Chapter 2

Homotopy Asymptotic Method and Its Application

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/67876

Abstract

As we all know, perturbation theory is closely related to methods used in the numerical analysis fields. In this chapter, we focus on introducing two homotopy asymptotic methods and their applications. In order to search for analytical approximate solutions of two types of typical nonlinear partial differential equations by using the famous homotopy analysis method (HAM) and the homotopy perturbation method (HPM), we consider these two systems including the generalized perturbed Kortewerg-de Vries-Burgers equation and the generalized perturbed nonlinear Schrödinger equation (GPNLS). The approximate solution with arbitrary degree of accuracy for these two equations is researched, and the efficiency, accuracy and convergence of the approximate solution are also discussed.

Keywords: homotopy analysis method, homotopy perturbation method, generalized KdV-Burgers equation, generalized perturbed nonlinear Schrödinger equation, approximate solutions, Fourier transformation

1. Introduction

In the past decades, due to the numerous applications of nonlinear partial differential equations (NPDEs) in the areas of nonlinear science [1, 2], many important phenomena can be described successfully using the NPDEs models, such as engineering and physics, dielectric polarization, fluid dynamics, optical fibers and quantitative finance and so on [3–5]. Searching for analytical exact solutions of these NPDEs plays an important and a significant role in all aspects of this subject. Many authors presented various powerful methods to deal with this problem, such as inverse scattering transformation method, Hirota bilinear method, homogeneous balance method, Bäcklund transformation, Darboux transformation, the generalized Jacobi elliptic function expansion method, the mapping deformation method and so on [6–10]. But once people noticed the complexity of nonlinear terms of NPDEs, they could not find the exact analytic solutions for many of them, especially with disturbed terms. Researchers had to develop some approximate and numerical methods for nonlinear theory; a great deal of efforts has been proposed for these problems, such as the multiple-scale method, the variational iteration method, the indirect matching method, the renormalization method, the Adomian decomposition method (ADM), the generalized differential transform method and so forth [11–13], among them the perturbation method [14], including the regular perturbation method, the singular perturbation method and the homotopy perturbation method (HPM) and so on.

Perturbation theory is widely used in numerical analysis as we all know. The earliest perturbation theory was built to deal with the unsolvable mathematical problems in the calculation of the motions of planets in the solar system [15]. The gradually increasing accuracy of astronomical observations led to incremental demands in the accuracy of solutions to Newton's gravitational equations, which extended and generalized the methods of perturbation theory. In the nineteenth century, Charles-Eugène Delaunay discovered the problem of small denominators which appeared in the nth term of the perturbative expansion when he was studying the perturbative expansion for the Earth-Moon-Sun system [16]. These well-developed perturbation methods were adopted and adapted to solve new problems arising during the development of Quantum Mechanics in the twentieth century. In the middle of the twentieth century, Richard Feynman realized that the perturbative expansion could be given a dramatic and beautiful graphical representation in terms of what are now called Feynman diagrams [17]. In the late twentieth century, because the broad questions about perturbation theory were found in the quantum physics community, including the difficulty of the nth term of the perturbative expansion and the demonstration of the convergent about the perturbative expansion, people had to pay more attention to the area of non-perturbative analysis, and much of the theoretical work goes under the name of quantum groups and non-commutative geometry [18]. As we all know, the solutions of the famous Korteweg-de Vries (KdV) equation cannot be reached by perturbation theory, even if the perturbations were carried out. Now, we can divide the perturbation theory to regular and singular perturbation theory; singular perturbation theory concerns those problems which depend on a parameter (here called ε) and whose solutions at a limiting value have a non-uniform behavior when the parameter tends to a pre-specified value. For regular perturbation problems, the solutions converge to the solutions of the limit problem as the parameter tends to the limit value. Both of these two methods are frequently used in physics and engineering today. There is no guarantee that perturbative methods lead to a convergent solution. In fact, the asymptotic series of the solution is the norm. In order to obtain the perturbative solution, we involve two distinct steps in general. The first is to assume that there is a convergent power asymptotic series about the parameter ε expressing the solution; then, the coefficients of the nth power of ε exist and can be computed via finite computation. The second step is to prove that the formal asymptotic series converges for ε small enough or to at least find a summation rule for the formal asymptotic series, thus providing a real solution to the problem.

The homotopy analysis method (HAM) was firstly proposed in 1992 by Liao [19], which yields a rapid convergence in most of the situations [20]. It also showed a high accuracy to solutions of the nonlinear differential systems. After this, many types of nonlinear problems were solved with HAM by others, such as nonlinear Schrödinger equation, fractional KdV-

Burgers-Kuramoto equation, a generalized Hirota-Satsuma coupled KdV equation, discrete KdV equation and so on [21–24]. With this basic idea of HAM (as $\hbar = -1$ and $H(x, t) = 1$), Jihuan He proposed the homotopy perturbation method(HPM) [25] which has been widely used to handle the nonlinear problems arising in the engineering and mathematical physics [26, 27].

In this chapter, we extend the applications of HAM and HPM with the aid of Fourier transformation to solve the generalized perturbed KdV-Burgers equation with power-law nonlinearity and a class of disturbed nonlinear Schrödinger equations in nonlinear optics. Many useful results are researched.

1.1. The homotopy analysis method (HAM)

Let us consider the following nonlinear equation

$$N[u(x, t)] = 0, \tag{1}$$

where N is a nonlinear operator, $u(x, t)$ is an unknown function and $x$and t denote spatial and temporal independent variables, respectively.

With the basic idea of the traditional homotopy method, we construct the following zero-order deformation equation

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] = q\hbar H(x, t)N[\phi(x, t; q)] \tag{2}$$

where $\hbar \neq 0$ is a non-zero auxiliary parameter, $q \in [0, 1]$ is the embedding parameter, $H(x, t)$ is an auxiliary function, L is an auxiliary linear operator, $\tilde{u}_0(x, t)$ is an initial guess of $u(x, t)$ and $\phi(x, t; q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x, t; 0) = u_0(x, t), \phi(x, t; 1) = u(x, t). \tag{3}$$

Thus, as q increases from 0 to 1, the solution $\phi(x, t; q)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Expanding $\phi(x, t; q)$ in Taylor series with respect to q, we have

$$\begin{aligned} \phi(x, t; q) &= u_0 + \sum_{m=1}^{\infty} u_m q^m \\ &= u_0 + qu_1 + q^2 u_2 + \cdots; u_0 = \tilde{u}_0(x, t), u_m = u_m(x, t). \end{aligned} \tag{4}$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m}{\partial q^m} \phi(x, t; q)|_{q = 0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are so properly chosen such that they are smooth enough, the Taylor's series (4) with respect to q converges at $q = 1$, and we have

$$u = \phi(x, t; 1) = \sum_{m=0}^{\infty} u_m, \tag{6}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao. As $\hbar = -1$ and $H(x, t) = 1$, Eq. (2) becomes

