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Emergence of Classical Distributions from Quantum Distributions: The Continuous Energy Spectra Case

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Abstract

We explore the properties of quantum states and operators that are conjugate to the Hamiltonian eigenstates and operator when the Hamiltonian spectrum is continuous, i.e., we find time-like operators $\widehat{T}$ such that $[\widehat{T}, \widehat{H}] = i\hbar$. This is a property expected for a time operator. We explicitly unfold the momentum sign degeneracy of energy states. We consider the free-particle case, and we find, among other things, that the time states are also the solution of the quantized version of the classical motion of the particle.

Keywords: time operator, time eigenstates, conjugate states, free-particle time eigenstates

1. Introduction

The problem of the time operator in quantum mechanics has been studied by numerous researchers for many years and remains a subject of current research. There are many instances in which a time variable is useful. An example of such a situation is calculating the tunneling time of a particle passing through a barrier. This time was recently measured, and it was shown to vanish [1, 2].

There are several approaches in this area that were developed by Kijowski [3], Hegerfeldt et al. [4], Weyl [5], Galapon [6], Arai and Yasumichi [7, 8], Strauss et al. [9, 10], and Hall [11], among others. The work by these authors may appear to be in four differing approaches; however, we shall show that they are simply different approaches to the same theme, approximated ones.

Some of these approaches are similar to the work of Weyl on periodic functions [5]. Weyl defined the Hermitian form

$$-i\sum_{n\neq m}\frac{(-1)^{n-m}}{n-m}x_m x_n,\tag{1}$$

where $\{x_m\}$ are the components of a vector in the basis $e^{i2\pi m/n}/\sqrt{n}$, $m = 0,1,\ldots, n - 1$. Galapon, Arai et al., Straus et al., and Hall used a similar expression but with a factor of one instead of the $(-1)^{n-m}$ factor. Their results are valid in a limited region of the Hilbert space for the expression of Galapon and Arai. Strauss wanted to obtain a Lyapunov function; instead, he obtained a function that only gives the sign of time, as was shown by Hall. A different factor might result in a time operator that would be valid over the entire Hilbert space. In this chapter, we find a proper factor to obtain sensible time-like kets and operators that are valid over the entire Hilbert space, for the purely continuous energy spectrum case.

We introduce time-like kets and operators following a different route. We search for the states that are conjugate to the energy eigenstates, which is a natural approach to this subject. We find time kets and operators that are valid over the entire Hilbert space. We also find that we can make contact with the operators defined by other authors. These operators lack the oscillatory function found in this work.

Time is typically viewed as a parameter and not as a dynamical variable in classical and quantum mechanics. However, the characteristics of the time variable depend on the representation being considered. In classical mechanics, we have shown that we can talk of translations along the energy direction; in that case, the energy variable becomes a parameter, and time becomes a dynamical variable, a function of the phase-space variables [12].

For comparison, let us consider the coordinate representation of quantum mechanics. If a variable, s, with units of length is the parameter used in the shifting along the coordinate direction through the displacement operator, $e^{is\hat{P}/\hbar}$, in the momentum representation, s becomes the coordinate operator and the momentum $\hat{P}$ becomes a parameter. A similar behavior is expected when considering energy-time representations. However, the problem is to define a time representation in quantum mechanics, and we use the conjugacy concept in this chapter to find such a representation.

The basis for this work is that time is another coordinate that has to be determined. The conjugate pair coordinate-momentum is a pair of conjugate coordinates that are used to define representations of wave functions and operators. Similarly, energy and time can be used as an alternative coordinate set, but the time coordinate has to be defined. As coordinate and momentum eigenstates, the time eigenstates will also be nonnormalizable, and their peculiarities originate from the type of coordinate that energy is a semibounded quantity.

In Section 2, we use the rewriting of the identity operator in terms of energy eigenstates to define the states that are conjugate to the energy eigenstates and subsequently determine some of their properties and several time-like operators. We define time states for negative and positive momentum values.

Section 3 is devoted to time-like operators and their properties. Time operators are written in three different forms. We verify that the time kets are eigenkets of the time operators. We find "evolution equations" for time kets and note that the time operators are the generators for translations along the energy direction. We also discuss how a wave packet is shifted along the energy direction.

In numerical calculations, we have to address finite regions of variables and not infinite intervals. Therefore, we focus our attention on approximate expressions for time operators in Section 4. We find approximate expressions of time operators that can be used in numerical calculations and are of help in the understanding of the expressions found by other authors.

The free-particle problem is analyzed in Section 5. We find expressions for the time kets for the free particle. The coordinate matrix elements of the time operators are also found, and we learn that the time states are also a solution to the quantum analog of the classical motion. The support of the time states embodies the classical trajectories, and as $\hbar \to 0$, we recover the classical motion.

We conclude the chapter with some concluding remarks.

2. Time eigenstates

In this section, we define the states that are conjugate to the energy eigenstates and the corresponding conjugate operator to a given quantum Hamiltonian $\widehat{H}$. We also derive some of their properties. The definition of conjugacy between the operators $\widehat{T}$ and $\widehat{H}$ that we will use here is the usual one, i.e., that these operators should comply with the constant commutator relationship $[\widehat{T}, \widehat{H}] = i\hbar$. We will consider the case of a purely continuous energy spectrum with a Hamiltonian operator $\widehat{H}$ of the form $\widehat{H} = \widehat{P}^2/2m + \widehat{V}(\widehat{Q})$, where $\widehat{P}$ is the momentum operator, $\widehat{Q}$ is the coordinate operator, and $\widehat{V}(\widehat{Q})$ is the potential energy operator. We will also consider that the sign of the momentum operator commutes with the Hamiltonian. The continuous eigenvalues of the Hamiltonian are denoted by $E \in [0, \infty)$ and correspond to the eigenkets $\{|E\rangle\}$.

We will base our definition of time states on rewriting the identity operator in terms of energy eigenstates and using the integral representation of the Dirac delta function. We assume that the Hamiltonian is self-adjoint. Thus, we will work on the span of the Hamiltonian eigenstates, denoted by

$$D = \left\{ |\psi\rangle \Big| |\psi\rangle = \int_0^\infty dE \psi(E)|E\rangle, \quad \psi(E) = \langle E|\psi\rangle \right\} \tag{2}$$

We assume that the closure relationship for the energy eigenstates holds, $\widehat{I} = \int_0^{E_m} dE|E\rangle\langle E|$. The i times the derivative is self-adjoint in a finite interval and hence will work in the subspace $E \in [0, E_m]$, $E_m < \infty$, which implies that $p \in [-p_m, p_m]$, $p_m < \infty$.

