$

From the equation $x(\eta) = 0$, we find $\eta : \eta \sim \sqrt[3]{\varepsilon/3}$.

We prove that $x(\xi) + \varepsilon y(\xi) \neq 0$ on the interval $[\eta, 1]$.

Really,

$$x(\xi) + \varepsilon y(\xi) \sim \xi + \varepsilon \xi^{-2} \neq 0, \quad \xi \in [\eta, 1].$$

3.4. It is construction explicit form of the solution of the model Lighthill equation

We will consider the problem [57], i.e., (41) again

$$(x + \varepsilon y(x))y'(x) + q(x)y(x) = r(x), \quad y(1) = b \quad (76)$$

where b is given constant, $x \in [0, 1]$, $y'(x) = dy/dx$. Given functions are subjected to the conditions $U: q(x), r(x) \in C^{(\infty)}[0, 1]$.

Here, we consider the case $q_0 = -1$; this is done to provide a detailed illustration of the idea of the application of the method. We search for the solution of problem (76) in the form

$$y(x) = \mu^{-1}\pi_{-1}(t) + \sum_{k=0}^{\infty} (\pi_k(t) + u_k(x))\mu^k, \quad (77)$$

where $t = x/\mu$, $\varepsilon = \mu^2$, $u_k(x) \in C^{(\infty)}[0, 1]$ and $\pi_k(t) \in C^{(\infty)}[0, \mu_0]$, $\mu_0 = 1/\mu$.

Note that $\pi_k(t) = \pi_k(t, \mu)$, i.e., $\pi_k(t)$ depends also on μ , but this dependence is not indicated.

The initial conditions for the functions $\pi_j(t)$ are taken as

$$\pi_{-1}(1/\mu) = b\mu, \quad b = u^0 - \sum_{k=0}^{\infty} \mu^k u_k(1), \quad \pi_k(\mu_0) = 0, \quad k = 0, 1, \dots \quad (78)$$

Substituting Eq. (77) into Eq. (76), we obtain to determine the functions $\pi_k(t)$, $k = -1, 0, 1, \dots$, $u_n(x)$, $n = 0, 1, \dots$,

we have the following equations:

$$(t + \pi_{-1}(t))\pi'_{-1}(t) = q(\mu t)\pi_{-1}(t), \quad \pi_{-1}(\mu_0) = b\mu, \quad (79.-1)$$

$$Lu_0(x) := xu'_0(x) - q(x)u_0(x) = r(x), \quad u_0(x) \in C^{(\infty)}[0, 1] \quad (80.0)$$

$$D\pi_0(t) := (t + \pi_{-1}(t))\pi'_0(t) + (\pi'_{-1}(t) - q(\mu t))\pi_0(t) = -u_0(t\mu)\pi'_{-1}(t), \quad \pi_0(\mu_0) = 0 \quad (79.0)$$

$$Lu_1(x) = 0, \quad u_1(x) \in C^{(\infty)}[0, 1], \quad (80.1)$$

$$D\pi_1(t) = -u_0(t\mu)\pi'_0(t) + \pi_0(t)\pi'_0(t) - u_1(t\mu)\pi'_{-1}(t), \quad \pi_1(\mu_0) = 0 \quad (79.1)$$

$$Lu_2(x) := -u_0(x)u'_0(x), \quad u_2(x) \in C^{(\infty)}[0, 1] \quad (80.2)$$

$$D\pi_2(t) := -u_0(t\mu)\pi'_{-1}(t) - \pi_0(t)\pi'_1(t) - u_1(t\mu)\pi'_0(t) - \pi_1(t)\pi'_0(t) - u_2(t\mu)\pi'_{-1}(t), \quad \pi_2(\mu_0) = 0 \quad (79.2)$$

$$Lu_3(x) := -u_0(x)u'_1(x) - u'_0(x)u_1(x), \quad u_3(x) \in C^{(\infty)}[0, 1], \quad (80.3)$$

$$D\pi_3(t) = \sum_{\substack{i+j=2 \\ i \geq 0, j \geq -2}} u_i(\mu t)\pi'_j(t) + \sum_{\substack{i+j=2 \\ i, j \geq 0}} \pi_i(t)\pi'_j(t), \quad \pi_3(\mu_0) = 0, \quad (79.3)$$