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] + qN[\phi(x, t; q)] = 0. \tag{7}$$

Eq. (7) is used mostly in the HPM, whereas the solution is obtained directly, without using Taylor's series. As $H(x, t) = 1$, Eq. (2) becomes

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] = q\hbar N[\phi(x, t; q)], \tag{8}$$

which is used in the HAM when it is not introduced in the set of base functions. According to definition (5), the governing equation can be deduced from Eq. (2). Define the vector

$$\vec{u}_m(x, t) = \{u_0, u_1, u_2, \cdots, u_m\}. \tag{9}$$

Differentiating Eq. (2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(x, t) R_{m-1}(\vec{u}_{m-1}, x, t), \tag{10}$$

where

$$R_{m-1}(\vec{u}_{m-1}, x, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N[\phi(x, t; q)]|_{q=0}. \tag{11}$$

And

$$\chi_m = \begin{cases} 0, x \leq 1 \\ 1, x \geq 2 \end{cases}. \tag{12}$$

It should be emphasized that $u_m(x, t)$ for $m \geq 1$ is governed by the linear Eq. (10) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Mathematica and Matlab.

1.2. The homotopy perturbation method

To illustrate the basic concept of the homotopy perturbation method, consider the following nonlinear system of differential equations with boundary conditions

$$\begin{cases} A(u) = f(r), r \in \Omega, & (13.1) \\ B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma = \partial\Omega & (13.2) \end{cases}, \tag{13}$$

where B is a boundary operator and Γ is the boundary of the domain Ω, $f(r)$ is a known analytical function. The differential operator A can be divided into two parts, L and N, in general, where L is a linear and N is a nonlinear operator. Eq. (13) can be rewritten as follows:

$$L(u) + N(u) = f(r). \tag{14}$$

We construct the following homotopy mapping $H(\phi, q){:}\Omega \times [0, 1] \rightarrow R$, which satisfies

$$H(\phi, q) = (1 - q)[L(v) - L(\tilde{u}_0)] + q[A(v) - f(r)] = 0, q \in [0, 1], r \in \Omega, \tag{15}$$

where $\tilde{u}_0$is an initial approximation of Eq. (13), and is the embedding parameter; we have the following power series presentation for ϕ,

$$\phi = \sum_{i=0}^{\infty} u_i(x, t) q^i = u_0 + q u_1 + q^2 u_2 + \cdots. \tag{16}$$

The approximate solution can be obtained by setting $q = 1$, that is

$$u = \lim_{q \to 1} \phi = u_0 + u_1 + u_2 + \cdots. \tag{17}$$

If we let $u_0(x, t) = \tilde{u}_0(x, t)$,notice the analytic properties of f, L, $\tilde{u}_0$ and mapping (15), we know that the series of (17) is convergence in most cases when $q \in [0, 1]$ [28]. We obtain the solution of Eq. (13).

To study the convergence of the method, let us state the following theorem.

Theorem (Sufficient Condition of Convergence).

Suppose that X and Y are Banach spaces and $N : X \rightarrow Y$ is a contract nonlinear mapping that is

$$\forall u, u* \in X : \|N(u) - N(u*)\| \leq \gamma \|u - u*\|, 0 < \gamma < 1. \tag{18}$$

Then, according to Banach's fixed point theorem, N has a unique fixed point u, that is $N(u) = u$. Assume that the sequence generated by homotopy perturbation method can be written as

$$U_n = N(U_{n-1}), U_n = \sum_{i=0}^{n} u_i, u_i \in X, n = 1, 2, 3, \cdots, \tag{19}$$

and suppose that

$$U_0 = u_0 \in B_r(u), B_r(u) = \{u* \in X | \|u* - u\| < \gamma\} \tag{20}$$

$$\text{then, we have (i) } U_n \in B_r(u), \text{ (ii)} \lim_{n \to \infty} U_n = u. \tag{21}$$

Proof. (i) By inductive approach, for $n = 1$, we have

$\|U_1 - u\| = \|N(U_0) - N(u)\| \le \gamma \|U_0 - u\|$ and then

$$\|U_n - u\| = \|N(U_{n-1}) - N(u)\| \le \gamma^n \|U_0 - u\| \le \gamma^n r \Rightarrow U_n \in B_r(u)$$

(ii) Because of $0 < \gamma < 1$, we have $\lim_{n\to\infty} \|U_n - u\| = 0$ that is $\lim_{n\to\infty} U_n = u$.

2. Application to the generalized perturbed KdV-Burgers equation

Consider the following generalized perturbed KdV-Burgers equation

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xx} + \delta u_{xxx} = f(t, x, u). \tag{22}$$

where $\alpha, \beta, \gamma, \delta, p$ are arbitrary constants, and $f = f(t, x, u)$ is a disturbed term, which is a sufficiently smooth function in a corresponding domain.

This equation with $p \ge 1$ is a model for long-wave propagation in nonlinear media with dispersion and dissipation. Eq. (22) arises in a variety of physical contexts which include a number of equations, and many valuable results about Eq. (22) have been studied by many authors in [29–31]. In fact, if one takes different value of $\alpha, \beta, \gamma, \delta, p$ and f, Eq.(22) represents a large number of equations, such as KdV equation, MKdV equation, CKdV equation, Burgers equation, KdV-Burgers equation and the equations as the following forms.

Fitzhugh-Nagumo equation [32]:

$$u_t - u_{xx} = f = u(u - \alpha)(1 - u), \tag{23}$$

Burgers-Huxley equation [33]

$$u_t + \alpha u^\delta u_x - \lambda u_{xx} = f = \beta u(1 - u^\delta)(\eta u^\delta - \gamma) \tag{24}$$

Burgers-Fisher equation [34]

$$u_t + \alpha u^\delta u_x - u_{xx} = f = \beta u(1 - u^\delta) \tag{25}$$

It’s significant for us to handle Eq. (22).

2.1. The generalized KdV-Burgers equation

If we let $f = 0$ in Eq. (22), we can obtain the famous generalized KdV-Burgers equation with nonlinear terms of any order [35, 36].