We start with the rewriting of the identity operator in terms of the energy eigenkets,

$$\hat{I} = \int_0^{E_m} dE |E\rangle\langle E| = \int_0^{E_m} dE'dE\delta(E - E')|E'\rangle\langle E| = \int_0^{E_m} dE'dE \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt\, e^{it(E-E')/\hbar}|E'\rangle\langle E|$$

$$\tag{3a}$$

$$= \int_{-\infty}^{\infty} dt \int_0^{E_m} dE'dE \frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}}|E'\rangle\langle E| \frac{e^{i\tau E/\hbar}}{\sqrt{2\pi\hbar}},$$

where we have made use of the properties of the Dirac delta function. We can separate the negative and positive momentum parts of the above expression by means of the closure relationship for the momentum states, obtaining

$$\hat{I} = \int_{-p_m}^{p_m} dp \int_0^{E_m} dE |E\rangle\langle E|p\rangle\langle p|E\rangle\langle E| = \int_{-p_m}^{p_m} dp \int_0^{E_m} dE'dE\delta(E - E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$= \int_{-p_m}^{p_m} dp \int_0^{E_m} dE'dE \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt e^{it(E-E')/\hbar}|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E| \tag{3b}$$

$$= \int_{-\infty}^{\infty} dt \int_{-p_m}^{p_m} dp \int_0^{E_m} dE'dE \frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}}|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E| \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}.$$

Thus, we define time-like kets as

$$|t\rangle := \int_0^{E_m} dE \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}}|E\rangle, \quad |t(p)\rangle := \int_0^{E_m} dE \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}}|E\rangle\langle E|p\rangle. \tag{4}$$

With these kets, the identity operator is written as

$$\hat{I} = \int_{-\infty}^{\infty} dt|t\rangle\langle t| = \int_{-\infty}^{\infty} dt \int_{-p_m}^{p_m} dp|t(p)\rangle\langle t(p)| = \hat{I}_- + \hat{I}_+, \tag{5a}$$

where

$$\hat{I}_- := \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp|t(p)\rangle\langle t(p)|, \quad \hat{I}_+ := \int_{-\infty}^{\infty} dt \int_0^{p_m} dp|t(p)\rangle\langle t(p)|. \tag{5b}$$

Then, the identity operator is written in terms of the time evolution of some bras and kets, which are composed of all the energy eigenstates.

Now, we define time-like operators $\widehat{T}$ and $\widehat{T}_\pm$ by introducing a factor t in the integrand of Eq. (5):

$$\widehat{T} = \int_{-\infty}^{\infty} dt\, t|t\rangle\langle t|, \tag{6a}$$

and

$$\widehat{T}_- := \int_{-\infty}^{\infty} dt \, t \int_{-p_m}^{0} dp |t(p)\rangle\langle t(p)|, \quad \widehat{T}_+ := \int_{-\infty}^{\infty} dt \, t \int_{0}^{p_m} dp |t(p)\rangle\langle t(p)|. \tag{6b}$$

The function $e^{itE/\hbar}$ exists only for $E \in [0, E_m]$, so that, for the sake of simplicity of notation, we, sometimes, will include explicitly the function $\Theta(E) - \Theta(E - E_m)$, where Θ is the step function, when necessary, otherwise we will omit this factor.

The commutator between these operators and the Hamiltonian operator is

$$\begin{aligned}
[\widehat{T}, \widehat{H}] &= \int_{-\infty}^{\infty} dt \, t \int_{0}^{E_m} dE' dE \frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}} \left[|E'\rangle\langle E|, \widehat{H}\right] \\
&= \int_{0}^{E_m} dE' dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, t \, e^{it(E-E')/\hbar}(E - E')|E'\rangle\langle E| \\
&= \int_{0}^{E_m} dE' dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \left[\left(-i\hbar\frac{\partial}{\partial E}[\Theta(E) - \Theta(E - E_m)]\right.\right. \\
&\quad \left.\left. + i\hbar[\delta(E) - \delta(E - E_m)]\right)e^{it(E-E')/\hbar}\right](E - E')|E'\rangle\langle E| \\
&= \int_{0}^{E_m} dE' dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar}[[\Theta(E) - \Theta(E - E_m)]i\hbar\frac{\partial}{\partial E} \\
&\quad + i\hbar[\delta(E) - \delta(E - E_m)]](E - E')|E'\rangle\langle E| \\
&\quad - i\hbar \int_{0}^{E_m} dE' \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar}(E - E')|E'\rangle\langle E|\Big|_{E=0}^{E_m} \\
&= \int_{0}^{E_m} dE' dE \delta(E - E')\left[i\hbar\frac{\partial}{\partial E} + i\hbar[\delta(E) - \delta(E - E_m)]\right](E - E')|E'\rangle\langle E| \\
&\quad - i\hbar \int_{0}^{E_m} dE' \delta(E - E')(E - E')|E'\rangle\langle E|\Big|_{E=0}^{E_m} \\
&= \int_{0}^{E_m} dE' dE \, \delta(E - E')i\hbar\frac{\partial}{\partial E}(E - E')|E'\rangle\langle E| \\
&\quad + i\hbar \int_{0}^{E_m} dE' dE \, \delta(E - E')[\delta(E) - \delta(E - E_m)](E - E')|E'\rangle\langle E| \\
&= i\hbar \int_{0}^{E_m} dE' dE \, \delta(E - E')|E'\rangle\langle E| \\
&\quad + \int_{0}^{E_m} dE' dE \delta(E - E')(E - E')|E'\rangle i\hbar\frac{\partial}{\partial E}\langle E| = i\hbar \int_{0}^{E_m} dE |E\rangle\langle E| \\
&= i\hbar\widehat{I},
\end{aligned} \tag{7a}$$

where we have made use of the integration by parts. This is one of the properties that a time operator should comply with—the constant commutator with the Hamiltonian. We also have that

$$\left[\widehat{T}_-,\widehat{H}\right] = \int_{-\infty}^{\infty} dt\, t \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE \frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}} \langle E'|p\rangle\langle p|E\rangle \left[|E'\rangle\langle E|,\widehat{H}\right]$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt\, t e^{it(E-E')/\hbar}(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt[((-i\hbar\frac{\partial}{\partial E}[\Theta(E)-\Theta(E-E_m)]$$

