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# An Operational Approach to Conceptual Understanding Using Semiotic Theory

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## Abstract

Duval suggests that understanding of a mathematical concept is accessed through the commonality in its associated registers of representation. In this chapter, we present two studies where students in treatment (with a broader experience using registers of representations and comparison (with more limited experience using registers of representation) populations were interviewed to assess their ability to perform both familiar and unfamiliar treatments and conversions. As most mathematical concepts include a range of associated registers of representations, we assess the importance of using a broader range of treatments and conversions among these registers and suggest an operational approach to using these treatments and conversions to gain insight into the understanding of the concept.

**Keywords:** semiotics, registers of representation, treatments, conversions, conceptual understanding, multivariable calculus

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## 1. Introduction

In mathematics, representations are commonly used from the algebraic, geometric, numerical and verbal registers when concepts are presented and discussed. Movement between and within these registers of representation is well recognized as an important part of understanding these concepts [4]. Duval [1] takes this a step further by defining a mathematical object (i.e., concept) as the commonality of all its associated registers of representation. He goes on to indicate that, as seeing this commonality requires various registers of representation, “a two-register synergy, and sometimes a three-register synergy” (p. 126), is required to understand mathematical objects (concepts). “Synergy” of registers can be considered “simultaneous awareness” of the registers of representations.

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Based on Duval's assertion that the understanding of mathematical concepts can only be achieved through simultaneous awareness of associated representations, McGee and colleagues [2, 3] implied, without an explicit presentation, that an operational framework might become accessible by associating the comprehension of a mathematical object with the ability to fluidly move between its associated registers of representations. It was found [2, 3] that promoting the ability to move fluidly across three registers of representation throughout topics of integration and differentiation significantly improved students' problem-solving abilities. McGee and Moore-Russo [4] also found that a similar multi-representational perspective on conceptual understanding appears to positively impact teaching and learning with preservice teachers as well.

This chapter will present an explicit operational approach to using semiotic theory to assess students' understanding and will summarize data obtained from two studies that provide insight into its implications, applications and methodology.

## 2. Theoretical framework

The semiotic basis for mathematical understanding lies in movement among and within the semiotic registers associated with a mathematical concept. These transformations (movements involving different registers of representations for the same mathematical object) fall into two categories:

- Conversions describe a movement from a representation within a given register to another representation within a different register where both registers are associated with the exact same mathematical concept. For example, moving from the representation within the verbal register, "we start with 20 and increase by 10 each year" to the formula in the symbolic register  $y = 20 + 10x$  would represent a conversion.
- Treatments describe movement from a representation within a given register to another representation within the same register where both registers are associated with the exact same mathematical concept. For example, simplifying the formula within the symbolic register  $2y = 20 + 4x$  to the formula  $y = 10 + 2x$  within the same symbolic register would represent a treatment.

Duval [1] asserts that a mathematical concept can only be understood by seeing that which is common to all of its representations. For example, the number "3" can only be fully understood if we see the commonality of several registers of representations including groupings containing three items, the number 3 on a number line and numerical operations such as " $2 + 1$ ," to name a few.

While Duval [1] emphasizes on the need to harness various registers of representation when understanding mathematical concepts, others [2–6] studied the nature of how registers of representation are used. The initial introduction to a mathematical concept most often begins with an established order of representations associated with the concept known as a semiotic chain [2, 6]. For example, when presenting a line, a presentation might begin with the formula  $y = 2x + 3$  (symbolic register), proceed to a table of values associated with the formula (numeric register) and conclude with a graph of a line with slope two and intercept 3 (geometric register). McGee and Martinez-Planell [2] found that as a concept is better understood, students would progress toward simultaneous awareness of the concept's representations which would be associated

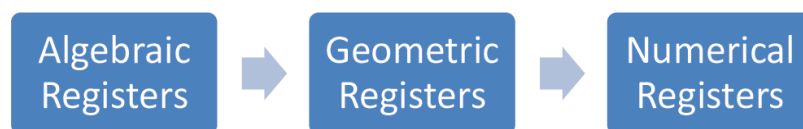
with the ability to perform treatments and conversions that are not in the initial semiotic chain. An example of this evolution is shown in **Figures 1** and **2**.