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xx} + \delta u_{xxx} = 0. \tag{26}$$

Eq. (26) is solved on the infinite line $-\infty < x < \infty$ together with the initial condition $u(x, 0) = f(x),\ -\infty < x < \infty$ by using the HAM. We first introduce the traveling wave transform

$$\xi = x + ct + \xi_0. \tag{27}$$

where c are constants to be determined later and $\xi_0 \in C$ are arbitrary constants. Secondly, we make the following transformation:

$$u(\xi) = v^{1/p}(\xi). \tag{28}$$

Eq. (26) is reduced to the following form:

$$\begin{aligned} &p(p+1)(2p+1)\delta v(\xi)v''(\xi) + (p+1)(2p+1)\delta(1-p)v'^2(\xi) \\ &+p(p+1)(2p+1)\gamma v(\xi)v'(\xi) + cp^2(p+1)(2p+1)v^2(\xi) \\ &+p^2(2p+1)\alpha v^3(\xi) + p^2(p+1)\beta v^4(\xi) = 0 \end{aligned} \tag{29}$$

where the derivatives are performed with respect to the coordinate ξ. We can conclude that Eq. (26) has the following solution, by using the deformation mapping method:

$$\tilde{u}_0 = \left\{ -\frac{c(1+p)}{2\alpha} + \frac{d(1+p)\gamma}{p\alpha}\sqrt{\frac{c^2p^2}{4d^2\gamma^2}}\tanh(d\sqrt{\frac{c^2p^2}{4d^2\gamma^2}}(x+ct+\xi_0)) \right\}^{\frac{1}{p}}. \tag{30}$$

2.2. The approximate solutions by using HAM

To solve Eq. (22) by means of HAM, we choose the initial approximation

$$u_0(x,t) = \tilde{u}_0(x,t)\Big|_{t=0} = g(x), \tag{31}$$

where $\tilde{u}_0(x,t)$ is an arbitrary exact solution of Eq. (23).

According to Eq. (1), we define the nonlinear operator

$$N[\phi] = \phi_t + \alpha\phi^p\phi_x + \beta\phi^{2p}\phi_x + \gamma\phi_{xx} + \delta\phi_{xxx} - f(\phi), \phi = \phi(x,t;q). \tag{32}$$

It is reasonable to express the solution $u(x,t)$ by set of base functions $g_n(x)t^n, n \geq 0$, under the rule of solution expression; it is straightforward to choose $H(x,t) = 1$ and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} \tag{33}$$

with the property

$$L[c(x)] = 0. \tag{34}$$

From Eqs. (10, 11 and 32), we have

$$\begin{aligned} R_{m-1}(\vec{u}_{m-1},x,t) = u_{m-1,t} + \gamma u_{m-1,xx} + \delta u_{m-1,xxx} + \alpha D_{m-1}(\phi^p\phi_x) \\ + \beta D_{m-1}(\phi^{2p}\phi_x) - F(u_0,u_1,\cdots,u_{m-1}), \end{aligned} \tag{35}$$

where

$$D_{m-1}(\phi^n\phi_x)=\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{m-1}=0}^{k_{m-2}}\sum_{i=0}^{m-1}C_n^{k_1}C_{k_1}^{k_2}C_{k_2}^{k_3}\cdots C_{k_{m-2}}^{k_{m-1}}u_0^{n-k_1}u_1^{k_1-k_2}\cdots u_{m-1}^{k_{m-1}}u_{i\xi} \tag{36}$$

and $n\geq k_1\geq k_2\geq\cdots\geq k_{m-1}\geq 0\in N$, with

$$\begin{gathered}\sum_{j=1}^{m-1}k_j+i=m-1, i=0,\cdots,m-1\\ F(u_0,u_1,\cdots,u_{m-1})=\frac{1}{(n-1)!}\frac{\partial^{(m-1)}}{\partial q^{m-1}}f(x,t,u)\bigg|_{q=0}.\end{gathered} \tag{37}$$

Now, the solution of the mth-order deformation in Eq. (10) with initial condition $u_m(x,t)=0$ for $m\geq 1$ becomes

$$u_m=\chi_m u_{m-1}+L^{-1}[\hbar R_{m-1}(\vec{u}_{m-1},x,t)], \tag{38}$$

Thus, from Eqs. (31, 35 and 38), we can successively obtain

$$u_0=\tilde{u}_0(x,0)=g(x), \tag{39}$$

$$u_1=-\hbar t[\tilde{u}_{0t}+f(u_0)], \tilde{u}_{0t}=\frac{\partial}{\partial t}\tilde{u}_0(x,t)|_{t=0}, \tag{40}$$

$$u_2=(1+\hbar)u_1+\hbar(\alpha u_0^p u_{1,x}+\beta u_0^{2p}u_{1,x}+\gamma u_{1,xx}+\delta u_{1,xxx}-f_u(u_0)u_1)t \tag{41}$$

$$\vdots$$

$$u_m=(1+\hbar)u_{m-1}+\hbar[\gamma u_{1,xx}+\delta u_{1,xxx}+\alpha D_{m-1}(\phi^p\phi_x)+\beta D_{m-1}(\phi^{2p}\phi_x)-F(u_0,u_1,\cdots,u_{m-1})]t \tag{42}$$

$$\vdots$$

We obtain the mth-order approximate solution and exact solution of Eq. (22) as follows

$$u_{m,appr}=\sum_{k=0}^{m}u_k, u_{exact}=\phi(x,t;1)=\lim_{m\to\infty}\sum_{k=0}^{m}u_k \tag{43}$$

if we choose

$$\tilde{u}_0(x,0)=\left\{-\frac{c(1+p)}{2\alpha}+\frac{d(1+p)\gamma}{p\alpha}\sqrt{\frac{c^2p^2}{4d^2\gamma^2}}\tanh(d\sqrt{\frac{c^2p^2}{4d^2\gamma^2}}x)\right\}^{\frac{1}{p}}. \tag{44}$$

From Eqs. (39–44), we can obtain the corresponding approximate solution of Eq. (22).

2.3. Example

In the following, three examples are presented to illustrate the effectiveness of the HAM. We first plot the so-called $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$ to discover the valid region of $\hbar$, which corresponds to the line segment nearly parallel to the horizontal axis. The simulate comparison between the initial exact solution, exact solution and the fourth order of approximation solution is given.

Now, we consider the small perturbation term $f = \varepsilon\tilde{f}$ in Eq. (22).