$$+i\hbar[\delta(E)-\delta(E-E_m)]e^{it(E-E')/\hbar})(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt\, e^{it(E-E')/\hbar}\Big[[\Theta(E)-\Theta(E-E_m)]i\hbar\frac{\partial}{\partial E}$$

$$+i\hbar[\delta(E)-\delta(E-E_m)](E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$-i\hbar\int_{-p_m}^{0} dp \int_{0}^{E_m} dE' \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt\, e^{it(E-E')/\hbar}(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E||_{E=0}^{E_m}$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE\delta(E-E')\left[i\hbar\frac{\partial}{\partial E} + i\hbar[\delta(E)-\delta(E-E_m)]\right] \tag{7b}$$

$$(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$- i\hbar\int_{-p_m}^{0} dp \int_{0}^{E_m} dE'\delta(E-E')(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|_{E=0}^{E_m}$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE\delta(E-E')i\hbar\frac{\partial}{\partial E}(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$+i\hbar\int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE\delta(E-E')[\delta(E)-\delta(E-E_m)]$$

$$(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$= i\hbar\int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE\delta(E-E')|E'\rangle\langle E'|p\rangle\langle p|E\rangle\langle E|$$

$$+\int_{-p_m}^{0} dp \int_{0}^{E_m} dE'dE\delta(E-E')(E-E')|E'\rangle\langle E'|p\rangle i\hbar\frac{\partial}{\partial E}\langle p|E\rangle\langle E|$$

$$= i\hbar\int_{-p_m}^{0} dp \int_{0}^{E_m} dE|E\rangle\langle E|p\rangle\langle p|E\rangle\langle E| = i\hbar\hat{I}_-,$$

$$\left[\widehat{T}_+,\widehat{H}\right] = i\hbar\int_{0}^{p_m} dp \int_{0}^{E_m} dE|E\rangle\langle E|p\rangle\langle p|E\rangle\langle E| = i\hbar\hat{I}_+. \tag{7c}$$

The operator

$$-i\hbar\frac{\partial}{\partial E} + i\hbar[\delta(E)-\delta(E-E_m)] \tag{8}$$

is a time-like operator in the energy representation, which is symmetric in the interval $[0, E_m]$ regardless of the boundary conditions at $E = 0, E_m$, when the functions exist only in the interval $[0, E_m]$.

Thus, we can say that the kets

$$|t(p)\rangle := \int_0^{E_m} dE \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}} |E\rangle\langle E|p\rangle = \frac{e^{-it\widehat{H}/\hbar}}{\sqrt{2\pi\hbar}} |p\rangle, \tag{9a}$$

$$|t\rangle := \int_0^{E_m} dE \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}} |E\rangle, \tag{9b}$$

can be considered as time-like kets. We will study some of their properties in what follows.

The inner product between time states is

$$\langle t'(p')|t(p)\rangle = \frac{1}{2\pi\hbar} \langle p'|e^{-i(t-t')\hat{H}/\hbar}|p\rangle = \frac{1}{2\pi\hbar} \langle p'|p(t-t')\rangle, \tag{10a}$$

$$\langle t'|t\rangle = \int_0^{E_m} dE \langle t'|E\rangle \langle E|t\rangle = \int_0^{E_m} dE \frac{e^{it'E/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}} = \frac{1}{2\pi\hbar} \int_0^{E_m} dE \, e^{i(t'-t)E/\hbar}$$
$$= \frac{1}{\pi(t'-t)} e^{i(t'-t)E_m/2\hbar} \sin\left(\frac{E_m}{2\hbar}(t'-t)\right), \tag{10b}$$

with limit

$$\lim_{E_m\to\infty} \langle t'|t\rangle = \frac{1}{2}\delta(t'-t) + \frac{i}{2\pi(t'-t)}. \tag{10c}$$

Thus, the time states are not orthogonal due to the bounded nature of the Hamiltonian operator.

2.1. Properties of the transformation function between energy and time states

The transformation function between energy and time representations is given by

$$\langle E|t\rangle = \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}}, \quad E \in [0, E_m], \quad t \in (-\infty, \infty). \tag{11}$$

A property of this transformation function is that it is a sort of eigenfunction of the time-like operator, $i\hbar(d/dE)[\Theta(E) - \Theta(E - E_m)] - i\hbar[\delta(E) - \delta(E - E_m)]$, when the functions exists in the interval $E \in [0, E_m]$, in the energy representation,

$$\left[i\hbar[\delta(E) - \delta(E - E_m)] - i\hbar\frac{\partial}{\partial E}[\Theta(E) - \Theta(E - E_m)]\right]\langle t|E\rangle = t[\Theta(E) - \Theta(E - E_m)]\langle t|E\rangle, \tag{12}$$

and it is also an eigenfunction of the energy operator, $i\hbar \, d/dt$,

$$i\hbar\frac{\partial}{\partial t}\langle E|t\rangle = i\hbar\frac{\partial}{\partial t}\frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}} = E\langle E|t\rangle. \tag{13}$$

This is similar to the corresponding properties of the transformation function between coordinate and momentum representations. The squared modulus of the transformation function is constant for all values of t and E, as is desired for coordinate variables.

Time kets can be used as a coordinate system for quantum systems and are similar to coordinate or momentum eigenkets. The norm of a wave packet in the time representation is (see Eq. (5))

$$\begin{aligned}
\langle\psi|\psi\rangle &= \int_{-\infty}^{\infty} dt\langle\psi|t\rangle \ \langle t|\psi\rangle = \int_{-\infty}^{\infty} dt\int_{0}^{E_m} dE'dE\frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}}\langle\psi|E'\rangle\frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}\langle E|\psi\rangle \\
&= \int_{0}^{E_m} dE'dE \ \langle\psi|E'\rangle \ \langle E|\psi\rangle\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt \ e^{it(E-E')/\hbar} \\
&= \int_{0}^{E_m} dE'dE\langle\psi|E'\rangle \ \langle E|\psi\rangle\delta(E-E') \\
&= \int_{0}^{E_m} dE|\langle E|\psi\rangle|^2.
\end{aligned} \tag{14}$$

Thus, we will obtain well-defined quantities if the wave packet $|\psi\rangle$ is normalized in the energy representation, i.e., if $\int_{0}^{\infty} dE|\langle E|\psi\rangle|^2 = 1$. We also note that the transformation from energy to time representations is norm preserving, i.e., it is unitary.