We solve these problems successively. We write problem (79.-1) as

$$tz'(t) - q(\mu t)z(t) = -z(t)z'(t), \quad z(\mu_0) = b\mu,$$

where

$$z = \pi_{-1}(t), \quad \mu_0 = \mu^{-1}.$$

The fundamental solution of the homogeneous equation corresponding to this equation is of the form

$$z^0(t) = \exp \left\{ \int_{\mu_0}^t q(\mu s) \frac{ds}{s} \right\} = \exp \left\{ \int_{\mu_0}^t (q(\mu s) + 1) \frac{ds}{s} - \int_{\mu_0}^t \frac{ds}{s} \right\} = \frac{p(t, \mu)}{\mu t},$$

where

$$p(t, \mu) = \exp \left\{ \int_{\mu_0}^t (q(\mu s) + 1) \frac{ds}{s} \right\}.$$

Using the expression for $z^0(t)$, the solution of the inhomogeneous equation for $z(t)$ can be written as

$$z(t) = \frac{p(t, \mu)}{\mu t} [z(\mu_0) + \mu \int_{\mu_0}^t p^{-1}(s, \mu) z(s) z'(s) ds],$$

$$\text{Or } tz(t) = p(t, \mu)b - p(t, \mu) \int_{\mu_0}^t p^{-1}(s, \mu) z(s) z'(s) ds.$$

After integrating by parts, we reduce the last expression to the following equation:

$$tz(t) = p(t, \mu)b - \frac{z^2(t)}{2} + p(t, \mu) \frac{b^2 \mu^2}{2} + \frac{p(t, \mu)}{2} \int_{\mu_0}^t \frac{1 + q(\mu s)}{s} p^{-1}(s, \mu) z^2(s) ds$$

or

$$z^2(t) + 2tz(t) - p(t, \mu)b_0 = p(t, \mu) \int_{\mu_0}^t \phi(s, \mu) p^{-1}(s, \mu) z^2(s) ds := p(t, \mu)T(t, z^2) \quad (81)$$

where $\phi(s, \mu) = (1 + q(\mu s))/s$, $b_0 = 2b + b^2 \mu^2$.

Let $b_0 > 0$. Let us introduce the notation $z_0(t) = -t + \sqrt{t^2 + b_0 p(t, \mu)}$. This function satisfies the inequality $0 < z_0(t) \leq Mt^{-1}$ ($t > 0$) and is a strictly decreasing bounded function on the closed interval $[0, \mu_0]$. Here and elsewhere, all constants independent of the small parameter μ are denoted by M . Let S_μ be the set of functions $z(t)$ satisfying the condition

$$\|z - z_0\| \leq M\mu, \quad \text{where } \|z\| = \max_{0 \leq t \leq \mu_0} |z(t)|,$$

Theorem 10. If $b_0 > 0$, then there exists a unique constraint of the solution of problem (79.-1) from the set S_μ .

Proof. Equation (81) is equivalent to the equation $z = F[t, z]$, where

$$F[t, z] = -t + \sqrt{t^2 + bp(t, \mu) + p(t, \mu)T(t, z^2)}.$$

Suppose that $\|\varphi(t, \mu)\| \leq M\mu$, $0 < m \leq p(t, \mu) \leq M$, $\|p^{-1}(t)\| \leq M$. First, let us estimate $T(t, z^2)$ on the set S_μ . We have

$$\begin{aligned} |T(t, z^2)| &\leq \int_t^{\mu_0} |\varphi(s, \mu)| |p^{-1}(s, \mu)| |z(s)|^2 ds \leq M\mu \int_t^{\mu_0} |z(s)|^2 ds \leq M\mu \int_0^{\mu_0} |z(s)|^2 ds \leq \\ &\leq M\mu \int_0^1 |z(s)|^2 ds + M\mu \int_1^{\mu_0} |z(s)|^2 ds \leq M\mu. \end{aligned}$$