Example 1. Consider the CKdV equation with small disturbed term

$$u_t + 6uu_x - 6u^2u_x + u_{xxx} = \varepsilon u^2, 0 < \varepsilon \ll 1 \tag{45}$$

with the initial exact solution

$$\tilde{u}_0(x,t) = \frac{1}{2} - \frac{1}{2}\tanh[\frac{1}{2}(x-t)]. \tag{46}$$

From Section 2.2, we have

$$u_0 = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right), \tilde{u}_{0t} = \frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right), \tag{47}$$

$$u_1 = -\hbar\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\}t \tag{48}$$

$$\begin{aligned}
u_2 = &-(1+\hbar)\hbar t\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\} \\
&- \hbar^2t^2\left\{6\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\}_x \\
&+ 6\hbar^2t^2\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]2\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\}_x \\
&- \hbar^2t^2\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\}_{xxx} \\
&+ 2\varepsilon\hbar^2t^2\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]\left\{\frac{1}{4}\mathrm{sech}^2\left(\frac{1}{2}x\right) + \varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\} \\
= &\frac{\hbar t}{32}\left[\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right)\right]\sec h^5\left(\frac{x}{2}\right)\left\{\hbar(5t - 3 - 3\varepsilon) - 3 - 3\varepsilon\right. \\
&+ 2\hbar t\varepsilon(1+\varepsilon) + 2\cosh(x)\left[2\varepsilon - 2 - 2\hbar(1+\varepsilon) + \hbar t\left(2\varepsilon^2 + 7\varepsilon - 3\right)\right] \\
&+ \left[\hbar\left(t - \varepsilon - 1 + 2t\varepsilon^2\right) - \varepsilon - 1\right]\cosh(2x) - 2\sinh\left(\frac{x}{2}\right)[1 - \varepsilon + \hbar - \varepsilon\hbar \\
&\left.+ \hbar t\left(2 - 3\varepsilon + 2\varepsilon^2\right) + (1-\varepsilon)\cosh x + \hbar\left(1 - t - \varepsilon + 2t\varepsilon^2\right)\cosh x)]\right\}
\end{aligned} \tag{49}$$

$$\cdots$$

$$
\begin{aligned}
u_{appr} = {} & \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right) - \hbar\left\{\frac{1}{4}\text{sech}^2\left(\frac{1}{2}x\right) + -\varepsilon\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}x\right)\right]^2\right\}t \\
& + \frac{\hbar t}{32}\left[\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right)\right]\sec h^5\left(\frac{x}{2}\right)\left\{\hbar(5t - 3 - 3\varepsilon) - 3 - 3\varepsilon + \right. \\
& 2\hbar t\varepsilon(1+\varepsilon) + 2\cosh(x)\left[2\varepsilon - 2 - 2\hbar(1+\varepsilon) + \hbar t\left(2\varepsilon^2 + 7\varepsilon - 3\right)\right] \\
& + \left[\hbar\left(t - \varepsilon - 1 + 2t\varepsilon^2\right) - \varepsilon - 1\right]\cosh(2x) - 2\sinh\left(\frac{x}{2}\right)[1 - \varepsilon + \hbar - \varepsilon\hbar \\
& \left. + \hbar t\left(2 - 3\varepsilon + 2\varepsilon^2\right) + (1-\varepsilon)\cosh x + \hbar\left(1 - t - \varepsilon + 2t\varepsilon^2\right)\cosh x)\right]\right\} + \cdots
\end{aligned}
\tag{50}
$$

The $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$ in Eq. (45) are shown in **Figure 1(a)**, and the comparison between the initial exact solution and the fourth order of approximation solution is shown in **Figure 1(b)**.

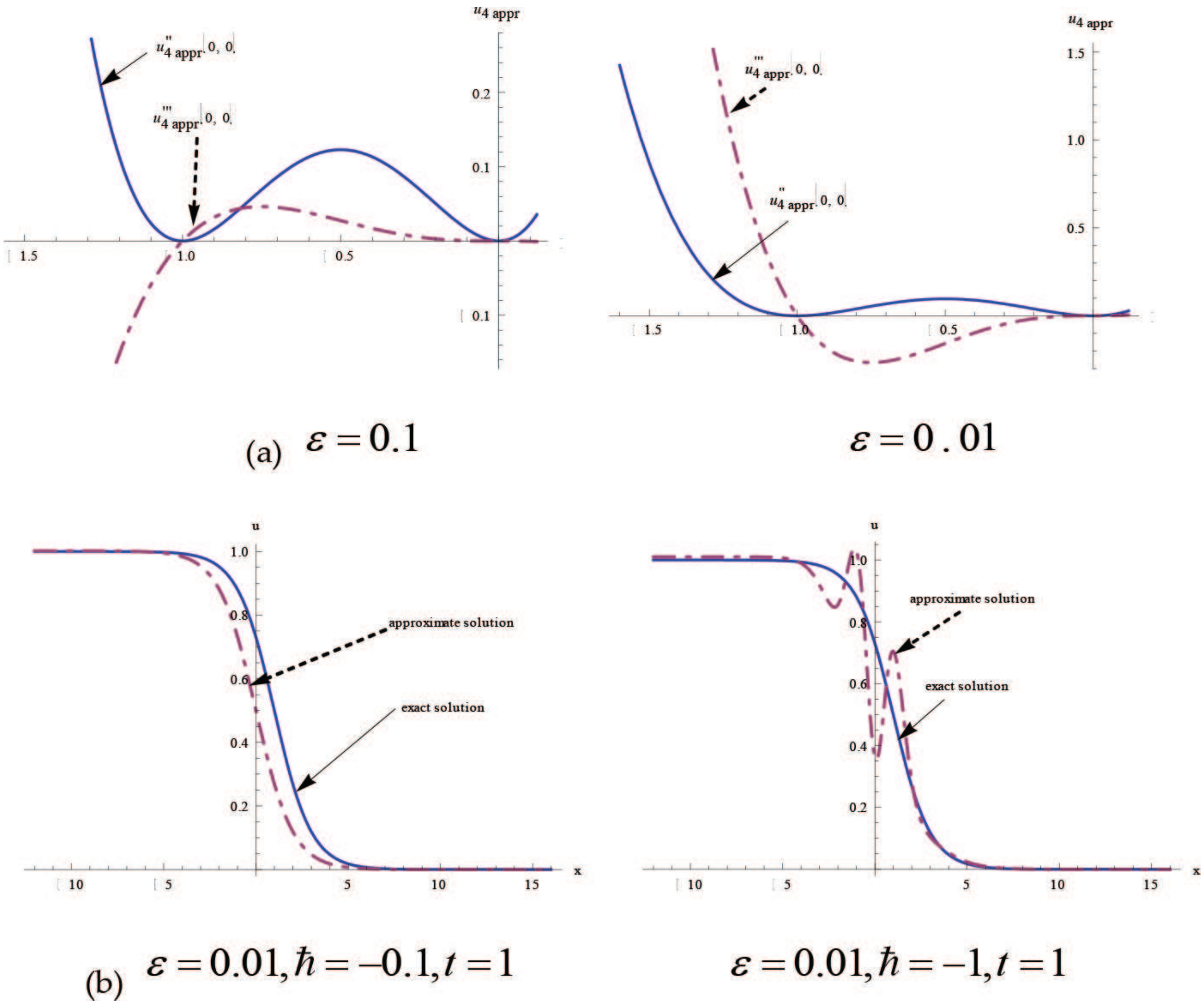


Figure 1. (a) The $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$at the fourth order of approximation. (b) The initial exact solution and the fourth order of approximation solution.