**Figure 1** presents a semiotic chain containing the geometric, numerical and symbolic registers that might be associated with the initial presentation of a mathematical concept. McGee and Martinez-Planell [2] would consider a more procedural understanding to be associated with limited movement among these registers. For example, if we assume that a student can only replicate the two conversions found in the semiotic chain of **Figure 1**:

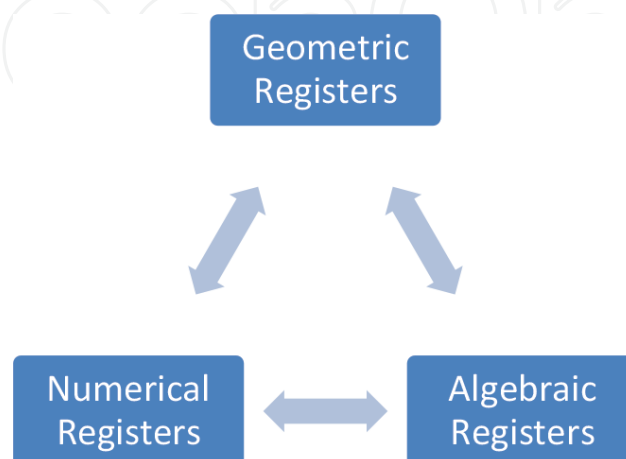
- geometric  $\rightarrow$  numeric register and
- numeric  $\rightarrow$  symbolic register.

Conceptual understanding, on the other hand, would be associated with the ability to perform up to all six possible conversions associated with the geometric, numerical, and symbolic registers:

- geometric  $\rightarrow$  numeric register,
- numeric  $\rightarrow$  symbolic register,
- geometric  $\rightarrow$  symbolic register,
- numeric  $\rightarrow$  geometric register,
- symbolic  $\rightarrow$  numeric register, and
- symbolic  $\rightarrow$  geometric register.



**Figure 1.** An example of a semiotic chain.



**Figure 2.** An example of simultaneous awareness of registers of representation.

**Figure 2** provides an illustration of what simultaneous awareness of all registers might look like.

The evolution from performing only the two conversions in the semiotic chain shown in **Figure 1** to performing up to six conversions found in **Figure 2** is the basis for the operational approach to conceptual understanding that is outlined in this chapter.

### 3. Overview of the operational approach and methodology

Our operational approach using semiotics to assess the conceptual understanding of a concept is based on the assumption that a procedural approach to solving a problem without conceptual understanding will likely be restricted to treatments and conversions associated with a semiotic chain (see **Figure 1**). If conceptual understanding is perceived as understanding the commonality of various registers of representation as Duval suggests, then this produces simultaneous awareness of registers that would be consistent with **Figure 2**.

Our operational approach to assessing conceptual understanding can loosely be summarized as follows:

- Identify the semiotic chain or common treatments and conversions that would best be associated with a procedural approach to solving a problem or presenting a concept.
- Create written or verbal assessments that incorporate both familiar and unfamiliar treatments and conversions that involve the same registers of representations observed in step one.

In step two, it should be noted that if there are  $n$  representations associated with a concept then there are  $n-1$  treatments and conversions in an associated semiotic chain and  $n!$  total possible treatments and conversions that pass among these representations. While certain contexts may make some of these impractical, we would suggest that all  $n!$  possibilities should be considered. It should be noted that conversions that present a real-world situation from the verbal register as the target are among our most successful instruments in interviews that are seeking to assess students' understanding. For example, given the representation from the symbolic register  $y = 10 \times 2^x$ , asking students to find a representation from the verbal register (real-world situation) that could be represented by this formula can provide considerable insight into students' thinking processes.

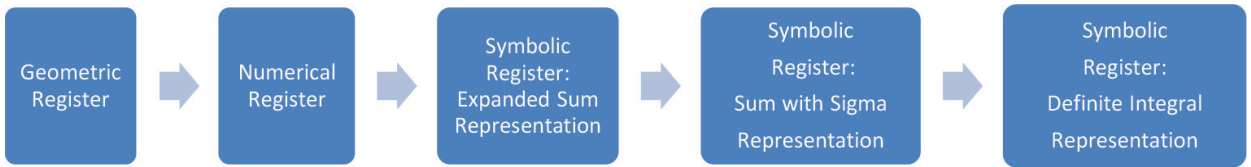
In this chapter, we present the data from two studies [2, 3] that had previously not observed their data in this context to determine what insight this operational approach might provide into students' understanding. In particular, to what degree is it necessary to be able to move fluidly among familiar and unfamiliar treatments and conversions in order to understand mathematical concepts when conceptual understanding is measured using other standard classroom instruments?