Here, we have used the triangle inequality

$$|z(t)| \leq |z(t) - z_0(t)| + |z_0(t)|,$$

as well as the inequality

$$|z_0(t)| \leq Mt^{-1} \quad (t > 0).$$

The Fréchet derivative of the operator $F(t, z)$ with respect to z at the point $z_0(t)$ is a linear operator:

$$F'_z(t, z_0)h = -p(t, \mu) \int_t^{\mu_0} \varphi(s, \mu) p^{-1}(s, \mu) z_0(s) h(s) \frac{ds}{\sqrt{t^2 + p(t, \mu)(b + T(t, z^2))}},$$

where $h(t)$ is a continuous function on the closed interval $[0, \mu_0]$. Note that, in view of $T(t, z_0^2) = O(\mu)$, the denominator of this expression is strictly positive on the closed interval $[0, \mu_0]$. For $F'_z(t, z_0)$, we can obtain the estimate $\|F'_z(t, z_0)\| \leq M\mu \ln \mu^{-1}$ in the same way as the estimate for $T(t, z^2)$. Hence, in turn, it follows from the Lagrange inequality that the operator is a contraction operator in the set S_μ . Therefore, by the fixed-point principle, Eq. (81) has a unique solution from the class S_μ . The theorem is proved.

Corollary. The following inequalities hold:

1. $z(t) = \pi_{-1}(t) \geq M > 0$ for all $t \in [0, \mu_0]$;
2. $\pi_{-1}(t) \leq Mt^{-1} \quad (t > 0)$.

The other function $\pi_j(t)$, $u_j(x)$, $j = 0, 1, 2, \dots$ is determined from the inhomogeneous linear equations; therefore, the following lemmas are needed.

Lemma 9. For any function $f(x) \in C^{(\infty)}[0, 1]$, the equation $L\xi = f(x)$ has a unique bounded solution $\xi(x) \in C^{(\infty)}[0, 1]$ expressible as

$$\xi(x) = Q(x) \int_0^x Q^{-1}(s) f(s) \frac{ds}{x}, \quad Q(x) = \exp \left\{ \int_1^x (q(s) + 1) \frac{ds}{s} \right\}.$$

Proof. The proof follows from the fact that the general solution of the equation under consideration is expressed as

$$\xi(x) = Q(x)x^{-1}[\xi(1) + \int_1^x Q^{-1}(s)f(s)ds].$$

If we choose

$$\xi(1) = \int_0^1 Q^{-1}(s)f(s)ds.$$

then we obtain the required result.

This lemma implies that all the functions $u_k(x)$, $k = 0, 1, \dots$ are uniquely determined and belong to the class $C^\infty[0, 1]$.

Lemma 10. *The problem*

$$(t + \pi_{-1}(t))\eta'(t) + (\pi'_{-1}(t) - q(\mu t))\eta(t) = k(t), \quad \eta(\mu_0) = 0, \quad (82)$$

where the function $k(t)$ belongs to $C^\infty[0, 1]$ is continuous and bounded, and if $|k(t)| \leq Mt^{-2}$, $t \rightarrow \infty$, has a unique uniformly bounded solution $\eta(t) = \eta(t, \mu)$ on the closed interval $t \in [0, \mu_0]$ for a small μ .

Proof. The fundamental solution of the homogeneous equation (82) is of the form

$$\Phi(t) = \frac{(1 + \mu^2 b)g(t, \mu)}{\mu(t + \pi_{-1}(t))}, \quad g(t, \mu) = \exp \left\{ - \int_t^{\mu_0} (1 + q(\mu s)) \frac{ds}{s + \pi_{-1}(s)} \right\}.$$

Obviously, $\|g(t, \mu)\| \leq M$ and $g^{-1}(t, \mu) \leq M$ for $t \in [0, \mu_0]$ and μ is small. The solution of problem (82) can be expressed as

$$\eta(t) = \frac{g(t, \mu)}{t + \pi_{-1}(t)} \int_{\mu_0}^t g^{-1}(s, \mu) k(s) ds. \quad (83)$$