Example 2. Consider the KdV-Burgers equation with small disturbed term

$$u_t + 6uu_x + u_{xx} - u_{xxx} = \varepsilon \sin u \tag{51}$$

with the initial exact solution

$$\tilde{u}_0(x,t) = \frac{1}{50}\left\{1 - \coth[-\frac{1}{10}(x - \frac{6}{25}t)]\right\}^2 \tag{52}$$

From Section 2.2, we have

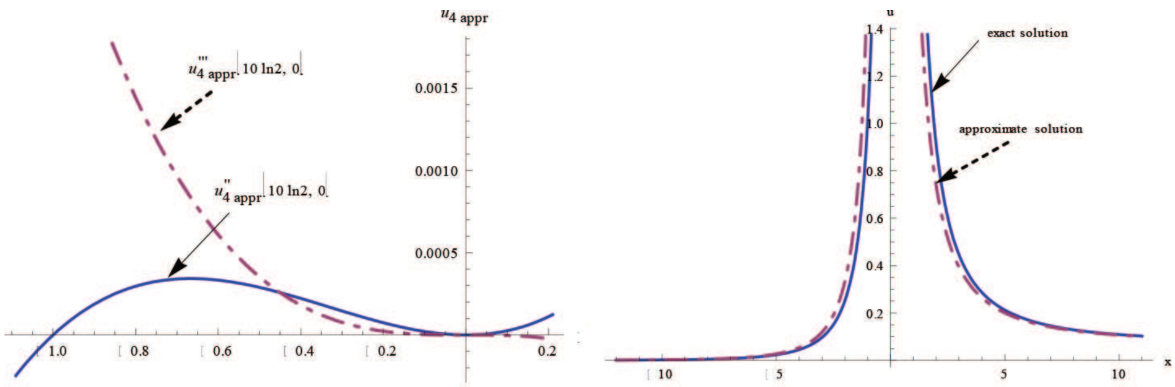
$$u_0 = \frac{1}{50}[1 - \coth(-\frac{1}{10}x)]^2, \tilde{u}_{0t} = \frac{3}{3125}\operatorname{csch}^2(\frac{1}{10}x)[1 + \coth(\frac{1}{10}x)] \tag{53}$$

$$u_1 = -\hbar\varepsilon \sin\left\{\frac{1}{50}[1 - \coth(\frac{-1}{10}x)]^2\right\}t - \frac{3\hbar t}{3125}\operatorname{csch}^2(\frac{1}{10}x)[1 + \coth(\frac{1}{10}x)] \tag{54}$$

$$u_2 = (1+\hbar)u_1 + \hbar t(6u_0u_{1,x} + u_{1,xx} - u_{1,xxx} - \varepsilon u_1 \cos u_0) \tag{55}$$

$$\begin{aligned} u_{appr} = {} & \frac{1}{50}\left[1 - \coth\left(-\frac{1}{10}x\right)\right]^2 - \hbar\varepsilon \sin\left\{\frac{1}{50}\left[1 - \coth\left(-\frac{1}{10}x\right)\right]^2\right\}t \\ & - \frac{3}{3125}\hbar t \operatorname{csch}^2\left(\frac{1}{10}x\right)\left[1 + \coth\left(\frac{1}{10}x\right)\right] + u_2 + \cdots \end{aligned} \tag{56}$$

The $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$ in Eq. (51) are shown in **Figure 2(a)**; the comparison between the initial exact solution and the fourth order of approximation solution is shown in **Figure 2(b)**.



(a) $\varepsilon = 0.02, t = 0$ (b) $\varepsilon = 0.05, \hbar = 0.1, t = 1$

Figure 2. (a) The $\hbar$ curves of $u''_{appr}(10\ln 2, 0)$ and $u'''_{appr}(10\ln 2, 0)$ at the fourth order of approximation. (b) The initial exact solution and the fourth order of approximation solution.

Example 3. Consider the Burgers-Fisher equation

$$u_t + u^2 u_x - u_{xx} = \varepsilon u(1 - u^2) \tag{57}$$

with the exact solution and the initial exact solution

$$u_{1_{exact}} = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{3}x - \frac{1+9\varepsilon}{9}t + \xi_0\right]} \tag{58}$$

$$u_{2_{exact}} = \sqrt{\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{3}x - \frac{1+9\varepsilon}{9}t + \xi_0\right]} \tag{59}$$

$$\tilde{u}_0(x,t) = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{3}x - \frac{1}{9}t + \xi_0\right]} \tag{60}$$

From Section 2.2, we have

$$u_0 = \sqrt{\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{3}x\right)}, \qquad \tilde{u}_{0t} = \operatorname{sech}^2\left(\frac{1}{3}x\right)/18\sqrt{2 - 2\tanh\left(\frac{1}{3}x\right)} \tag{61}$$

$$u_1 = -\frac{\hbar t \operatorname{sech}^2\left(\frac{1}{3}x\right)}{18\sqrt{2 - 2\tanh\left(\frac{1}{3}x\right)}} - \hbar t\varepsilon\sqrt{\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{3}x\right)}\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{3}x\right)\right) \tag{62}$$

$$u_2 = (1+\hbar)u_1 + \hbar t(\alpha u_0 u_{1,x} - u_{1,xx} - \varepsilon u_1 + 3\varepsilon u_0^2 u_1) \tag{63}$$