With single, double and triple integrals, the registers of representation can be seen (for single integrals) in **Table 1**. The semiotic chain most commonly used in university classrooms

Representation in geometric register for an approximation of the area under $y = x^2$ between $x = 1$ and $5$ using four rectangles and the midpoint rule for the height of each rectangle	
Representation in the numerical register for the above approximation of the area	$(1.5)^2 \times 1 + (2.5)^2 \times 1 + (3.5)^2 \times 1 + (4.5)^2 \times 1$
Expanded sum representation in the symbolic register for the above approximation of the area	$x_1^2 \Delta x + x_2^2 \Delta x + x_3^2 \Delta x + x_4^2 \Delta x$
Sum with sigma representation in the symbolic register for the above approximation of the area	$\sum_{i=1}^4 x_i^2 \Delta x$
Sum with sigma representation in the symbolic register for the precise value of the area	$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n x_i^2 \Delta x \right)$
Definite integral representation in the symbolic register for the precise value of the area	$\int_1^5 x^2 dx$

**Table 1.** The registers of representation associated with an integral.

that traces the path from a numerical Riemann sum approximating the area under a curve to a definite integral representing the precise area is shown in **Figure 3**: While the precise details associated with the registers of representations as we trace paths to the area under a curve, a volume under a surface and the mass associated with a volume change to reflect single, double, and triple integrals, the overarching semiotic chain can remain the same. Our first study provides insight into the ability of our operational approach to assess conceptual understanding with this semiotic chain when used with double and triple integrals.



**Figure 3.** Semiotic chain associated with an integral.

The second study we present involved slopes and derivatives. It was interesting in that more registers of representation were involved and, unlike our first study where a single semiotic chain was most commonly used, instructors used a variety of semiotic chains. **Table 2** presents some registers associated with constant slope, and **Table 3** presents some registers of representation associated with variable rates of change. Instructors invariably presented semiotic chains when discussing slopes and derivatives; however, the semiotic chains varied. For example, some began with a geometric representation, others began with a table of values and so on. While the semiotic chains associated with their presentations varied, we nonetheless felt that our operational approach could be modified to determine whether greater flexibility in performing treatments and conversions could be associated with greater conceptual understanding without assuming a unique semiotic chain as the starting point.

For each study in this chapter, we will look at two demographically similar populations where neither academic nor demographic factors distinguish them: One population studied the topics associated with the study with greater access to a broad range of experiences involving registers of representation and a more exploratory approach that would potentially facilitate less-common treatments and conversions. This population will be referred to as the experimental population. The second population will be referred to as the comparison population, where students had a more procedural background where they sometimes used fewer registers of representation and were less likely to explore less-common treatments and conversions. We will then present outcomes for these two populations involving traditional

Verbal

John earns \$10 an hour

Numerical

Hours Worked	1	2	3
Money Earned	\$10	\$20	\$30

Geometric

Algebraic

$t = \text{hour of the day}, f(t) = \text{cumulative money earned up to that hour } m = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$

**Table 2.** Common representations of constant slope in different registers.

Verbal	At 1:00, the height of a river is 2 feet and at 3:00, the height of a river is 8 feet.										
Numerical	<table><tr><td>Time of Day</td><td>1:00</td><td>2:00</td><td>3:00</td></tr><tr><td>Height of River (ft)</td><td>2</td><td>4</td><td>8</td></tr></table>			Time of Day	1:00	2:00	3:00	Height of River (ft)	2	4	8
Time of Day	1:00	2:00	3:00								
Height of River (ft)	2	4	8								
Algebraic	$t = \text{time of day}, h(t) = \text{height of the river at time } t, m = \frac{h(t_2) - h(t_1)}{t_2 - t_1}$										
Geometric											

Table 3. Registers of representation associated with variable slope.

problems that are often seen in multivariable calculus classes and interviews to assess students’ abilities to perform treatments and conversions. From these data, we will assess to what degree the ability to flexibly perform treatments and conversions is necessary to conceptual understanding where conceptual understanding is measured by performance-solving standard calculus questions.

4. Results

In our first study on topics from integration, Table 4 presents the results from interviews of the experimental and comparison populations with treatments and conversions that were not likely to have been seen by the comparison group. With every single treatment and conversion, the experimental group performed significantly better (Student's *t*-test, *p* < 0.05) than the comparison group. Table 5 presents the results from interviews of the experimental and comparison populations with treatments and conversions that were likely seen by both populations. The experimental group did better with all transformations and significantly (Student's *t*-test, *p* < 0.05) better than the comparison group with all treatments and conversions except the conversion of the geometric register to definite integral representation of the symbolic register.

**Table 6** presents the results of common examination questions that were considered to be appropriate for both groups and would likely be appropriate for most multivariable calculus classes. The experimental group did significantly (Student's  $t$ -test,  $p < 0.05$ ) better than the comparison group on all questions.