The estimate of the integral term in Eq. (83) shows that it is bounded by the constant M . Hence, it also follows that $|\eta(t)| \leq Mt^{-1}$ ($t > 0$). The solution of problem (79.0) is defined by the integral Eq. (83), where

$$k(t) = -u_0(t\mu)\pi_{-1}(t) = -u_0(t\mu)q(\mu t) \frac{\pi_{-1}(t)}{t + \pi_{-1}(t)},$$

satisfies the assumptions of the lemma. Therefore, the function $\pi_0(t)$ is bounded on $[0, \mu_0]$. The boundedness of the other functions $\pi_k(t)$, $k = 1, 2, \dots$ is proved in a similar way, because the

right-hand sides of the equations defining these functions satisfy the assumptions of Lemma 10. The estimate of the asymptotic behavior of the series (77) is also carried out using Lemma 10.

Let us introduce the notation

$$y(x) = \mu^{-1}\pi_{-1}(t) + \sum_{k=0}^n \mu^k (\pi_k(t) + u_k(x)) + \mu^{n+1} R_{n+1}(x, \mu). \quad (84)$$

The following statement holds.

Theorem 11. *Let $b_0 > 0$ (for this, it suffices that the condition $b_0 := b - y_0(1) > 0$ holds). Then the solution of problem (76) exists on the closed interval $[0, 1]$ and its asymptotics can be expressed as Eq. (84) and $|R_{n+1}(x, \mu)| \leq M$ for all $x \in [0, 1]$.*

Example. Consider the equation

$$(x + \varepsilon y(x))y'(x) + y(x) = 1, \quad y(1) = b,$$

This equation is integrated exactly

$$y(x) = \varepsilon^{-1} \left[-x + \sqrt{x^2 + 2b_0\varepsilon + \varepsilon^2 (y^{(0)})^2 + 2\varepsilon x} \right],$$

where $b_0 = b - 1$. If $b_0 > 0$, then the solution of problem (1) exists on the closed interval $[0, 1]$, which is confirmed by Theorem 11. The equation for $\pi_{-1}(t)$ is of the form

$$(t + \pi_{-1}(t))\pi'_{-1}(t) + \pi_{-1}(t) = 0, \quad \pi_{-1}(\mu_0) = b\mu.$$

The solution of this problem can be expressed as

$$\pi_{-1}(t) = -t + \sqrt{t^2 + 2b + b^2\mu^2}.$$

The equation for $u_0(x)$ has the solution $y_0(x) = 1 \in C^\infty[0, 1]$. Further,

$$\pi_0(t) = \frac{-\pi_{-1}(t) + b\mu}{t + \pi_{-1}(t)}, \quad u_k(x) = 0, \quad k = 1, 2, \dots,$$

where $b = b_0$. The asymptotics of the solutions of problem (76) can be expressed as

$$y(x) = \mu^{-1}\pi_{-1}(x/\mu) + 1 + \pi_0(x/\mu) + o(\mu) \text{ for all } x \in [0, 1], \quad \mu \rightarrow 0.$$

4. Lagerstrom model problem

The problem [32]

$$v''(r) + \frac{k}{r}v'(r) + v(r)v'(r) = \beta[v'(r)]^2, \quad v(\varepsilon) = 0, \quad v(\infty) = 1, \quad (85)$$

where $0 < \beta$ is constant, $k \in \mathbb{N}$.

It has been proposed as a model for Lagerstrom Navier-Stokes equations at low Reynolds numbers. It can be interpreted as a problem of distribution of a stationary temperature $v(r)$.

The first two terms in Eq. (1) is $(k+1)$ dimensional Laplacian depending only on the radius, and the other two members—some nonlinear heat loss.

It turns out that not only the asymptotic solution but also convergent solutions of Eq. (1) can be easily constructed by a fictitious parameter [70]. The basic idea of this method is as follows. The initial problem is entered fictitious parameter $\lambda \in [0, 1]$ with the following properties:

1. $\lambda = 0$, the solution of the equation satisfies all initial and boundary conditions;
2. The solution of the problem can be expanded in integral powers of the parameter λ for all $\lambda \in [0, 1]$.