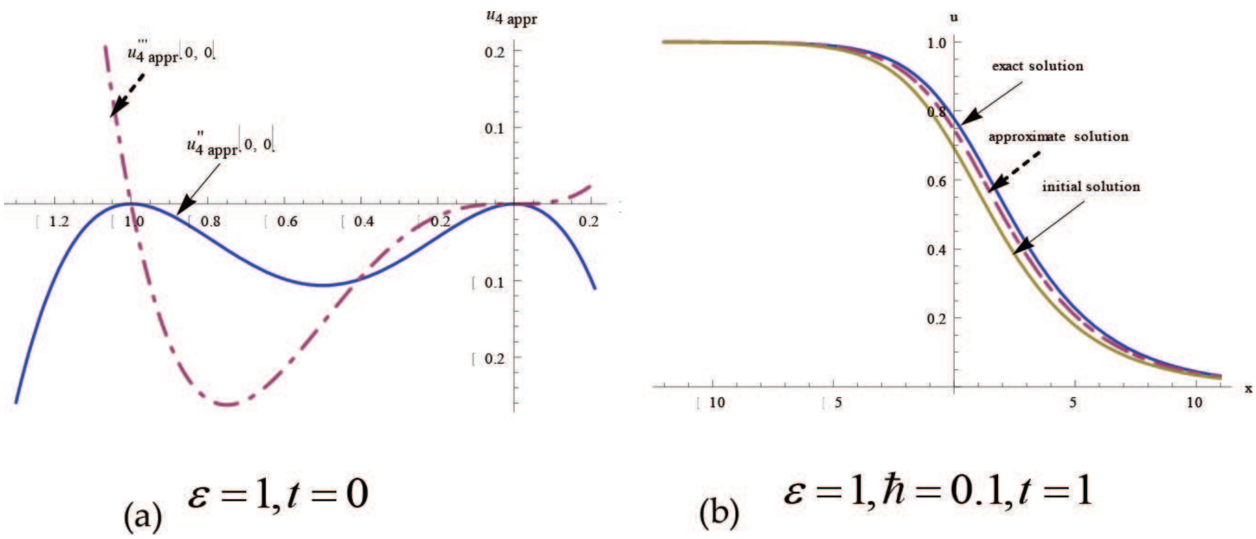


Figure 3. (a) The $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$ at the fourth order of approximation. (b) The exact solution, initial exact solution and the fourth order of approximation solution.

$$u_{appr}=\sqrt{\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{3}x\right)}-\frac{\hbar t\,\mathrm{sech}^2\left(\frac{1}{3}x\right)}{18\sqrt{2-2\tanh\left(\frac{1}{3}x\right)}}$$
$$-\hbar t\varepsilon\sqrt{\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{3}x\right)}\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{1}{3}x\right)\right)+u_2+\cdots \tag{64}$$

The $\hbar$ curves of $u''_{appr}(0,0)$ and $u'''_{appr}(0,0)$ in Eq. (57) are shown in **Figure 3(a)**, the comparison between the initial exact solution and the fourth order of approximation solution is shown in **Figure 3(b)**.

3. Application to the generalized perturbed NLS equation

In this section, we will use the HPM and Fourier's transformation to search for the solution of the generalized perturbed nonlinear Schrödinger equation (GPNLS)

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2 u}{\partial t^2}+\delta(z)u|u|^2-i\alpha(z)u=\beta(z)f(u,z,t). \tag{65}$$

If we let $t\to x, z\to t$,Eq. (65) turns to the following form

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\beta(t)\frac{\partial^2 u}{\partial x^2}+\delta(t)u|u|^2-i\alpha(t)u=\beta(t)f(u,t,x). \tag{66}$$

where disturbed term f is a sufficiently smooth function in a corresponding domain. $\alpha(t)$ represents the heat-insulating amplification or loss. $\beta(t)$ and $\delta(t)$ are the slowly increasing dispersion coefficient and nonlinear coefficient, respectively. The transmission of soliton in the real communication system of optical soliton is described by Eq. (66) with $f=0$ [37–39].

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\beta(t)\frac{\partial^2 u}{\partial x^2}+\delta(t)u|u|^2-i\alpha(t)u=0. \tag{67}$$

We make the transformation

$$u=A(t)\varphi(\xi)e^{i\eta},\ \xi=k_1x+c_1(t),\ \eta=k_2x+c_2(t) \tag{68}$$

With the following consistency conditions,

$$A(t)=ce^{\int_0^t\alpha(\tau)d\tau},\ c_1(t)=-k_1k_2\int_0^t\beta(\tau)d\tau,\ c_2(t)=\frac{1}{2}(a_2k_1^2-k_2^2)\int_0^t\beta(\tau)d\tau,\ \delta(t)=\frac{-a_4k_1^2}{c^2}\beta(t)e^{-2\int_0^t\alpha(\tau)d\tau} \tag{69}$$

where k_1, k_2, a_2, a_4, c are arbitrary non-zero constants.

If we let $f(u,t,x)=\frac{1}{2}k_1^2f(\varphi)e^{i\eta}$, substituting Eq. (68) into Eq. (67), we have

$$\varphi''_{\xi\xi}-a_2\varphi-2a_4\varphi^3=f(\varphi). \tag{70}$$

By using the general mapping deformation method [10, 40], we can obtain the following solutions of the corresponding undisturbed Eq. (70) when $f=0$.

$$\tilde{\varphi}_0=cn[k_1x-k_1k_2\int_0^t\beta(\tau)d\tau]. \tag{71}$$

In order to obtain the solution of Eq. (70), we introduce the following homotopic mapping $H(\varphi,p)$: $R\times I\to R$,

$$H(\varphi,p)=L\varphi-L\tilde{\varphi}_0+q\left(L\tilde{\varphi}_0-2a_4\varphi^3-f(\varphi)\right). \tag{72}$$

where $R=(-\infty,\ +\infty)$, $I=[0,1]$, $\tilde{\varphi}_0$ is an initial approximate solution to Eq. (70), and the linear operator L is expressed as

$$L(u)=\varphi''_{\xi\xi}-a_2\varphi. \tag{73}$$

Obviously, from mapping Eq. (72), $H(\varphi,1)=0$ is the same as Eq. (70). Thus, the solution of Eq. (70) is the same as the solution of $H(\varphi,q)$ as $q\to 1$.