In our second study on slopes and derivatives, **Table 7** presents the results from interviews of the experimental and comparison populations. With every single treatment and conversion, the experimental group performed significantly better (Student's  $t$ -test,  $p < 0.05$ ) than the comparison group.

**Table 8** presents the results from interviews of the experimental and comparison population. In questions 1 and 3, the experimental group performed significantly better (Student's  $t$ -test,  $p < 0.05$ ) than the comparison group, and the difference between the two groups was not statistically significant in question 2.

Conversions and treatments	Comparison group	Experimental group
Geometric register to numerical register	42%	100%
Numerical register to the expanded sum or sum with sigma representation of the symbolic register	17%	80%
Sum with sigma representation of the symbolic register to the definite integral representation of the symbolic register	0%	80%
Verbal register to numerical register	0%	50%

**Table 4.** Treatments and conversions involving less commonly seen treatments and conversions.

Conversion	Comparison group	Experimental group
Geometric register to definite integral representation of the symbolic register	67%	80%
Definite integral representation of the symbolic register to verbal register	17%	80%
Verbal register to definite integral representation of the symbolic register	33%	80%

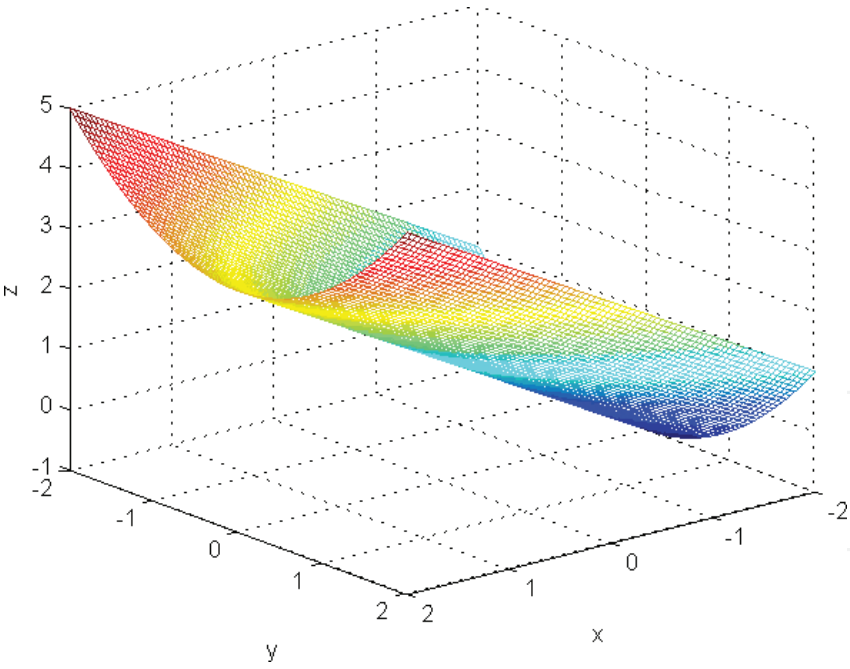
**Table 5.** Treatments and conversions involving more commonly seen treatments and conversions.

Question	Comparison Group ( $n = 68$ )	Experimental Group ( $n = 36$ )
Find the volume over the $xy$ -plane and between the surfaces $y = 0$ and $z = 10 - x^2 - y$ .	26%	53%
Find the volume over the plane $z = 1$ , below the surface $z = 10 - x^2$ and bounded by the planes $y = 1$ and 5	48%	73%

**Table 6.** Results on common examination questions for the experimental and control groups.

Treatment/conversion	Comparison group	Experimental group
Geometric register to the numerical register for the slope between two points.	8%	75%
Geometric register to the algebraic register for the slope between two points.	0%	67%
Geometric register to the numerical register for the directional slope on a plane.	0%	50%
Algebraic register to numerical register for the formula of a plane.	17%	58%
Verbal register to the algebraic register for a situation associated with a plane.	0%	67%
Geometric register to algebraic register for a physical plane presented using the 3D kit.	0%	33%

**Table 7.** Success rates for students in the control and experimental groups on tasks involving conversions between semiotic registers that the control group had not encountered previously.