It is convenient in Eq. (85) to make setting $r = \varepsilon x$, $v = 1 - u$, then

$$u''(x) + (kx^{-1} + \varepsilon)u'(x) - \lambda \varepsilon u(x)u'(x) = [u'(x)]^2, \quad u(1) = 1, \quad u(\infty) = 0. \quad (86)$$

We have the following

Theorem 12. For small $\varepsilon > 0$, the solution of problem (86) can be represented in the form of absolutely and uniformly convergent series

$$u(x) = u_0(x, \varepsilon) + v_k(\varepsilon)u_1(x, \varepsilon) + \dots + v_k^n(\varepsilon)u_n(x, \varepsilon) + \dots,$$

for the sufficiently small parameter ε , where

$$v_1(\varepsilon) \sim \left(\ln \frac{1}{\varepsilon}\right)^{-1}, \quad v_2 \sim \varepsilon \ln \frac{1}{\varepsilon}, \quad v_k \sim \frac{k-1}{k-2} \varepsilon (j > 2); \quad u_k(x, \varepsilon) = O(1), \quad \forall x \in [1, \infty)$$

Note that the function $u_n(x, \varepsilon)$ also depends on k , but for simplicity, this dependence is not specified.

Proof. We introduce Eq. (86) parameter λ , i.e., consider the problem

$$u''(x) + (kx^{-1} + \varepsilon)u'(x) - \beta[u'(x)]^2 = \lambda \varepsilon u(x)u'(x), \quad u(1) = 1, \quad u(\infty) = 0 \quad (87)$$

Here, we will prove this Theorem 12 in the case $\beta = 0$ only for simplicity.

Setting $\lambda = 0$ in Eq. (87), we have

$$u''_0 + (x^{-1}k + \varepsilon)u'_0 = 0, \quad u_0(1) = 1, \quad u_0(\infty) = 0. \quad (88)$$

It has a unique solution

$$u_0 = X(x, \varepsilon) := 1 - X_1(X, \varepsilon), \quad X_1 = C_0 \int_1^x s^{-k} e^{-\varepsilon s} ds, \quad C_0^{-1} = \int_1^\infty s^{-k} e^{-\varepsilon s} ds.$$

Therefore, Eq. (88) with zero boundary conditions is the Green's function

$$K(x, s, \varepsilon) = \begin{cases} C_0^{-1} X_1(x, \varepsilon) X(s, \varepsilon), & 1 \leq x \leq s, \\ C_0^{-1} X_1(s, \varepsilon) X(x, \varepsilon), & s < x < \infty. \end{cases}$$

Hence, the problem (87) is reduced to the system of integral equations

$$\begin{aligned} u(x) &= X(x, \varepsilon) + \lambda \varepsilon \int_1^\infty G(x, s, \varepsilon) u(s) u'(s) ds, \\ u'(x) &= X'(x, \varepsilon) + \lambda \varepsilon \int_1^\infty G_x(x, s, \varepsilon) u(s) u'(s) ds, \end{aligned} \quad (89)$$

where

$$G(x, s, \varepsilon) = \begin{cases} X_1(x, \varepsilon) X(s, \varepsilon) / X'(s, \varepsilon), & 1 \leq x \leq s, \\ X_1(s, \varepsilon) X(x, \varepsilon) / X'(s, \varepsilon), & s < x < \infty. \end{cases}$$

In Eq. (89), we make the substitution $u = X(x, \varepsilon)\varphi(x)$, $u' = X'(x, \varepsilon)\psi(x)$, and then we have

$$\begin{aligned} \varphi(x) &= 1 + \lambda \varepsilon \int_1^\infty Q_1(x, s, \varepsilon) \varphi(s) \psi(s) ds := 1 + \lambda \varepsilon Q_1(\varphi\psi), \\ \psi(x) &= 1 + \lambda \varepsilon \int_1^\infty Q_2(x, s, \varepsilon) \varphi(s) \psi(s) ds := 1 + \lambda \varepsilon Q_2(\lambda\psi), \end{aligned} \quad (90)$$

where

$$\begin{aligned} Q_1 &= X^{-1}(x, \varepsilon) G(x, s, \varepsilon) X(s, \varepsilon) X'(s, \varepsilon), \\ Q_2 &= X_x^{-1}(x, \varepsilon) G_x(x, s, \varepsilon) X(s, \varepsilon) X'(s, \varepsilon). \end{aligned}$$