3.1. Approximate solution

In order to obtain the solution of Eq. (70), set

$$\varphi=\sum_{i=0}^{\infty}\varphi_i(\xi)q^i=\varphi_0+q\varphi_1+q^2\varphi_2+\cdots \tag{74}$$

If we let $\varphi_0=\tilde{\varphi}_0$,notice the analytical properties of f, $\tilde{\varphi}_0$, and mapping Eq. (72), we can deduce that the series of Eq. (74) are uniform convergence when $q\in[0,1]$. Substituting expression (74) into $H(u,q)=0$ and expanding nonlinear terms into the power series in powers of q, we compare the coefficients of the same power of q on both sides of the equation and we have

$$q^0: L\varphi_0=L\tilde{\varphi}_0, \tag{75}$$

$$q^1: L\varphi_1=f(\varphi_0), \tag{76}$$

$$q^2: L\varphi_2=6a_4\varphi_0^2\varphi_1+f_\varphi(\varphi_0)\varphi_1, \tag{77}$$

$$\cdots$$

$$\begin{aligned} q^n: L\varphi_n=F(\varphi_0,\varphi_1,\cdots,\varphi_{n-1})+2a_4\sum_{k_1=0}^{3}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}\cdots \\ \sum_{k_{n-1}=0}^{k_{n-2}}C_3^{k_1}C_{k_1}^{k_2}C_{k_2}^{k_3}\cdots C_{k_{n-2}}^{k_{n-1}}\varphi_0^{3-k_1}\varphi_1^{k_1-k_2}\varphi_2^{k_2-k_3}\cdots\varphi_{n-2}^{k_{n-2}-k_{n-1}}\varphi_{n-1}^{k_{n-1}} \end{aligned}. \tag{78}$$

$$\cdots$$

where $3 \geq k_1 \geq k_2 \geq \cdots \geq k_{n-1} \geq 0 \in N$, $\sum_{j=1}^{n-1} k_j = n-1, n \in N^+$ and $F(\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) = \frac{1}{(n-1)!}\frac{\partial^{(n-1)}}{\partial p^{n-1}}$ $f(\varphi_0, \varphi_1, \cdots, \varphi_{n-1})|_{p=0}$.

From Eq. (75) we have $\varphi_0(\xi) = \tilde{\varphi}_0(\xi)$. If we select $\varphi_1|_{\xi=0} = 0$, by using Fourier transformation and from Eq. (76), we have

$$\varphi_1 = \frac{1}{\sqrt{a_2}}\int_0^{\xi} f(\varphi_0)(e^{\sqrt{a_2}(\xi-\tau)} - e^{-\sqrt{a_2}(\xi-\tau)})d\tau, \quad a_2 \neq 0, \quad f(\varphi_0) = f(\varphi_0(\tau)). \tag{79}$$

If we select $\varphi_2|_{\xi=0} = 0$, from Eq. (77) we have

$$\varphi_2 = \frac{1}{\sqrt{a_2}}\int_0^{\xi} [6a_4\varphi_0^2\varphi_1 + f_\varphi(\varphi_0)\varphi_1](e^{\sqrt{a_2}(\xi-\tau)} - e^{-\sqrt{a_2}(\xi-\tau)})d\tau. \tag{80}$$

where $a_2 \neq 0, \varphi_0 = \varphi_0(\tau), \varphi_1 = \varphi_1(\tau)$.

We obtain the first- and second-order approximate solutions $u_{1\text{hom}}(x,t)$ and $u_{2\text{hom}}(x,t)$ of the Eq. (70) as follows:

$$\varphi_{1\text{hom}}(x,t) = \tilde{\varphi}_0 + \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi} f(\varphi_0)(e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau \tag{81}$$

$$u_{1\text{hom}}(x,t) = ce^{\int_0^t \alpha(\tau)d\tau + i[k_2x + \frac{1}{2}\int_0^t((2m^2-1)k_1^2 - k_2^2)\beta(\tau)d\tau]}\varphi_{1\text{hom}}(x,t) \tag{82}$$

$$\begin{aligned}\varphi_{2\text{hom}}(x,t) = \tilde{\varphi}_0 &+ \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi} f(\varphi_0)(e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau \\ &+ \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi}[-6m^2\varphi_0^2\varphi_1 + f_\varphi(\varphi_0)\varphi_1](e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau\end{aligned} \tag{83}$$

$$u_{2\text{hom}}(x,t) = ce^{\int_0^t \alpha(\tau)d\tau + i[k_2x + \frac{1}{2}\int_0^t((2m^2-1)k_1^2 - k_2^2)\beta(\tau)d\tau]}\varphi_{2\text{hom}}(x,t) \tag{84}$$

With the same process, we can also obtain the N-order approximate solution

$$\begin{aligned}\varphi_{n\text{hom}}(x,t) = \tilde{\varphi}_0 &+ \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi} f(\varphi_0)(e^{\sqrt{a_2}(\xi-\tau)} - e^{-\sqrt{a_2}(\xi-\tau)})d\tau \\ &+ \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi}[-6m^2\varphi_0^2\varphi_1 + f_\varphi(\varphi_0)\varphi_1](e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau \\ &+ \cdots + \frac{1}{\sqrt{2m^2-1}}\int_0^{\xi}(e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})[F(\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) - 2m^2 \\ &\sum_{k_1=0}^{3}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{n-1}=0}^{k_{n-2}} C_3^{k_1}C_{k_1}^{k_2}C_{k_2}^{k_3}\cdots C_{k_{n-2}}^{k_{n-1}}\varphi_0^{3-k_1}\varphi_1^{k_1-k_2}\varphi_2^{k_2-k_3}\cdots\varphi_{n-2}^{k_{n-2}-k_{n-1}}\varphi_{n-1}^{k_{n-1}}]d\tau\end{aligned} \tag{85}$$

$$u_{n\text{hom}}(x,t) = ce^{\int_0^t \alpha(\tau)d\tau + i[k_2x + \frac{1}{2}\int_0^t((2m^2-1)k_1^2 - k_2^2)\beta(\tau)d\tau]}\varphi_{n\text{hom}}(x,t) \tag{86}$$

where $3 \geq k_1 \geq k_2 \geq \cdots \geq k_{n-1} \geq 0 \in N$, $\sum_{j=1}^{n-1} k_j = n-1, n \in N^+$ and

$$F(\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) = \frac{1}{(n-1)!} \frac{\partial^{(n-1)}}{\partial p^{n-1}} f(\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) \Big|_{p=0} \tag{87}$$

3.2. Comparison of accuracy

In order to explain the accuracy of the expressions of the approximate solution represented by Eq. (86), we consider the small perturbation term

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\beta(t)\frac{\partial^2 u}{\partial x^2} + \delta(t)u|u|^2 - i\alpha(t)u = \frac{1}{2}\varepsilon k_1^2 \beta(t) e^{i\eta} \sin^n \varphi, \tag{88}$$

where $n \in N^+, \varphi = e^{-\int_0^t \alpha(\tau)d\tau - i(k_2 x + \frac{1}{2}(a_2 k_1^2 - k_2^2)\int_0^t \beta(\tau)d\tau)} u/c, 0 < \varepsilon \ll 1$.