2.2. The time eigenstates are conjugate to the energy eigenstates

Now, the Fourier transform of the time states is

$$\begin{aligned}
\int_{-\infty}^{\infty} dt\frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}|t\rangle &= \int_{-\infty}^{\infty} dt\frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}\int_{0}^{E_m} dE'\frac{e^{-itE'/\hbar}}{\sqrt{2\pi\hbar}}|E'\rangle = \int_{0}^{E_m} dE'|E'\rangle\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt \ e^{it(E-E')/\hbar} \\
&= \int_{0}^{E_m} dE'|E'\rangle\delta(E-E') = |E\rangle,
\end{aligned} \tag{15}$$

Thus, the kets $|t\rangle$ and $|E\rangle$ are conjugate indeed, i.e., the definition (9) is consistent; $|t\rangle$ and $|E\rangle$ are the Fourier transforms of each other, and then an eigenstate contains all the conjugate eigenstates with the same weight.

3. Time operators

We now focus on the time operators obtained from the time kets of the previous section and on their properties. Time operators for negative, positive, and any value of the momentum are defined as

$$\hat{T}_- = \int_{-\infty}^{\infty} dt\int_{-p_m}^{0} dp|t(p)\rangle t\langle t(p)|, \hat{T}_+ = \int_{-\infty}^{\infty} dt\int_{0}^{p_m} dp|t(p)\rangle t\langle t(p)|, \tag{16a}$$

and

$$\hat{T} = \int_{-\infty}^{\infty} dt |t\rangle t \langle t|. \tag{16b}$$

The last construction was also introduced, from another perspective, by Hegerfeldt et al. [4]. Our construction is different from that of Hegerfeldt et al. because it involves all the energy eigenstates and not only those that are time reflection invariant. Our time operator exhibits the time reversal property already.

Time operators can be written in three equivalent forms in the energy representation. One form is

$$
\begin{aligned}
\widehat{T}_- &= \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp |t(p)\rangle t \langle t(p)| \\
&= \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE \frac{e^{-it\,E'/\hbar}}{\sqrt{2\pi\hbar}} |E'\rangle\langle E'|p\rangle \, t \, \langle p|E\rangle \, \langle E| \frac{e^{it\,E/\hbar}}{\sqrt{2\pi\hbar}} \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE \Big(i\hbar \frac{\partial}{\partial E'} [\Theta(E') - \Theta(E' - E_m)] e^{-it\,E'/\hbar} \\
&\quad - i\hbar [\delta(E') - \delta(E' - E_m)] e^{-it\,E'/\hbar} \Big) |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| e^{it\,E/\hbar} \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dT \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE \Big(- [\Theta(E') - \Theta(E' - E_m)] e^{-it\,E'/\hbar} i\hbar \frac{\partial}{\partial E'} \\
&\quad - i\hbar [\delta(E') - \delta(E' - E_m)] e^{-it\,E'/\hbar} \Big) |E'\rangle\langle E'|p\rangle\langle p|E\rangle \, \langle E| e^{it\,E/\hbar} \\
&\quad + \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_{0}^{E_m} dE \, i\hbar \, [\Theta(E') \\
&\quad - \Theta(E' - E_m)] e^{-itE'/\hbar} |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle \, \langle E| e^{itE/\hbar} \Big|_{E'=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE \delta(E - E') \Big(- [\Theta(E') - \Theta(E' - E_m)] i\hbar \frac{\partial}{\partial E'} \\
&\quad - i\hbar [\delta(E') - \delta(E' - E_m)] \Big) |E'\rangle\langle E'|p\rangle\langle p|E\rangle \, \langle E| \\
&\quad + i\hbar \int_{-p_m}^{0} dp \int_{0}^{E_m} dE \, \delta \, (E - E') [\Theta(E') - \Theta(E' - E_m)] |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| \Big|_{E'=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE [(-[\Theta(E) - \Theta(E - E_m)] i\hbar \frac{\partial}{\partial E} \\
&\quad - i\hbar \, [\delta(E) - \delta(E - E_m)]) |E\rangle\langle E|p\rangle] \langle p|E\rangle\langle E| \\
&\quad + i\hbar \int_{-p_m}^{0} dp |E'\rangle\langle E'|p\rangle \, \langle p|E'\rangle\langle E'| \Big|_{E'=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE \Big(-i\hbar \frac{\partial}{\partial E} |E\rangle\langle E|p\rangle \Big) \langle p|E\rangle\langle E| - i\hbar \int_{-\infty}^{0} dp |E\rangle |\langle p|E\rangle|^2 \langle E| \Big|_{E=0}^{E_m} \\
&\quad + i\hbar \int_{-p_m}^{0} dp |E'\rangle |\langle p|E'\rangle|^2 \langle E'| \Big|_{E'=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE \Big(-i\hbar \frac{\partial}{\partial E} |E\rangle\langle E|p\rangle \Big) \langle p|E\rangle\langle E|,
\end{aligned}
\tag{17a}
$$

where we have performed an integration by parts. We also have that

$$\hat{T}_+ = \int_{-\infty}^{\infty} dt \int_0^{p_m} dp |t(p)\rangle t \langle t(p)| = \int_0^{p_m} dp \int_0^{E_m} dE \left(-i\hbar \frac{\partial}{\partial E} |E\rangle\langle E|p\rangle \right) \langle p|E\rangle\langle E|, \tag{17b}$$

and

$$\hat{T} = \int_{-\infty}^{\infty} dt |t\rangle \, t \, \langle t| = \int_0^{E_m} dE \left(-i\hbar \frac{\partial}{\partial E} |E\rangle \right) \langle E|. \tag{17c}$$

These are the forms in which the time operators act on energy eigenkets, but they take a different form when they act on states or on both, eigenstates and wave packets.