To prove the theorem, we need next

Lemma 11. The following estimate holds

$$\int_1^\infty |Q_j(x, s, \varepsilon)| ds \leq \int_1^\infty X(s, \varepsilon) ds \quad (j = 1, 2) \quad (91)$$

Given that, we have $0 \leq X_1(x, \varepsilon) \leq 1$, $|X'(x, \varepsilon)| = X'(x, \varepsilon)$, $X'(x, \varepsilon) \leq 0$, $x \in [1, \infty)$, we have

$$\begin{aligned} \int_1^\infty |Q_1(x, s, \varepsilon)| ds &\leq \int_1^x \frac{X_1(s, \varepsilon)}{X'_1(s, \varepsilon)} |X'(s, \varepsilon)| X(s, \varepsilon) ds + \\ &+ \int_x^\infty X^{-1}(x, \varepsilon) \frac{X^2(s, \varepsilon) |X'(s, \varepsilon)|}{X'_1(s, \varepsilon)} ds \leq \int_1^x X(s, \varepsilon) ds + \int_x^\infty X(s, \varepsilon) ds = \int_1^\infty X(s, \varepsilon) ds. \end{aligned}$$

Inequality Eq. (91) for $j = 2$ is proved similarly.

Further, by integrating by parts, we have

$$\int_1^\infty X(s, \varepsilon) ds = -1 + C_0 \int_1^\infty s^{-k+1} e^{-\varepsilon s} ds \leq \int_1^\infty s^{-k+1} e^{-\varepsilon s} ds / \int_1^\infty s^{-k} e^{-\varepsilon s} ds := \frac{v_k(\varepsilon)}{\varepsilon}.$$

Consequently,

$$\varepsilon \int_1^\infty X(x, \varepsilon) ds \leq v_k(\varepsilon). \quad (92)$$

It is from integral expressing of $v_k(\varepsilon)$ we can obtain the asymptotic behavior such as indicated in the theorem.

With the solution of Eq. (90), we can expand in series

$$\begin{aligned} \varphi(x) &= 1 + \varphi_1(x, \varepsilon)\lambda + \varphi_2(x, \varepsilon)\lambda^2 + \dots, \\ \Psi(x) &= 1 + \Psi_1(x, \varepsilon)\lambda + \Psi_2(x, \varepsilon)\lambda^2 + \dots \end{aligned}$$

The coefficients of this series are uniquely determined from the equations

$$\begin{aligned} \varphi_0 &= \Psi_0 = 1, \quad \varphi_1 = \varepsilon Q_1(1), \quad \Psi_1 = Q_2(1), \\ \varphi_n &= \varepsilon Q_1(\varphi_{n-1}) + \varepsilon Q_1(\Psi_{n-1}) + \varepsilon Q_1(\varphi_1 \Psi_{n-2}) + \dots + \varepsilon Q_1(\varphi_{n-2} \Psi_1), \\ \Psi_n &= \varepsilon Q_2(\varphi_{n-1}) + \varepsilon Q_2(\Psi_{n-1}) + \varepsilon Q_2(\varphi_1 \Psi_{n-2}) + \dots + \varepsilon Q_2(\varphi_{n-2} \Psi_1), \quad (n = 2, 3, \dots). \end{aligned}$$

Let $z = \sup_{1 \leq x < \infty} \{|\varphi(x)|, |\Psi(x)|\}$, then by using Eq. (92) we have a Majorant equation:

$z = 1 + \lambda v_k(\varepsilon) z^2$. The solution of this equation can be expanded in powers λ the under condition $8v_k(\varepsilon) \leq 1$ for all $\lambda \in [0, 1]$.

If we call $u_n(x, \varepsilon) = \frac{X(x, \varepsilon)\varphi_n(x, \varepsilon)}{v_k^n(\varepsilon)}$, we get the proof of the theorem.

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