From the discussion of Section 3.1, we obtain the second-order approximate Jacobi-like elliptic function solution of Eq. (88) as follows

$$\begin{aligned}\varphi_{2\text{hom}}(x,t) = {} & cn[k_1 x - k_1 k_2 \int_0^t \beta(\tau)d\tau] + \frac{\varepsilon}{\sqrt{2m^2-1}} \int_0^\xi \sin^n(\varphi_0)(e^{\sqrt{2m^2-1}(\xi-\tau)} \\ & - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau + \frac{1}{\sqrt{2m^2-1}} \int_0^\xi [-6m^2\varphi_0^2\varphi_1 + \varepsilon n \sin^{n-1}(\varphi_0) \\ & \cos(\varphi_0)\varphi_1](e^{\sqrt{2m^2-1}(\xi-\tau)} - e^{-\sqrt{2m^2-1}(\xi-\tau)})d\tau\end{aligned} \tag{89}$$

$$u_{2\text{hom}}(x,t) = ce^{\int_0^t \alpha(\tau)d\tau + i[k_2 x + \frac{1}{2}\int_0^t ((2m^2-1)k_1^2 - k_2^2)\beta(\tau)d\tau]} \varphi_{2\text{hom}}(x,t). \tag{90}$$

Set $\varphi_{exa}(x,t) = \sum_{i=0}^{\infty} \varphi_i(x,t)$ to be an exact solution of Eq. (88), notice that

$$\begin{aligned}L(\varphi_{exa} - \varphi_{2\text{hom}}) = {} & f(\varphi) + 2a_4 {\varphi_{exa}}^3 - [2a_4 {\varphi_0}^3 + f(\varphi_0) + 6a_4\varphi_0^2\varphi_1 \\ & + f_\varphi(\varphi_0)\varphi_1] = \varepsilon \sin^n\left(\sum_{i=0}^{\infty}\varphi_i\right) + 2a_4\left(\sum_{i=0}^{\infty}\varphi_i\right)^3 - [2a_4{\varphi_0}^3 + \varepsilon \sin^n(\varphi_0), \\ & + 6a_4\varphi_0^2\varphi_1 + \varepsilon n \sin^{n-1}(\varphi_0)\cos(\varphi_0)\varphi_1] = O(\varepsilon^2)\end{aligned} \tag{91}$$

where $0 < \varepsilon \ll 1$, selecting arbitrary constants such that $\varphi_{exa}(0) = \varphi_{2\text{hom}}(0)$, from the fixed point theorem [41], we have $\varphi_{exa} - \varphi_{2\text{hom}} = O(\varepsilon^2)$, then

$$\begin{aligned}|u_{exa} - u_{2\text{hom}}| & = |A(t)e^{i\eta}[\varphi_{exa} - \varphi_{2\text{hom}}]| \\ & = \left| \frac{\varepsilon^2 An \sin^{n-1}(\varphi_0)\cos(\varphi_0)}{\sqrt{2m^2-1}} \int_0^\xi \sin^n(\varphi_0)(e^{\sqrt{a_2}(\xi-\tau)} - e^{-\sqrt{a_2}(\xi-\tau)})d\tau \right| = O(\varepsilon^2).\end{aligned} \tag{92}$$

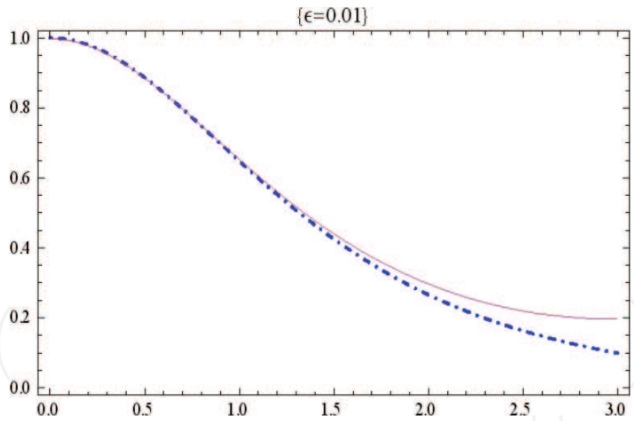


Figure 4. A comparison between the curves of solutions $|u_{1\text{hom}}(\xi)|$ (solid line) and $|u_0(\xi)|$ (dashed line) with $\varepsilon = 0.01$.

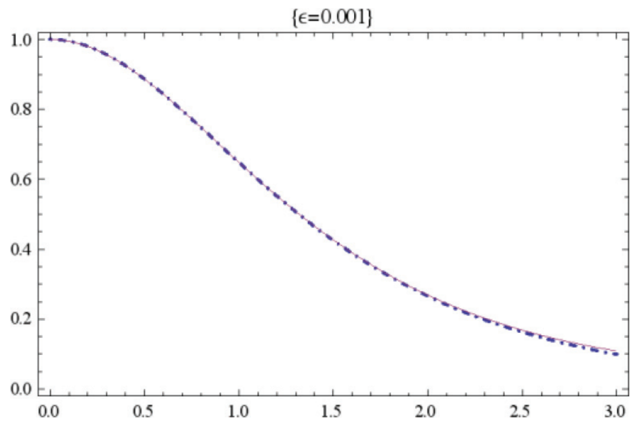


Figure 5. A comparison between the curves of solutions $|u_{1\text{hom}}(\xi)|$ (solid line) and $|u_0(\xi)|$ (dashed line) with $\varepsilon = 0.001$.

Therefore, from the above result, we know that the approximate solution,$u_{2\text{hom}}$, obtained by asymptotic method and possesses better accuracy.

Set $A(t) = 1, k_1 = k_2 = 1, \beta(t) = 1, m \to 1, n = 1, \xi \in [0, 3]$ and $\varepsilon = 0.01, 0.001$ for Eq. (90), and then, we will have the curves of solutions $|u_{1\text{hom}}(\xi)|$ and $|u_0(\xi)|$ and be able to compare them; see **Figures 4** and **5**. From **Figures 4** and **5**, it is easy to see that as $0 < \varepsilon \ll 1$ is a small parameter, and the solutions $|u_{1\text{hom}}(\xi)|$ and $|u_0(\xi)|$ are very close to each other. This behavior is coincident with that of the approximate solution of the weakly disturbed evolution in Eq. (88).

4. Conclusions

We research the generalized perturbed KdV-Burgers equation and GPNLS equation by using the HAM and HPM; these two powerful straightforward methods are much more simple and efficient than some other asymptotic methods such as perturbation method and Adomian decomposition method and so on. The Jacobi elliptic function and solitary wave approximate solution with arbitrary degree of accuracy for the disturbed equation are researched, which shows that these two methods have wide applications in science and engineering and also can be used in the soliton equation with complex variables, but it is still worth to research whether or not these two methods can be used in the system with high dimension and high order.

Acknowledgements

The work is supported by the Scientific Research Foundation of Nanjing Institute of Technology (Grant No. ZKJ201513,2016YB22).

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