A second energy representation of time operators is

$$\begin{aligned}
\widehat{T}_- &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_0^{E_m} dE'dE \, e^{-it\,E'/\hbar} |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| \left(-i\hbar\frac{\partial}{\partial E} [\Theta(E) - \Theta(E - E_m)] \right. \\
&\quad \left. +i\hbar[\delta(E) - \delta(E - E_m)] \right) e^{it\,E/\hbar} \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_0^{E_m} dE'dE \, e^{-it\,E'/\hbar} |E'\rangle\langle E'|p\rangle \Big[\Big(\Theta(E) \\
&\quad - \Theta(E - E_m)\Big] i\hbar\frac{\partial}{\partial E} + i\hbar[\delta(E) - \delta(E - E_m)]\rangle \langle p|E\rangle\langle E|] e^{it\,E/\hbar} \\
&\quad - \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp \int_0^{E_m} dE' \, e^{-it\,E'/\hbar} |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| e^{it\,E/\hbar} \big|_{E=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_0^{E_m} dE'dE \, \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar} |E'\rangle \, \langle E'|p\rangle \, i\hbar\frac{\partial}{\partial E} \langle p|E\rangle\langle E| \\
&\quad + i\hbar \int_{-p_m}^{0} dp \int_0^{E_m} dE'dE \, \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar} |E'\rangle \, \langle E'|p\rangle [\delta(E) \\
&\quad - \delta(E - E_m)]\langle p|E\rangle\langle E| \\
&\quad - i\hbar \int_{-p_m}^{0} dp \int_0^{E_m} dE' \, \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar} |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| \big|_{E=0}^{E_m} \\
&\quad - i\hbar \int_{-p_m}^{0} dp \int_0^{E_m} dE' \, \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \, e^{it(E-E')/\hbar} |E'\rangle\langle E'|p\rangle \, \langle p|E\rangle\langle E| \big|_{E=0}^{E_m} \\
&= \int_{-p_m}^{0} dp \int_0^{E_m} dE'dE \, \delta(E - E')|E'\rangle\langle E'|p\rangle i\hbar\frac{\partial}{\partial E} \langle p|E\rangle\langle E| \\
&= \int_{-p_m}^{0} dp \int_0^{E_m} dE |E\rangle \, \langle E|p\rangle i\hbar\frac{\partial}{\partial E} \langle p|E\rangle\langle E|,
\end{aligned}$$
$$\tag{18a}$$

$$\hat{T}_+ = \int_{-\infty}^{\infty} dt \int_0^{p_m} dp |t(p)\rangle \, t \, \langle t(p)| = \int_0^{p_m} dp \int_0^{E_m} dE |E\rangle\langle E|p\rangle i\hbar\frac{\partial}{\partial E} \langle p|E\rangle\langle E|, \tag{18b}$$

and

$$\hat{T} = \int_{-\infty}^{\infty} dt |t\rangle\, t\, \langle t| = \int_0^{E_m} dE |E\rangle i\hbar \frac{d}{dE} \langle E|. \tag{18c}$$

These are the various forms in which the time operators can act on states in the energy representation. The difference with the time operators when acting on energy eigenkets is a minus sign.

Other symmetric expressions for the time operators can also be obtained:

$$
\begin{aligned}
\hat{T}_- &= \int_{-\infty}^{\infty} dt \int_{-p_m}^0 dp |t(p)\rangle\, t\, \langle t(p)| \\
&= \int_{-\infty}^{\infty} dt \int_{-p_m}^0 dp \int_0^{E_m} dE' dE \frac{e^{-it\,E'/\hbar}}{\sqrt{2\pi\hbar}} |E'\rangle\, \langle E'|p\rangle\, t\, \langle p|E\rangle\, \langle E| \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}} \\
&= \int_{-p_m}^0 dp \int_0^{E_m} dE' dE |E'\rangle\, \langle E'|p\rangle\, \langle p|E\rangle\, \langle E| \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt\, t\, e^{it(E-E')/\hbar} \\
&= -i\hbar \int_{-p_m}^0 dp \int_0^{E_m} dE' dE |E'\rangle\, \langle E'|p\rangle\, \langle p|E\rangle\, \langle E|\delta'(E'-E),
\end{aligned}
\tag{19a}
$$

$$\hat{T}_+ = -i\hbar \int_0^{p_m} dp \int_0^{E_m} dE' dE |E'\rangle\, \langle E'|p\rangle\, \langle p|E\rangle\, \langle E|\delta'(E'-E), \tag{19b}$$

and

$$\hat{T} = -i\hbar \int_0^{E_m} dE' dE |E'\rangle\, \langle E|\delta'(E'-E). \tag{19c}$$

The domain of our time operators is D, defined in Eq. (2). The convergence of quantities depends on the type of wave packet that these operators act on. A wave packet of type $L^2(0, E_m)$ in the energy representation is a good choice (see Eq. (14)). Thus, the domain is invariant under the action of the time operators, and the commutator between the Hamiltonian and the time operators is thus valid in the entire domain D.

3.1. Time matrix elements of the Hamiltonian

The matrix elements of the Hamiltonian in the time representation are given by

$$
\begin{aligned}
\langle t'|\hat{H}|t\rangle &= \int_0^{E_m} dE' dE \frac{e^{it'E'/\hbar}}{\sqrt{2\pi\hbar}} \langle E'|\hat{H}|E\rangle \frac{e^{-it\,E/\hbar}}{\sqrt{2\pi\hbar}} = \frac{1}{2\pi\hbar} \int_0^{E_m} dE' dE\, E\, e^{it'\,E'/\hbar} e^{-itE/\hbar} \langle E'|E\rangle \\
&= \frac{1}{2\pi\hbar} \int_0^{E_m} dE\, E\, e^{i(t'-t)E/\hbar} = i\hbar \frac{d}{dt}\langle t'|t\rangle = -i\hbar \frac{d}{dt'}\langle t'|t\rangle.
\end{aligned}
\tag{20}
$$

This is the Schrödinger equation for time kets in the time representation.

3.2. The time ket is the eigenstate of the time operator

We can find the characteristic operator of the commutators $[\cdot, \widehat{H}]$ and $[\widehat{T}, \cdot]$. Because $[\widehat{T}, \widehat{H}] = i\hbar$ (see Eq. (7a)), the commutator between the operator $e^{-i\varepsilon \widehat{T}/\hbar}$, $\varepsilon \in [0, E_m]$, and the Hamiltonian is

$$[e^{-i\varepsilon\widehat{T}/\hbar}, \widehat{H}] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\varepsilon}{\hbar}\right)^n [\widehat{T}^n, \widehat{H}] = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-i\frac{\varepsilon}{\hbar}\right)^n i\hbar\, n\, \widehat{T}^{n-1} = \varepsilon e^{-i\varepsilon\widehat{T}/\hbar}. \tag{21}$$

Similarly, the commutator between the time operator and the time propagator is

$$[\widehat{T}, e^{-it\widehat{H}/\hbar}] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{t}{\hbar}\right)^n [\widehat{T}, \widehat{H}^n] = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-i\frac{t}{\hbar}\right)^n i\hbar\, n\, \widehat{H}^{n-1} = t e^{-it\widehat{H}/\hbar}, \tag{22}$$

where $t \in \mathbb{R}$.