Question	Control group (n = 32)	Experimental group (n = 36)
<div></div> <div><p>1. If <math>f</math> is represented by the above surface,</p><p>a. Draw the cross sections <math>x = 0</math> and <math>y = 0</math></p><p>b. Identify the signs of the following derivatives where <math>\vec{u}</math> is in the direction of <math>-\vec{i} - \vec{j}</math>. a. <math>f_x(2,2)</math> b. <math>f_y(2,-1)</math> c. <math>D_{\vec{u}}f(2, 2)</math></p><p>2. If <math>f(x,y) = \sin(x^2y)</math>, find formulas for the following: <math>f_x(x,y)</math> b. <math>f_y(x,y)</math></p></div>	53%	76%
	88%	83%

Question			Control group ( $n = 32$ )	Experimental group ( $n = 36$ )
	$x = 1$	$x = 3$	37%	61%
$y = 1$	3	5		
$y = 3$	7	4		

3. If the function  $f$  is represented by the above table:  
a. Find the best approximations for  $f_x(1,1)$  and  $f_y(1,1)$   
b. Find the formula for the tangent plane to  $f$  at the point  $(1,1,3)$  and use it to approximate  $f(1.1, 1.2)$

**Table 8.** The average scores on common examination questions for the experimental and control groups.

## 5. Discussion

Our operational approach uses the breadth of conversion and treatment capacity with associated registers of representation as an indication of conceptual understanding. So we begin by checking whether those that are able to navigate less-common treatments and conversions are likely to manifest greater understanding in other aspects of assessment.

**Table 4** shows a startling inability to navigate less-common treatments and conversions among students in the comparison group that were taught in a traditional stand-and-deliver manner as compared to students in the experimental group that were in an active learning environment where they were encouraged to explore and make sense of associations between registers of representation. The comparison group had an average success rate of less than 15% performing these treatments and conversions while the experimental group had an average success rate greater than 75%.

Our fundamental question was whether this sharp difference in the ability levels of these two populations with less-common treatments and conversions would be equally manifest with common treatments, conversions and problems. **Table 5** showed that with commonly seen treatments and conversions, the comparison group did far better but still had an average success rate of less than 40%. The experimental group, however, showed only a modest improvement moving from less-common to more-common transformations with an average success rate of 80%. The similar rate of success in the experimental group with both familiar and unfamiliar treatments and conversions would be consistent with simultaneous awareness of registers that we associate with conceptual understanding. While the comparison group did better, there was still a sharp and significant difference between the two populations with the experimental population achieving twice the rate of success than the comparison group.

Using standard classroom instruments with the populations of our first study, the greater capacity of the experimental group was manifest on common examination questions shown in **Table 5** where the experimental group averaged 63% and the comparison group averaged 37%. So the data from our first study would indicate that if we measure conceptual

understanding by assessing students' ability to perform familiar and unfamiliar treatments and conversions among registers of representation associated with a mathematical concept, our assessment is not inconsistent with assessment of students' understanding using traditional calculus problems.

In our second study, **Table 7** shows a startling contrast between the experimental and comparison groups with the comparison group obtaining a success rate of 4% with treatments and conversions and the experimental group obtaining a success rate of 58%. So, as with our first study, the experimental population demonstrated a far greater capacity to perform treatments and conversions among registers of representation associated with the given topic.

**Table 8** shows the results of the two populations in our second study using traditional calculus assessment instruments. With the most procedural question (question 2 on finding derivatives), the comparison population did slightly (but not significantly) better than the experimental population. With the questions that would be considered less procedural (questions 1 and 3), the comparison population averaged 45% and the experimental population averaged 68.5%. These results would once again reinforce that if we use students' ability to perform a broad range of treatments and conversions with registers of representations associated with a concept to assess their conceptual understanding, these results will not be inconsistent with traditional assessment instruments when the traditional instruments are associated with conceptual understanding. Interestingly, we did not find this to be the case, in this particular instance, with procedural problems and processes.

While we have focused on the role of harnessing various registers of representation in understanding concepts as suggested by Duval [1], it is worth noting that students that were successful performing treatments and conversions were over 11 times more likely (34 occurrences for the experimental group vs. three occurrences for the comparison group) to use intermediary registers (that were neither the source nor the target registers of the conversion) than students that were unsuccessful. For example, if one is asked to perform a conversion from a symbolic register (formula) to a geometric register (graph), a numerical register (table or set of coordinate points) is a reasonable intermediary register that is neither the source nor the target of the conversion. McGee and colleagues [2, 3] found the spontaneous use of intermediary registers in problem solving to be associated with student success. So this multi-register approach to understanding is both helpful in terms of providing registers from which one can glean the commonality and to provide intermediary registers which can be useful to solve problems and perform treatments and conversions.

The implications of this operational approach are twofold. As Duval [1] indicated that conceptual understanding lies in understanding the commonality of registers of representations, these studies of our operational approach provide data and insight into the importance that a broad range of treatments and conversions has in student understanding and provide an applied format to further research with this and associated concepts. The second is that this approach provides a context for conceptual understanding that encourages teachers and professors to harness various registers of representation simultaneously when promoting students' understanding.

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