The time ket $\hat{|t\rangle}$ is the time propagation of a zero time ket $\hat{|0\rangle}$,

$$\hat{|t\rangle} = \int_0^{\infty} dE \frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}} |E\rangle = e^{-it\widehat{H}/\hbar}\hat{|0\rangle}, \quad \hat{|0\rangle} = \frac{1}{\sqrt{2\pi\hbar}}\int_0^{\infty} dE |E\rangle. \tag{23}$$

Thus, according to Eq. (22), we can say that the time ket is an eigenstate of the time operator

$$\widehat{T}|t\rangle = \widehat{T}e^{-it\widehat{H}/\hbar}|0\rangle = e^{-it\widehat{H}/\hbar}\widehat{T}|0\rangle + t\, e^{-it\widehat{H}/\hbar}|0\rangle = t|t\rangle, \tag{24}$$

where we have set $\widehat{T}|0\rangle = 0$ because $|0\rangle$ is the zero-time state.

An "evolution equation" for the energy eigenstate is (see Eq. (15))

$$\begin{aligned}
\widehat{T}|E\rangle &= \int_{-\infty}^{\infty} dt \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}\widehat{T}|t\rangle = \int_{-\infty}^{\infty} dt \frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}t|t\rangle \\
&= \int_{-\infty}^{\infty} dt\left(-i\hbar\frac{\partial}{\delta E}[\Theta(E) - \Theta(E - E_m)]\frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}} + i\hbar[\delta(E) - \delta(E - E_m)]\frac{e^{itE/\hbar}}{\sqrt{2\pi\hbar}}\right)|t\rangle \\
&= \left(-i\hbar\frac{d}{dE}[\Theta(E) - \Theta(E - E_m)] + i\hbar[\delta(E) - \delta(E - E_m)]\right)|E\rangle \\
&= [\Theta(E) - \Theta(E - E_m)]\left(-i\hbar\frac{d}{dE}\right)|E\rangle.
\end{aligned} \tag{25}$$

Thus, the time operator is the generator of translations along the energy direction. All quantities are well defined as long as E and t belong to the allowed set of values for them. For other values of E and $E + \varepsilon$, we will get a linear combination of the energy eigenstates [14].

3.3. Shifting of operators

The shifting of the Hamiltonian along the energy direction is (see Eq. (21))

$$\widehat{H}(\mathcal{E}) := e^{-i\varepsilon\hat{T}/\hbar}\widehat{H}e^{i\varepsilon\hat{T}/\hbar} = (\widehat{H}\,e^{-i\varepsilon\hat{T}/\hbar} + \mathcal{E}\,e^{-i\varepsilon\hat{T}/\hbar})e^{i\varepsilon\hat{T}/\hbar} = \widehat{H} + \mathcal{E}, \qquad (26)$$

where $0 \leq E + \varepsilon$. For the translation of the time operator (see Eq. (22)), we have

$$\hat{T}(t) := e^{it\widehat{H}/\hbar}\hat{T}e^{-it\widehat{H}/\hbar} = e^{it\widehat{H}/\hbar}(e^{-it\widehat{H}/\hbar}\hat{T} + te^{-it\widehat{H}/\hbar}) = \hat{T} + t. \qquad (27)$$

These operations are well defined as long as $E + \varepsilon \geq 0$ [6, 14]. The derivative with respect to t of the time-shifted operator (27) is

$$\frac{d}{dt}\widehat{T}(t) = \hat{I}, \qquad (28)$$

that is, in the energy-time representations, t is the value that the time operator $\widehat{T}$ can take and not simply a parameter. Similarly, in the case of a translation of the Hamiltonian operator by the time operator, i.e., Eq. (26), we find that

$$\frac{d}{d\varepsilon}\widehat{H}(\varepsilon) = \hat{I}. \qquad (29)$$

Therefore, in the energy-time representations, ε is not simply a parameter, but it is related to the values that the Hamiltonian $\widehat{H}$ can take.

Thus, the use of energy and time eigenkets and operators instead of coordinate and momentum eigenkets and operators is similar to going from a parametric representation of curves, with time being the parameter of evolution, to a nonparametric representation in which time is now one of the coordinates.

4. Approximate expressions

In this section, we make contact with other expressions that have been used by other authors. Other works have not made use of the Sa(x;1) factor that appears in our results. The results in this section will allow us to obtain a better understanding of previous results.

4.1. Approximating the integral in an infinite interval

As an approximation, we replace the integral in an infinite interval $(2\pi)^{-1}\int_{-\infty}^{\infty} dt$ with the integral in the finite interval $t \in [-T/2, T/2]$, $\lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2} dt$. Then,

$$\widehat{T}_- = \int_{-\infty}^{\infty} dt \int_{-p_m}^{0} dp |t(p)\rangle t \langle t(p)|$$

$$\cong \frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE \, e^{-itE'/\hbar} |E'\rangle \, \langle E'|p\rangle t \langle p|E\rangle \, \langle E|e^{itE/\hbar}$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE |E'\rangle \, \langle E'|p\rangle \, \langle p|E\rangle \, \langle E| \frac{1}{T} \int_{-T/2}^{T/2} dt \, t \, e^{it(E-E')/\hbar} \qquad (30a)$$

$$= \int_{-p_m}^{0} dp \int_{0}^{E_m} dE' dE |E'\rangle \, \langle E'|p\rangle \, \langle p|E\rangle \, \langle E| \frac{i\hbar}{E - E'} \text{Sa}\left(\frac{T}{2\hbar}(E - E'); 1\right)$$

$$= \int_{0}^{E_m} dE' dE \frac{i\hbar}{E - E'} \text{Sa}\left(\frac{T}{2\hbar}(E - E'); 1\right) |E'\rangle \, \langle E'|\hat{I}_-|E\rangle \, \langle E|,$$

$$\widehat{T}_+ \cong \int_{0}^{E_m} dE' dE \frac{i\hbar}{E - E'} \text{Sa}\left(\frac{T}{2\hbar}(E - E'); 1\right) |E'\rangle \, \langle E'|\hat{I}_+|E\rangle \, \langle E|, \qquad (30b)$$
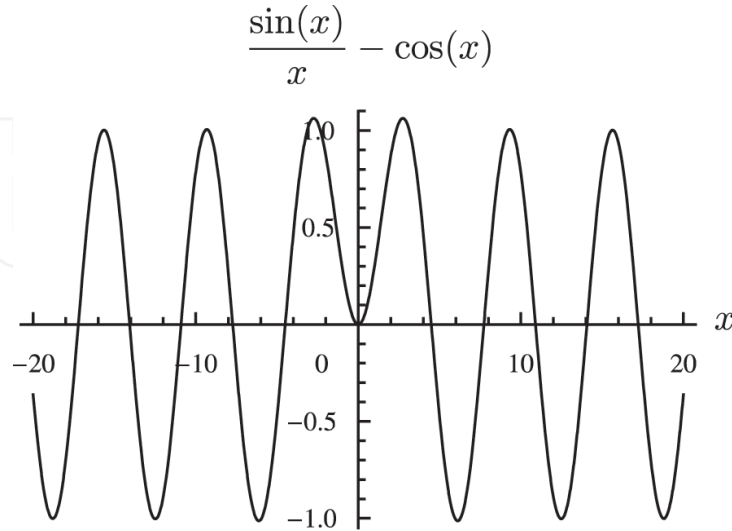
and

$$\widehat{T} \cong \int_{0}^{E_m} dE' dE |E'\rangle \, \langle E| \frac{i\hbar}{E - E'} \text{Sa}\left(\frac{T}{2\hbar}(E - E'); 1\right), \qquad (30c)$$

where the Sa function of type one is defined as

$$\text{Sa}(x; 1) := \frac{\sin(x)}{x} - \cos(x). \qquad (31)$$

A plot of this function can be found in **Figure 1**. This function is zero at $x = 0$ and oscillates between $\approx \pm 1$. The limit $T \to \infty$ of the integral of $\text{Sa}(Tx/2; 1)/Tx$ times a function $f(x)$ gives an approximation to the derivative of the latter at $x = 0$.

Figure 1. A plot of the function $\text{Sa}(x; 1) := \sin(x)/x - \cos(x)$.

Expressions that resemble Eq. (30c), but without the Sa factor, were used by other authors as a function that gives the sign of time in the continuous energy spectrum case [9–11].

5. The free particle

As an example of the time kets provided by our method, let us apply the derived results to the free-particle system. We find expressions for time eigenkets, including the case when a distinction of the sign of the momentum is needed. In this model, the momentum operator $\hat{P}$ commutes with the Hamiltonian operator $\hat{H}$, indicating a symmetry, allowing for some simplifications.

A set of energy eigenfunctions, in the coordinate representation, for the free-particle model is

$$\langle q|E_{\pm}\rangle = \frac{e^{\pm i\sqrt{2mE}q/\hbar}}{\sqrt{2\pi\hbar}}, \quad E \in [0,\infty). \tag{32}$$

The subscripts in these functions indicate the sign of the momentum of the particle. Thus, the zero-time eigenstate for the free particle is given as

$$\langle q|0_{\pm}\rangle := \int_0^\infty dE \frac{1}{\sqrt{2\pi\hbar}}\langle q|E_{\pm}\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty dE \frac{e^{\pm i\sqrt{2mE}\,q/\hbar}}{\sqrt{2\pi\hbar}} = \frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty \frac{p\,dp}{m}\frac{e^{\pm ip\,q/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= \frac{1}{m}\left(\mp i\,\hbar\frac{d}{dq}\right)\frac{1}{2\pi\hbar}\int_0^\infty dp\ e^{\pm i\,p\,q/\hbar} = \mp i\frac{\hbar}{m}\frac{d}{dq}\left(\frac{\delta(q)}{2} \pm \frac{i}{2\pi q}\right), \tag{33}$$

where we have made the change in variable $E = p^2/2m$. The unit of the last ket is time^{-1}. Various other authors have used kets obtained by direct quantization of the classical expression for the time variable and have obtained a time ket with units of time$^{1/2}$. However, our kets exhibit the properties discussed in this chapter.

Figure 2 shows a three-dimensional plot of the approximation of the squared modulus of the time states $\langle q|t_-\rangle$ and $\langle q|t_+\rangle$, obtained by not integrating from $E = 0$ to ∞ but up to a finite, large, value of E. They start highly localized at the origin and subsequently they move away from it and spread with time. The support of these functions resembles the classical motion curve $mq = pt$.
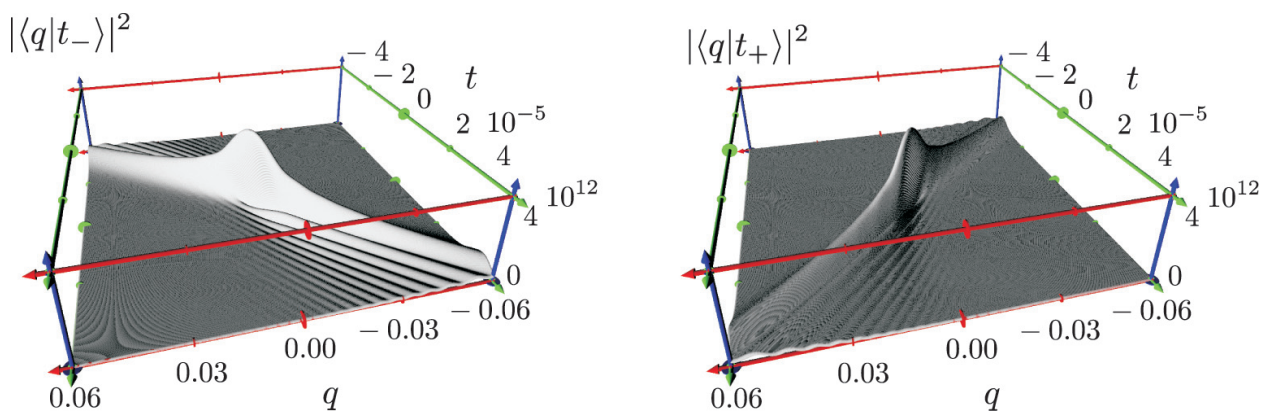
Figure 2. Three-dimensional plots of the squared modulus of the approximate time kets $|\langle q|t_-\rangle|^2$ and $|\langle q|t_+\rangle|^2$ for the free-particle model. The density is initially a highly localized density at $q = 0$ but subsequently it spreads and moves away from the origin. Dimensionless units.

For the sake of completeness, we write down the matrix elements of the time operators in the coordinate representation. They are

$$\langle q'|\widehat{T}_{\pm}|q\rangle = \int_0^{\infty} dE\langle q'|E_{\pm}\rangle\left(i\hbar\frac{\partial}{\partial E}\right)\langle E_{\pm}|q\rangle = \int_0^{\infty} dE\frac{e^{\pm i\sqrt{2mE}q'/\hbar}}{\sqrt{2\pi\hbar}}\left(i\hbar\frac{\partial}{\partial E}\right)\frac{e^{\mp i\sqrt{2mE}q/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= \pm\int_0^{\infty} dE\frac{e^{\pm i\sqrt{2mE}q'/\hbar}}{\sqrt{2\pi\hbar}}\frac{m}{\sqrt{2mE}}q\frac{e^{\mp i\sqrt{2mE}q/\hbar}}{\sqrt{2\pi\hbar}} = \pm\frac{1}{2\pi\hbar}q\int_0^{\infty} dp\, e^{\pm ip(q'-q)/\hbar} \qquad (34)$$

$$= \pm q\left[\frac{\delta(q-q')}{2} + \frac{i}{2\pi(q-q')}\right].$$

5.1. Solution to the quantized version of the classical motion of a free particle

The following calculation shows that the time states can also be the solution to the quantized classical expression for the motion of a free particle initially located at $q = 0$, i.e., the quantization of $mq = pt$. Let us rewrite the product $mq\langle q|t_-\rangle$ as follows:

$$mq\langle q|t_-\rangle = mq\int_0^{E_m} dE\frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}}\langle q|E_-\rangle = mq\int_0^{E_m} dE\frac{e^{-itE/\hbar}}{\sqrt{2\pi\hbar}}\frac{e^{-i\sqrt{2mE}q/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= mq\int_{-p_m}^0 dp\frac{p}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= m\int_{-p_m}^0 dp\frac{p}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}\left(-i\hbar\frac{\partial}{\partial p}\right)\frac{e^{i\,pq/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= m\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\left(i\hbar\frac{\partial}{\partial p}\right)\frac{p}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}} - i\hbar m\frac{p}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\Big|_{p=-p_m}^0$$

$$= i\hbar\, m\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\left(\frac{1}{m} - i\frac{p}{m}\frac{2tp}{2m\hbar}\right)\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}$$

$$+ i\hbar\frac{p_m}{2\pi\hbar}e^{-itp_m^2/2m\hbar}e^{ip_mq/\hbar} \qquad (35a)$$

$$= i\hbar\, m\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\frac{1}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}} + t\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\frac{p^2}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}$$

$$+ i\frac{p_m}{2\pi}e^{-itp_m^2/2m\hbar}e^{i\,p_mq/\hbar}$$

$$= t\left(-i\hbar\frac{d}{dq}\right)\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\frac{p}{m}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}} + i\hbar\int_{-p_m}^0 dp\frac{e^{ipq/\hbar}}{\sqrt{2\pi\hbar}}\frac{e^{-itp^2/2m\hbar}}{\sqrt{2\pi\hbar}}$$

$$+ i\frac{p_m}{2\pi}e^{-itp_m^2/2m\hbar}e^{ip_mq/\hbar}$$

$$= t\left(-i\hbar\frac{d}{dq}\right)\langle q|t_-\rangle + i\hbar\int_{-p_m}^0 dp\langle q|p\rangle\,\langle p|E\rangle + i\frac{p_m}{2\pi}e^{-itp_m^2/2m\hbar}e^{i\,p_mq/\hbar}$$

$$= t\langle q|\hat{P}|t_-\rangle + i\hbar(\langle q|\hat{I}_-|E\rangle + \langle q|\hat{P}|p_m\rangle\,\langle p_m|E\rangle),$$

$$mq\langle q|t_+\rangle = t\langle q|\hat{P}|t_+\rangle + i\hbar(\langle q|\hat{I}_+|E\rangle - \langle q|\hat{P}|p_m\rangle\,\langle p_m|E\rangle). \qquad (35b)$$

We can think of the last two terms in the above equations as quantum corrections to the classical trajectory of a free particle. These correction terms seem to vanish when $\hbar \to 0$.

On the other hand, the straightforward solution to the quantized version of the classical expression for the motion of a free particle gives a quite different function. The solution to the differential equation

$$mq\ f(q;\ t) = t\left(-i\hbar\frac{d}{dq}\right)f(q;\ t) \tag{36}$$

is

$$f(q;\ t) = N\ e^{i\ mq^2/2\hbar t}, \tag{37}$$

where N is a normalization constant. The squared modulus of this function is constant for all q and for all t. The squared modulus of the corresponding momentum function,

$$f(p;\ t) = N\sqrt{i\frac{t}{m}}\ e^{-i\ tp^2/2m\hbar}, \tag{38}$$

is not a localized function either; it actually is proportional to the transformation function between energy and time representations, in momentum representation. Thus, the route of forming conjugate states to the energy eigenstates seems to be a better path for obtaining appropriate time eigenstates.

6. Conclusions

We have introduced time-like states and time-like operators that are conjugate to the energy eigenstates and Hamiltonian operator, respectively. We have also given an interpretation of the obtained states and operators, and we have found that expressions obtained via other approaches to finding time eigenstates can be related to our expressions. However, the oscillatory Sa factor that we use solves many difficulties found in previous treatments. We have found the form of the time states for the free particle and a time operator that is valid for any L^2-type wave functions.

The approximation to time operators that we have introduced in this chapter uses expressions that can be adapted to the case of discrete energy spectra. We will explore this possibility in a later paper. From the literature on time operators, it might be believed that the treatment for a continuous energy spectrum is different from that for discrete energy spectrum systems. But, the results of this study suggest that both types of systems can be addressed in a similar manner.

Finally, we have found that the spectral measure $M(d\tau)$ of $\hat{T}$ is a nonorthogonal resolution of the identity defined by

$$\langle E'|\widehat{M_I}(d\tau)|E\rangle = \frac{e^{it(E'-E)/\hbar}}{2\pi\hbar}d\tau. \tag{39}$$

This measure exhibits the covariance property, as was previously stated by Holevo [13].

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